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On the existence of solution for a singular Riemann-Liouville fractional differential system by using measure of non-compactness

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Abstract

We investigate the existence of solution for a singular fractional differential system with Riemann-Liouville integral boundary conditions by using the measure of non-compactness.

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1. Introduction

Many works have been published on the existence of solutions for different singular fractional differential systems (see for example, [1], [6] and [7]-[9]). In 2012, the existence of positive solution for the singular equation $D^{\alpha}u(t) + f(t, u(t)) = 0$ with boundary conditions $u(1) = 0$ and $[I^{2-\alpha}u(t)]'_{t=0} = 0$ investigated, where $t \in [0, 1]$, $\alpha \in (1, 2)$ and D^{α} is the Riemann-Liouville fractional derivative ([3]). In 2013, the existence of positive solution for the system $D^{\alpha}u_i(t) + f_i(t, u_1(t), u_2(t)) = 0$ $(i = 1, 2)$ with boundary conditions $u_1(0) = u_1'(0) = 0$, $u_1(1) = \int_0^1 u_1(t)d\eta(t)$, $u_2(0) = u_2'(0) = 0$ and $u_2(1) = \int_0^1 u_2(t)d\eta(t)$ investigated, where $t \in [0, 1]$, $\alpha \in (2, 3]$, $f_1, f_2 \in C([0, 1] \times [0, \infty) \times [0, \infty), \mathbb{R})$, D^{α} is the Riemann-Liouville fractional derivative and $\int_0^1 u_i(t)d\eta(t)$ denotes the Riemann-Stieltjes integral ([10]). In 2014, the existence of solution for the problem $D^{\alpha}u(t) + f(t, u(t)) = 0$ with boundary conditions $u'(0) = \ldots = u^{(n-1)} = 0$ and $u(1) = \int_0^1 u(s)d\mu(s)$ investigated, where $n \geq 2$, $\alpha \in (n - 1, n)$, μ is bounded variation, f may have singularity at $t = 0$ and $\int_0^1 d\mu(s) < 1$ ([11]). By using the main idea of the above papers, we investigate the existence of solution for the singular system

$$\begin{cases} D^{\alpha_1}x(t) + f_1(t, x(t), y(t)) = 0, \\ D^{\alpha_2}y(t) + f_2(t, x(t), y(t)) = 0, \end{cases} \tag{1.1}$$

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with boundary conditions $x(0) = y(0) = 0$, $x^{(i)}(0) = y^{(i)}(0) = 0$ for $i = 2, \dots, n-1$, $x(1) = [I^{p_1}(h_1(t)x(t))]_{t=1}$ and $y(1) = [I^{p_2}(h_2(t)y(t))]_{t=1}$, where $n \geq 3$, $\alpha_1, \alpha_2 \in (n, n+1]$ $p_1, p_2 \geq 1$, $f_1, f_2 \in C((0,1] \times [0,\infty) \times [0,\infty))$, f_1, f_2 are singular at $t = 0$, $h_1, h_2 \in L^1[0,1]$ are non-negative and $[I^{p_j}(h_j(t))]_{t=1} \in [0, \frac{1}{2})$ for $j = 1, 2$ and f_1, f_2 satisfy the local Caratheodory condition on $(0,1] \times (0,\infty) \times (0,\infty)$.
We say that f satisfies the local Caratheodory condition on $[0,1] \times (0,\infty) \times (0,\infty)$ and denote it by $f \in Car([0,1] \times (0,\infty) \times (0,\infty))$, whenever the function $f(.,x,y) : [0,1] \to \mathbb{R}$ is measurable for all $(x,y) \in (0,\infty) \times (0,\infty)$, the function $f(t,.,.) : (0,\infty) \times (0,\infty) \to \mathbb{R}$ is continuous for almost all $t \in [0,1]$ and for each compact subset κ of $(0,\infty) \times (0,\infty)$ there exists a function $\varphi_\kappa \in L^1[0,1]$ such that $|f(t,x,y)| \leq \varphi_\kappa(t)$ for almost all $t \in [0,1]$ and all $(x,y) \in \kappa$.

Definition 1.1 ([5]). The Riemann-Liouville integral of order p for a function $f : (0,\infty) \to \mathbb{R}$ is defined by

$$I^p f(t) = \frac{1}{\Gamma(p)} \int_0^t (t-s)^{p-1} f(s) ds,$$

whenever the right-hand side is pointwise defined on $(0,\infty)$.

Definition 1.2 ([5]). The Caputo fractional derivative of order $\alpha > 0$ for a function $f : (a,\infty) \to \mathbb{R}$ is defined by

$${}^cD^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{f^n(s)}{(t-s)^{\alpha+1-n}} ds,$$

where $n = [\alpha] + 1$.

One can check that $\int_0^t (t-s)^{\alpha-1} s^\beta ds = B(\beta+1, \alpha) t^{\alpha+\beta}$ for all $\beta > 0$ and $\alpha > -1$, where $B(\beta, \alpha) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$ ([9]).
Suppose that X is a Banach space and m_X denotes the collection of all bounded subset of X.

