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Some common fixed point theorems for single valued mappings in G-ultrametric spaces

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Abstract

The purpose of this paper is to prove some common fixed point theorems for a single valued strongly contractive mapping having a pair of maps on a spherically complete G-ultrametric space.

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1. Introduction

Van Rooij [15] introduced the concept of ultrametric space as follows:
Let (X, d) be a metric space. Then (X, d) is called an ultrametric space, if the metric d satisfies the strong triangle inequality, i.e., for all $x, y, z \in X$:

$$d(x, y) \leq \max\{d(x, z), d(y, z)\}.$$

In this case, d is called to be ultrametric.

The fixed point theorems are used to determine conditions for the existence of solutions of polynomial differential equations of any order, or even of systems of such equations, see Priess-Crampe and Ribenboim [10, 11]. Methods of ultrametric dynamics also find applications in the study of differential equations over rings of power series, as in the work of van der Hoeven, for example see his lecture notes [14]. A very different and unexpected application of ultrametric dynamics is found in the determination of solutions of the famous Fermat equation in square matrices with entries in a p-adic field, see [13]. Programs with positive clauses were shown to have models by means of the fixed point theorem of Knaster and Tarski about monotonic

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self-maps in a complete lattice. More general programs, involving negation in clauses lead to the ultrametric space of maps from the Herbrand base with values 0, 1; in this space the values of the distance are the subsets of the Herbrand base. The fixed point of the immediate consequence operator gives conditions for the existence of models for the program, see Priess-Crampe and Ribenboim [9, 12] and Hitzler and Seda [3, 4].

As seen above, fixed point theory has a wide application in almost all fields of quantitative sciences such as economics, biology, physics, chemistry, computer science and many branches of engineering. It is quite natural to consider various generalizations of metric spaces in order to address the needs of these quantitative sciences. That's why in 2004, Mustafa and Sims introduced a new class of generalized metric spaces (see [7, 8]), which are called G-metric spaces, as generalization of a metric space (X, d). Subsequently, many fixed point results on such spaces appeared (see, for example, [2, 5, 6]). Here, we present the necessary definitions and results in G-metric spaces, which will be useful for the rest of the paper. However, for more details, we refer to [1, 2, 7, 8].

We start with basic definitions and a detailed overview of the essential results developed in the interesting works mentioned above.

Definition 1.1 ([8]). *. Let X be a nonempty set. Suppose that $G : X \times X \times X \to [0, +\infty)$ is a function satisfying the following conditions:*

G1) $G(x, y, z) = 0$, *if* $x = y = z$.
G2) $0 < G(x, x, y)$, *for all* $x, y, z \in X$ *with* $x \neq y$.
G3) $G(x, x, y) \leq G(x, y, z)$, *for all* $x, y, z \in X$ *with* $z \neq y$.
G4) $G(x, y, z) = G(x, z, y) = G(y, z, x) = \ldots$, *(symmetry in all three variables).*
G5) $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$, *for all* $x, y, z \in X$, *(rectangle inequality).*

then the function G is called a generalized metric, or more specifically a G-metric on X and the pair (X, G) is a G-metric space.

Definition 1.2 ([8]). *Let (X, G) be a G-metric space, then for $x_0 \in X$, $r > 0$, the G-ball (stripped ball) with center x_0 and radius r is*

$$B(x_0, r^-) = \{y \in X : G(x_0, y, y) < r\}$$

and the dressed ball of radius r and center x_0 is

$$B(x_0, r) = \{y \in X : G(x_0, y, y) \leq r\}.$$

Proposition 1 ([8]). *Let (X, G) be a G-metric space, then for any $x_0 \in X$ and $r > 0$, we have,*

(1) if $G(x_0, x, y) < r$, *then* $x, y \in B(x_0, r^-)$,
(2) if $y \in B(x_0, r^-)$, *then there exists* $\delta > 0$, *such that* $B(y, \delta^-) \subseteq B(x_0, r^-)$.

2. G-Ultrametric Spaces

First, we introduce a class of G-metric spaces, which are called G-ultrametric spaces and in the sequel give results which are required.

Definition 2.1. *A G-metric space (X, G) is called a G-ultrametric space, if the G-metric G satisfies the strong rectangle inequality, i.e., for all $a, x, y, z \in X$:*

$$G(x, y, z) \leq \max\{G(x, a, a), G(a, y, z)\}.$$

In this case, G is called a generalized ultrametric and the pair (X, G) is a G-ultrametric space.

Examples

(a) Let X be a nonempty set. The following function on X^3 defines a G-ultrametric on X:

$$G(x,y,z)=\begin{cases}0 & x=y=z,\\ 1 & \text{otherwise.}\end{cases}$$

In this case, (X,G) is called a discrete G-ultrametric space (or trivial G-ultrametric space).

(b) Every G-ultrametric on X defines an ultrametric d_G on X by,

$$d_G(x,y)=\max\{G(x,y,y),G(y,x,x)\},\qquad \text{for all } x,y\in X.$$

Conversely, for any d-ultrameric d on X,

$$G_1(x,y,z)=\max\{d(x,y),d(y,z),d(x,z)\},\qquad \text{for all } x,y\in X,$$

is readily seen to define an G-ultrametric on X^3.

(c) The mapping $G:\mathbb{N}\times\mathbb{N}\times\mathbb{N}\to[0,+\infty)$ is defined by,

$$G(m,n,l)=\begin{cases}0 & m=n=l,\\ \max\{1+\frac{1}{m},1+\frac{1}{n},1+\frac{1}{l}\} & \text{otherwise.}\end{cases}$$

is a G-ultrametric on $\mathbb{N}^3$.

2.1. The G-Ultrametric topology

Proposition 2. *Let (X,G) be a G-ultrametric space. Then the following statements hold.*

(a) Any point of a G-ball is a center of the ball.

(b) If two G-balls have a common point, one is contained in the other.

(c) The diameter of a G-ball is less than or equal to its radius.

Proof. (a) Let x_0 be a point in $B(x,r^-)$ and let w belongs to $B(x_0,r^-)$, it follows from Definitions 2.1 and 1.2 that

$$G(x,w,w)\le\max\{G(x,x_0,x_0),G(x_0,w,w)<r.$$

Hence $B(x_0,r^-)\subseteq B(x,r^-)$. Conversely, suppose that u be a point in $B(x,r^-)$, hence

$$\begin{aligned}G(x_0,u,u)&\le\max\{G(x_0,x,x),G(x,u,u)\}\\&=\max\{G(x,x_0,x),G(x,u,u)\}\\&<\max\{G(x_0,x_0,x),G(x,x_0,x_0),G(x,u,u)\}\\&<r,\end{aligned}$$

which implies that $B(x,r^-)\subseteq B(x_0,r^-)$. Thus $B(x,r^-)=B(x_0,r^-)$.

(b) Suppose that $B(x,r^-)$ and $B(y,s^-)$ are two G-balls such that, $B(x,r^-)\cap B(y,s^-)\neq\emptyset$ and $r\le s$. Now, let $w\in B(x,r^-