Definition 1.3 ([4]). A function $\mu : m_X \to [0,\infty)$ is called a measure of non-compactness, if it satisfies the following conditions:

(1) $\mu(Q) = 0$ if and only if Q is relatively compact.
(2) $\mu(Q_1) \leq \mu(Q_2)$ whenever $Q_1 \subset Q_2$,
(3) $\mu(\overline{conv(Q)}) = \mu(Q)$.
(4) $\mu(Q_1 \cup Q_2) = \max\{\mu(Q_1), \mu(Q_2)\}$.
(5) $\mu(Q_1 + Q_2) \leq \mu(Q_1) + \mu(Q_2)$.
(6) $\mu(\lambda Q) = |\lambda| \mu(Q)$ for all scalar λ,

for $Q \in m_X$.

The Kuratowski measure of non-compactness of Q is denoted by $K(Q)$ and defined by

$$K(Q) = \inf\{\epsilon > 0 : Q \subset \bigcup_{i=1}^n S_i \text{ and } diam(S_i) < \epsilon \text{ for } i = 1, \dots, n\},$$

([4]). If Q is unbounded, then put $K(Q) = \infty$ and $K(Q) = 0$ whenever $Q = \emptyset$ ([4]). Note that, $K(Q) \leq diam(Q)$ for all $Q \in m_X$ ([4]).

Lemma 1.4 ([9]). *Suppose that $0 < n-1 \leq \alpha < n$ and $x \in C[0,1] \cap L^1[0,1]$. Then, we have*

$$I^\alpha D^\alpha x(t) = x(t) + \sum_{i=0}^{n-1} c_i t^i,$$

for some real constants $c_0, \dots, c_{n-1}$.

Theorem 1.5 ([2]). *Let C be a nonempty, bounded, closed and convex subset of a Banach space X, K the Kuratowski measure of non-compactness on X and $T : C \to C$ a continuous operator. If there exists a constant $c \in [0, 1)$ such that $K(T(Q)) \leq c.K(Q)$ for all $Q \subset C$, then T has a fixed point.*

Now, we provide our first key result.

Lemma 1.6. *Let $y \in L^1[0, 1]$, $p \geq 1$ and $\alpha \geq 3$. Then $x(t) = \int_0^1 G(t, s)y(s)ds$ is a solution for the problem $D^\alpha x(t) + y(t) = 0$ with boundary conditions $x(0) = x^{(2)}(0) = \cdots = x^{(n-1)}(0) = 0$ and $x(1) = [I^p(h(t)x(t))]_{t=1}$, where $h \in L^1[0, 1]$,*

$$G(t, s) = G_1(t, s) + \frac{t}{\mu(p)} \int_0^1 (1 - t)^{p-1} h(t)G_1(t, s)dt,$$

$G_1(t, s) = \frac{t(1-s)^{\alpha-1}}{\Gamma(\alpha)}$ whenever $0 \leq t \leq s \leq 1$, $G_1(t, s) = \frac{t(1-s)^{\alpha-1}-(t-s)^{\alpha-1}}{\Gamma(\alpha)}$ whenever $0 \leq s \leq t \leq 1$ and $\mu(p) = \Gamma(p) - \int_0^1 t(1 - t)^{p-1}h(t)dt$.

Proof. By using Lemma 1.4, we have $x(t) = -\frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha-1} y(s)ds + c_0 + c_1 t + \ldots + c_n t^n$ for some real constants. Since $x(0) = x^{(i)}(0) = 0$ for $i \geq 2$, we get $c_0 = c_2 = c_3 = \ldots = c_n = 0$. Thus, $x(t) = -\frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha-1} y(s)ds + c_1 t$. Since $[I^p(h(t)x(t))]_{t=1} = \frac{1}{\Gamma(p)} \int_0^1 (1 - s)^{p-1} h(s)ds$, by using the boundary condition at $t = 1$ we obtain

$$-\frac{1}{\Gamma(\alpha)} \int_0^1 (1 - s)^{\alpha-1} y(s)ds + c_1 = \frac{1}{\Gamma(p)} \int_0^1 (1 - s)^{p-1} h(s)ds$$

and so $c_1 = \frac{1}{\Gamma(p)} \int_0^1 (1 - s)^{p-1} h(s)x(s)ds + \frac{1}{\Gamma(\alpha)} \int_0^1 (1 - s)^{\alpha-1} y(s)ds$. Thus,

$$\begin{aligned} x(t) &= -\frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha-1} y(s)ds + \frac{t}{\Gamma(\alpha)} \int_0^1 (1 - s)^{\alpha-1} y(s)ds + \frac{t}{\Gamma(p)} \int_0^1 (1 - s)^{p-1} h(s)ds \\ &= \int_0^1 G_1(t, s)y(s)ds + t[I^p(h(t)x(t))]_{t=1}, \end{aligned}$$

which implies

$$\begin{aligned} [I^p(h(t)x(t))]_{t=1} &= \frac{1}{\Gamma(p)} \int_0^1 \int_0^1 (1 - t)^{p-1} h(t)G_1(t, s)y(s)dsdt \\ &\quad + \frac{1}{\Gamma(p)} \int_0^1 (1 - t)^{p-1} th(t)[I^p(h(t)x(t))]_{t=1} dt. \end{aligned}$$

Since $[I^p(h(t)x(t))]_{t=1} = \int_0^1 [I^p(h(t)x(t))]_{t=1} dt$, we get

$$\int_0^1 (1 - \frac{1}{\Gamma(p)} (1 - t)^{p-1} th(t))[I^p(h(t)x(t))]_{t=1} dt = \frac{1}{\Gamma(p)} \int_0^1 (1 - t)^{p-1} h(t) \int_0^1 G_1(t, s)y(s)dsdt.$$

Hence,

$$[I^p(h(t)x(t))]_{t=1}(1 - \frac{1}{\Gamma(p)} \int_0^1 (1 - t)^{p-1} th(t)dt) = \frac{1}{\Gamma(p)} \int_0^1 (1 - t)^{p-1} h(t) \int_0^1 G_1(t, s)y(s)dsdt$$

and so $[I^p(h(t)x(t))]_{t=1} = \frac{\int_0^1 (1-t)^{p-1} h(t) \int_0^1 G_1(t,s)y(s)ds\ dt}{\Gamma(p)(1-\frac{1}{\Gamma(p)} \int_0^1 (1-t)^{p-1} th(t)dt)}$. This implies that

$$x(t) = \int_0^1 G_1(t, s)y(s)ds + \frac{t \int_0^1 (1 - t)^{p-1} h(t) \int_0^1 G_1(t, s)y(s)ds\ dt}{\Gamma(p) - \int_0^1 (1 - t)^{p-1} th(t)dt} = \int_0^1 G(t, s)y(s)ds,$$

where $G(t, s) = G_1(t, s) + \frac{t}{\mu(p)} \int_0^1 (1 - t)^{p-1} h(t)G_1(t, s)dt$. □

By using some calculations, one can see that $G(t,s) \geq 0$ and $G(t,s) \leq \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha-1)}(1+\Lambda(p))$ for all $t,s \in [0,1]$, where $\Lambda(p) = \frac{1}{\mu(p)} \int_0^1 (1-t)^{p-1} h(t)dt$.
Now for each natural number n, consider the map $f_{i,n}(t,x,y) = f_i(t,\chi_n(x),\chi_n(y))$, where $\chi_n(x) = x$ whenever $x \geq \frac{1}{n}$ and $\chi_n(x) = \frac{1}{n}$ whenever $x < \frac{1}{n}$. Here, we first investigate the regular system

$$\begin{cases} D^{\alpha_1} x + f_{1,n}(t,x,y) = 0, \\ \\ D^{\alpha_2} y + f_{2,n}(t,x,y) = 0, \end{cases} \tag{1.2}$$

with same boundary conditions in the problem (1). For each $n \geq 1$ and $i = 1,2$, consider the map $T_{n,i}(x,y)(t) = \int_0^1 G_{\alpha_i}(t,s) f_{n,i}(s,x(s),y(s))ds$, where $G_{\alpha_i}(t,s)$ is the Green function in Lemma 1.6 which replaced α and p by α_i and p_i. Put

$$T_n(x,y)(t) = (T_{n,1}(x,y)(t), T_{n,2}(x,y)(t))$$

and

$$\|T_n(x,y)(t)\|_* = \max\{T_{n,1}(x,y)(t), T_{n,2}(x,y)(t)\}.$$

Since $f_1, f_2 \in Car([0,1] \times \mathbb{R}^2)$, it is easy to check that $f_{n,1}, f_{n,2} \in Car([0,1] \times \mathbb{R}^2)$ for all n and so there exist $\varphi_1, \varphi_2 \in L^1[0,1]$ such that $|f_{n,i}(t,x(t),y(t))| \leq \varphi_i(t)$ for almost all $t \in [0,1]$, $n \geq 1$ and $i = 1,2$. Now, consider the set $C = \{(x,y) \in C[0,1] \times C[0,1] : \ \|(x,y)\|_* \leq \|\varphi\|_\infty^*\}$, where $\|\varphi\|_\infty^* = \max\{\|\varphi_1\|_\infty, \|\varphi_2\|_\infty\}$. Note that, C is closed, bounded and convex.

Lemma 1.7. *For each $n \geq 1$, T_n maps C into C and is equi-continuous on each bounded subset of $C([0,1],\mathbb{R}) \times C([0,1],\mathbb{R})$.*

Proof. Let $n \geq 1$ and $(x,y) \in C$ be given. First, we show that T_n maps C into C. Note that,

$$T_{n,i}(x,y)(t) \leq \int_0^1 \frac{(1-s)^{\alpha_i-1}}{\Gamma(\alpha_i-1)}(1+\frac{1}{\mu(p_i)}\int_0^1 (1-t)^{p_i-1}h_i(t)dt) f_{n,i}(s,x(s),y(s))ds,$$

for $i = 1,2$. Hence,

$$T_{n,i}(x,y)(t) \leq \int_0^1 \frac{(1-s)^{\alpha_i-1}}{\Gamma(\alpha_i-1)}(1+\frac{1}{\mu(p_i)}\int_0^1 (1-t)^{p_i-1}h_i(t)dt)\varphi_i(s)ds, \tag{1.3}$$

for $i = 1,2$. Since $[I^{p_i}(h_i(t))]_{t=1} \in [0,\frac{1}{2})$, $\frac{1}{\Gamma(p_i)}\int_0^1 (1-t)^{p_i-1}h_i(t)dt \in [0,\frac{1}{2})$. Also, we have

$$\frac{1}{\Gamma(p_i)}\int_0^1 (1-t)^{p_i-1} t h_i(t)dt \leq \frac{1}{\Gamma(p_i)}\int_0^1 (1-t)^{p_i-1}h_i(t)dt.$$

Thus, $\frac{1}{\Gamma(p_i)}\int_0^1 (1-t)^{p_i-1} t h_i(t)dt \in [0,\frac{1}{2})$ and so $1 - \frac{1}{\Gamma(p_i)}\int_0^1 (1-t)^{p_i-1} t h_i(t)dt) \in [0,\frac{1}{2})$. This implies that

$$\begin{aligned} \frac{1}{\mu(p_i)}\int_0^1 (1-t)^{p_i-1}h_i(t)dt &= \frac{\int_0^1 (1-t)^{p_i-1}h_i(t)dt}{\Gamma(p_i) - \int_0^1 (1-t)^{p_i-1} t h_i(t)dt} \\ &= \frac{\frac{1}{\Gamma(p_i)}\int_0^1 (1-t)^{p_i-1}h_i(t)dt}{1 - \frac{1}{\Gamma(p_i)}\int_0^1 (1-t)^{p_i-1} t h_i(t)dt} \in [0,1) \end{aligned}$$

and so $1 + \frac{1}{\mu(p_i)}\int_0^1 (1-t)^{p_i-1}h_i(t)dt \leq 2$. By using this inequality and (1.3), we get

$$\begin{aligned} T_{n,i}(x,y)(t) &\leq \frac{2}{\Gamma(\alpha_i-1)}\int_0^1 (1-s)^{\alpha_i-1}\varphi_i(s)ds \leq \frac{2\|\varphi_i\|_\infty}{\Gamma(\alpha_i-1\ )}\int_0^1 (1-s)^{\alpha_i-1}ds \\ &= \frac{2}{\Gamma(\alpha_i)}\|\varphi_i\|_\infty \leq \|\varphi_i\|_\infty \leq \|\varphi\|_\infty^* \end{aligned}$$

and so $\|T_n(x,y)\|_* \leq \|\varphi\|_\infty^*$. Now, we show that T is equi-continuous on each bounded subset F of $C([0,1],\mathbb{R}) \times C([0,1],\mathbb{R})$. Let $\{(x_k,y_k)\}_{k=1}^\infty$ be a bounded sequence in F and $0 \leq t_1 < t_2 \leq 1$. Then, we have

$$\begin{aligned}|T_{i,n}(x_k,y_k)(t_2) \;&-\; T_{i,n}(x_k,y_k)(t_1)| \leq \frac{1}{\Gamma(\alpha_i)}[\int_0^{t_1}[(t_2-s)^{\alpha_i-1}-(t_1-s)^{\alpha_i-1}] \\ &\times\; f_{n,i}(s,x_k(s),y_k(s))ds + \int_{t_1}^{t_2}(t_2-s)^{\alpha_i-1}f_{n,i}(s,x_k(s),y_k(s))]ds \\ &+\; (t_2-t_1)\int_0^1[\frac{(1-s)^{\alpha_i-1}}{\Gamma(\alpha_i)}+G_{2,i}(s)]f_{n,i}(s,x_k(s),y_k(s))ds \\ &\leq\; \frac{1}{\Gamma(\alpha_i)}[\int_0^1[(t_2-s)^{\alpha_i-1}-(t_1-s)^{\alpha_i-1}]\varphi_i(s)ds+(t_2-t_1)^{\alpha_i-1}\|\varphi_i\|_1 \\ &+\; (t_2-t_1)\|\varphi_i\|_1(\frac{1}{\Gamma(\alpha_i)}+\Lambda_i(p_i)),\end{aligned}$$

where for $i=1,2$, $\Lambda_i(p_i) = \frac{1}{\mu(p_i)}\int_0^1(1-t)^{p_i-1}h_i(t)dt$, $G_{2,i}(s) = \frac{1}{\mu(p_i)}\int_0^1(1-t)^{p_i-1}h_i(t)G_{1,i}(t,s)dt$ and $G_{1,i}(t,s)$ is defined as $G_1(t,s)$ by replacing α_i instead α. Let $0<\epsilon<1$, $0 \leq t_1 < t_2 \leq 1$ and $0 \leq s \leq t_1$. Choose $\delta > 0$ such that $t_1 - t_2 < \delta$ implies $(t_2-s)^{\alpha_i-1}-(t_1-s)^{\alpha_i-1} < \epsilon$ for $i=1,2$. Let $k \geq 1$ and $0 \leq t_1 < t_2 \leq 1$ with $t_1 - t_2 < \min\{\delta,\epsilon\}$ be given. Then, we have

$$|T_{i,n}(x_k,y_k)(t_2)-T_{i,n}(x_k,y_k)(t_1)| \leq \epsilon\|\varphi_i\|_1(\frac{3}{\Gamma(\alpha_i)}+\Lambda_i(p_i))$$

and so $\lim_{t_2\to t_1}\|T_n(x_k,y_k)(t_2)-T_n(x_k,y_k)(t_1)\|_*=0$. Also, we have

$$\begin{aligned}\|T_n(x_k,y_k)(t)\|_* &\leq \max\{\int_0^1\frac{(1-s)^{\alpha_1-1}}{\Gamma(\alpha_1-1)}(1+G_{2,1}(s))\varphi_1(s)ds, \int_0^1\frac{(1-s)^{\alpha_2-1}}{\Gamma(\alpha_2-1)}(1+G_{2,2}(s))\varphi_2(s)ds\} \\ &\leq \max\{\frac{\|\varphi_1\|_1(1+\Lambda_1(p_1))}{\Gamma(\alpha_1-1)}, \frac{\|\varphi_2\|_1(1+\Lambda_2(p_2))}{\Gamma(\alpha_2-1)}\}.\end{aligned}$$

Let $\{(x_k,y_k)\}_{k=1}^\infty$ be sequence in F and $(x_k,y_k) \to (x,y)$. Hence, $x_k \to x$, $y_k \to y$. Note that,

$$\begin{aligned}\|T_n(x_k,y_k)(t)-T_n(x,y)(t)\|_* \leq \max\{&\int_0^1 G_{\alpha_1}(t,s)|f_{1,n}(s,x_k(s),y_k(s))-f_{1,n}(s,x(s),y(s))|ds, \\ &\int_0^1 G_{\alpha_2}(t,s)|f_{2,n}(s,x_k(s),y_k(s))-f_{2,n}(s,x(s),y(s))|ds\} \\ &\leq 2\|\varphi\|_1^*(\frac{1}{\Gamma(\alpha_m-1)}(1+\Lambda_M)),\end{aligned}$$

where $\alpha_m = \min\{\alpha_1,\alpha_2\}$ and $\Lambda_M = \max\{\Lambda_1(p_1),\Lambda_2(p_2)\}$. Since

$$|f_{i,n}(s,x_k(s),y_k(s))-f_{i,n}(s,x(s),y(s))| \to 0,$$

for $i=1,2$, by using the Lebesgue dominated convergence Theorem, we conclude that T_n is equi-continuous on F for all n. □

2. Main Results

Theorem 2.1. *Let $n \geq 3$, $f_1,f_2 \in Car([0,1]\times(0,\infty)^2)$, $\alpha_1,\alpha_2 \in (n,n+1]$, $p_1,p_2 \geq 1$, $h_1,h_2 \in L^1[0,1]$ be nonnegative functions and $[I^{p_1}(h_1(t))]_{t=1}, [I^{p_1}(h_1(t))]_{t=1} \in [0,\frac{1}{2})$. Suppose that there exist $g_1,g_2 \in L^1([0,1])$ such that $\|g_i\|_1 < \frac{\Gamma(\alpha_i-1)}{2}$ for almost all $t \in [0,1]$ and $i=1,2$. Assume that $K(f_i(t,Q)) \leq g_i(t)K(Q)$ for all* *bounded subset Q of $C[0,1]\times C[0,1]$ and $i=1,2$, where K is the Kuratowski measure of non-compactness. Then, for each $n\geq 1$ the system*

$$\begin{cases} D^{\alpha_1}x + f_{1,n}(t,x,y)=0, \\ D^{\alpha_2}y + f_{2,n}(t,x,y)=0, \end{cases}$$

with boundary conditions $x(0)=y(0)=0$, $x^{(i)}(0)=y^{(i)}(0)=0$ for $i=2,\dots,n-1$, $x(1)=[I^{p_1}(h_1(t)x(t))]_{t=1}$ and $y(1)=[I^{p_2}(h_2(t)y(t))]_{t=1}$ has a solution.

Proof. Let Q be a bounded subset $C[0,1]\times C[0,1]$, $n\in\mathbb{N}$ and $i=1$ or 2. Choose bounded sets $F,S\subset C[0,1]$ such that $Q=(F,S)$. Put $F_1:=\{x\in F\ :\ x\geq\frac{1}{n}\}$ and $S_1:=\{x\in S: x\geq\frac{1}{n}\}$. Then, we have

$$\begin{aligned} K(f_{i,n}(t,Q)) &= K(f_{i,n}(t,F,S))=K(f_i(t,\chi_n(F),\chi_n(S)))\leq K(\chi_n(F),\chi_n(S)) \\ &= K(F_1\cup\{\frac{1}{n}\},S_1\cup\{\frac{1}{n}\})=K((F_1,S_1)\cup(\frac{1}{n},S_1)\cup(F_1,\frac{1}{n})) \\ &= \max\{K(F_1,S_1),K(S_1,\frac{1}{n}),K(F_1,\frac{1}{n})\}. \end{aligned}$$

If $K(S_1)=\rho$, then there exist $W_i\subset C[0,1]$ and $m\in\mathbb{N}$ such that $S_1\subset\bigcup_{i=1}^m W_i$ and $diam(W_i)<\rho$. Hence, $(\frac{1}{n},S_1)\subset\bigcup_{i=1}^m(\frac{1}{n},W_i)$,

$$diam(a,W_i)=\sup_{\xi,\eta\in W_i}\|(\frac{1}{n},\xi)-(\frac{1}{n},\eta)\|_*=\sup_{\xi,\eta\in E_i}|\xi-\eta|=diam(W_i),$$

and $K(\frac{1}{n},S_1)\leq K(S_1)$. By using a similar method, we conclude that $K(S_1)\leq K(\frac{1}{n},S_1)$. Thus, $K(S_1)=K(\frac{1}{n},S_1)$ and $K(F_1)=K(F_1,\frac{1}{n})$. Thus, there exist $m_0\in\mathbb{N}$ and $(E_i,H_i)\subset C[0,1]\times C[0,1]$ such that $(F_1,S_1)\subset\bigcup_{i=1}^{m_0}(E_i,H_i)$ and $diam(E_i,H_i)\leq\rho_0$ whenever $K(F_1,S_1)=\rho_0$. This implies that

$$\sup\{\|(e,h)-(e',h')\|_*:(e,h),(e',h')\in(E_i,H_i)\}\leq\rho_0$$

and so

$$\sup\{\max\{|e-e'|,|h-h'|\}\ :\ e,e'\in E_i,\ h,h'\in H_i\}\leq\rho_0.$$

Hence, $\sup_{e,e'\in E_i}|e-e'|\leq\rho_0$ and $\sup_{h,h'\in H_i}|h-h'|\leq\rho_0$. Thus, $F_1\subset\bigcup_{i=1}^{m_0}E_i$ with $diam(E_i)\leq\rho_0$ and $S_1\subset\bigcup_{i=1}^{m_0}H_i$ with $diam(H_i)\leq\rho_0$ for all i. This implies that $K(F_1)\leq K(F_1,S_1)$ and $K(S_1)\leq K(F_1,S_1)$. Hence, $\max\{K(F_1,S_1),K(\frac{1}{n},S_1),K(F_1,\frac{1}{n})\}=K(F_1,S_1)$ and so

$$K(f_{i,n}(t,Q))\leq g_i(t)K(F_1,S_1)\leq g_i(t)K(Q)$$

for all i. Also, we have $K(T_n(Q))=K(\int_0^1 G_{\alpha_1}(t,s)f_{1,n}(s,Q)ds,\int_0^1 G_{\alpha_2}(t,s)f_{2,n}(s,Q)ds)$. For each $s\in[0,1]$, $n\in\mathbb{N}$ and $i=1,2$, put $\rho_i(s):=K(f_{i,n}(s,Q))\leq g_i(s)K(Q)$. Choose a natural number k_0 and bounded sets $U_{i,j}\subset C[0,1]\times C[0,1]$ $(i=1,2)$ such that $f_{i,n}(s,Q)\subseteq\bigcup_{j=1}^{k_0}U_{i,j}$. Then, we have $diam(U_{i,j})\leq\rho_i(s)\leq g_i(s)K(Q)$ and

$$G_{\alpha_i}(t,s)f_{i,n}(s,Q)\subseteq\int_0^1\bigcup_{j=1}^{k_0}\theta_i(s)U_{i,j}ds=\bigcup_{j=1}^{k_0}\int_0^1\theta_i(s)U_{i,j}ds,$$

for $i=1,2$, where $\theta_i(s)=\frac{(1-s)^{\alpha_i}}{\Gamma(\alpha_i-1)}(1+\Lambda_i)$ and $\int_0^1\theta_i(s)U_{i,j}ds=\{\int_0^1\theta_i(s)u(s)ds: u\in U_{i,j}\}$. Thus,

$$\begin{aligned} diam(\int_0^1\theta_i(s)U_{i,j}ds) &= \sup_{u,u'\in U_{i,j}}|\int_0^1\theta_i(s)u(s)ds-\int_0^1\theta_i(s)u'(s)ds| \\ &= \sup_{u,u'\in U_{i,j}}|\int_0^1\theta_i(s)|u(s)-u'(s)|ds\leq\int_0^1\theta_i(s)diam(U_{i,j})ds\leq\int_0^1\theta_i(s)\rho_i(s)ds \end{aligned}$$

and so

$$K(\int_0^1 G_{\alpha_i}(t,s)f_{i,n}(s,Q)ds) \leq \int_0^1 \theta_i(s)K(f_{i,n}(s,Q))ds \leq \int_0^1 \theta_i(s)g_i(s)K(Q)ds$$
$$\leq K(Q)\|\theta_i\|_\infty \|g_i\|_1 \leq k_i K(Q),$$

where $k_i = \|\theta_i\|_\infty \|g_i\|_1$. It is easy to check that $k_i \in [0,1)$ for all $i = 1,2$. By using last inequality, we get $\max_{i=1,2}\{K(\int_0^1 G_{\alpha_i}(t,s)f_{i,n}(s,Q)ds)\} \leq kK(Q)$, where $k = \max\{k_1, k_2\}$.
Now, we show that $K(A,B) = \max\{K(A), K(B)\}$. As it proved in first part, $K(A) \leq K(A,B)$ and $K(B) \leq K(A,B)$, where $A, B \subset X := C[0,1] \times C[0,1]$ are bounded sets and $\|(.,.)\|_{**}$ defined on X^2 by $\|(e_1,e_2)\|_{**} = \max\{\|e_1\|_*, \|e_2\|_*\}$. It is known that $(X^2, \|(.,.)\|_{**})$ is a Banach space. Let $K(A) := r_1$, $K(B) := r_2$ and $r := \max\{r_1, r_2\}$. Choose natural numbers n_1 and n_2 such that $A \subset \bigcup_{i=1}^{n_1} Z_i$ and $B \subset \bigcup_{j=1}^{n_2} V_j$, where $Z_i, V_j \subset X, diam(Z_i) < r_1$ and $diam(V_j) < r_2$ for $i = 1, \ldots, n_1$ and $j = 1, \ldots, n_2$. Without less of generality suppose that $n_1 \geq n_2$ (in other case the proof is similar). Put $V_{n_2+1} = V_{n_2+2} = \ldots = V_{n_1} := V_{n_2}$. Then, $(A,B) \subset \bigcup_{i=1}^{n_1}(Z_i, V_i)$ and for each $i = 1, \ldots, n_1$, we have

$$diam(Z_i,V_i) = \sup_{z,z' \in Z_i, v,v' \in V_i} \|(z,v)-(z',v')\|_{**} = \sup_{z,z' \in Z_i, v,v' \in V_i} \|(z-z', v-v')\|_{**}$$
$$= \sup_{z,z' \in Z_i, v,v' \in V_i} \{\max\{\|(z-z')\|_*, \|(v-v')\|_*\}\} \leq \max\{r_1,r_2\} = r.$$

Hence, $K(A,B) \leq \max\{K(A), K(B)\}$ and so $K(A,B) = \max\{K(A), K(B)\}$. Thus,

$$K(T_n(Q)) = K(\int_0^1 G_{\alpha_1}(t,s)f_{1,n}(s,Q)ds, \int_0^1 G_{\alpha_2}(t,s)f_{2,n}(s,Q)ds)$$
$$= \max_{i=1,2}\{\int_0^1 G_{\alpha_i}(t,s)f_{i,n}(s,Q)ds\} \leq kK(Q).$$

By using the Darbo's fixed point theorem, T_n has a fixed point in C for all n. This implies that the system has a solution $(x_n, y_n) \in C$, that is,

$$x_n(t) = \int_0^1 G_{\alpha_1}(t,s)f_{1,n}(s, x_n(s), y_n(s))ds$$

and

$$y_n(t) = \int_0^1 G_{\alpha_2}(t,s)f_{2,n}(s, x_n(s), y_n(s))ds.$$

□

Now, we provide our main result.

Theorem 2.2. *Let $n \geq 3$, $f_1, f_2 \in Car([0,1] \times (0,\infty)^2)$, $\alpha_1, \alpha_2 \in (n, n+1]$, $p_1, p_2 \geq 1$ and $h_1, h_2 \in L^1[0,1]$ be non-negative functions with $[I^{p_1}(h_1(t))]_{t=1}, [I^{p_1}(h_1(t))]_{t=1} \in [0, \frac{1}{2})$. Suppose that there exist $g_1, g_2 \in L^1([0,1])$ such that $\|g_i\|_1 < \frac{\Gamma(\alpha_i - 1)}{2}$ and $K(f_i(t,Q)) \leq g_i(t)K(Q)$ for $i = 1,2$, where $K(Q)$ is the Kuratowski measure of non-compactness of a bounded set Q. Then the singular system*

$$\begin{cases} D^{\alpha_1}x + f_1(t,x,y) = 0, \\ \\ D^{\alpha_2}y + f_2(t,x,y) = 0, \end{cases}$$

with boundary conditions $x(0) = y(0) = 0$, $x^{(i)}(0) = y^{(i)}(0) = 0$ for $i = 2, \ldots, n-1$, $x(1) = [I^{p_1}(h_1(t)x(t))]_{t=1}$ and $y(1) = [I^{p_2}(h_2(t)y(t))]_{t=1}$ has a solution in C.

Proof. By using Theorem 2.1, the problem (1.2) has a solution $(x_n, y_n) \in C$ for all n. Since C is closed, there is $(x, y) \in C$ such that $\lim_{n\to\infty}(x_n, y_n) = (x, y)$. It is easy to check that (x, y) satisfies the boundary condition of the problem (1.1). Also, one can check that $\lim_{n\to\infty} f_{i,n}(t, x_n(t), y_n(t)) = f_i(t, x(t), y(t))$ for almost all $t \in [0, 1]$ and $i = 1, 2$. On the other hand, we have $G_{\alpha_i}(t, s)f_{i,n}(s, x_n(s), y_n(s)) \leq \frac{1+\Lambda_i(p_i)}{\Gamma(\alpha_i-1)}\varphi_i(s)$ for all n, $i = 1, 2$ and almost all $(t, s) \in [0, 1] \times [0, 1]$. Now by using the Lebesgue dominated convergence theorem, we obtain $x(t) = \int_0^1 G_{\alpha_1}(t, s)f_{1,n}(s, x(s), y(s))ds$ and $y(t) = \int_0^1 G_{\alpha_2}(t, s)f_{2,n}(s, x(s), y(s))ds$. This implies that, (x, y) is a solution for the problem (1.1). □

Here, we provide an example to illustrate our main result.

Example 2.3. Consider the singular fractional system

$$\begin{cases} D^{\frac{7}{2}}x(t) + \frac{0.3}{t^{\frac{1}{2}}}(\frac{1}{2}x(t) + \frac{1}{3}y(t)) = 0, \\ D^{\frac{10}{3}}x(t) + \frac{0.2}{t^{\frac{1}{3}}}(\frac{1}{4}x(t) + \frac{3}{5}y(t)) = 0, \end{cases} \tag{2.1}$$

with boundary conditions $x(0) = y(0) = x'(0) = y'(0) = x''(0) = y''(0) = 0$, $x(1) = [I^{\frac{3}{2}}(tx(t))]_{t=1}$ and $y(1) = [I^{\frac{5}{2}}(t^{\frac{1}{2}}y(t))]_{t=1}$. Now, consider the maps $f_1(t, x, y) = \frac{0.3}{t^{\frac{1}{2}}}(\frac{1}{2}x + \frac{1}{3}y)$, $f_2(t, x, y) = \frac{0.2}{t^{\frac{1}{3}}}(\frac{1}{4}x + \frac{3}{5}y)$, $g_1(t) = \frac{0.3}{t^{\frac{1}{2}}}$, $g_2(t) = \frac{0.2}{t^{\frac{1}{3}}}$, $u(x, y) = \frac{1}{2}x + \frac{1}{3}y$ and $v(x, y) = \frac{1}{4}x + \frac{3}{5}y$. Put $\alpha_1 = \frac{7}{2}$, $\alpha_2 = \frac{10}{3}$, $p_1 = \frac{3}{2}$, $p_2 = \frac{5}{2}$, $h_1(t) = t$, $h_2(t) = t^{\frac{1}{2}}$. It is easy to check that $f_1, f_2 \in Car([0, 1] \times (0, \infty)^2)$, $g_1, g_2 \in L^1[0, 1]$ are non-negative and $h_1, h_2 \in L^1[0, 1]$. Also, we have

$$[I^{p_1}(h_1(t)]_{t=1} = [I^{\frac{3}{2}}(t)]_{t=1} = \frac{1}{\Gamma(\frac{3}{2})}\int_0^1 (1-s)^{\frac{1}{2}}sds = \frac{1}{\Gamma(\frac{3}{2})}\frac{\Gamma(2)\Gamma(\frac{3}{2})}{\Gamma(2+\frac{3}{2})} = \frac{2}{\sqrt{\pi}}\frac{4}{15} \in [0, \frac{1}{2}),$$

$$[I^{p_2}(h_2(t)]_{t=1} = [I^{\frac{5}{2}}(t^{\frac{1}{2}})]_{t=1} = \frac{1}{\Gamma(\frac{5}{2})}\int_0^1 (1-s)^{\frac{3}{2}}s^{\frac{1}{2}}ds = \frac{1}{\Gamma(\frac{5}{2})}\frac{\Gamma(\frac{3}{2})\Gamma(\frac{5}{2})}{\Gamma(4)} = \frac{1}{\frac{3\sqrt{\pi}}{4}}\frac{\frac{\sqrt{\pi}}{2}\frac{3\sqrt{\pi}}{4}}{6} = \frac{\sqrt{\pi}}{12} \in [0, \frac{1}{2}),$$

$$\|g_1\|_1 = \int_0^1 \frac{0.3}{t^{\frac{1}{2}}}dt = 0.6 < \frac{3\sqrt{\pi}}{8} = \frac{\Gamma(\frac{7}{2}-1)}{2} = \frac{\Gamma(\alpha_1-1)}{2}$$

and $\|g_2\|_1 = \int_0^1 \frac{0.2}{t^{\frac{1}{3}}}dt = 0.3 < \frac{\Gamma(\frac{10}{3}-1)}{2} = \frac{\Gamma(\alpha_2-1)}{2}$. On the other hand, we have

$$\begin{aligned} K(u(Q)) &= K(u((A, B))) = K(\frac{1}{2}A + \frac{1}{3}B) \\ &= \max\{K(A), K(B)\}(\frac{1}{2} + \frac{1}{3}) = K(Q)(\frac{1}{2} + \frac{1}{3}) \leq K(Q), \end{aligned}$$

for all $Q = (A, B) \subset C[0, 1] \times C[0, 1]$. Since $f_1(t, x, y) = g(t)u(x, y)$, we get

$$K(f(t, Q)) = K(g_1(t)u(Q)) = g_1(t)K(u(Q)) \leq g_1(t)K(Q).$$

By using a similar method, we get $K(f(t, Q)) = K(g_1(t)u(Q)) = g_1(t)K(u(Q)) \leq g_1(t)K(Q)$. Now by using Theorem 2.2, the system (2.1) has a solution.

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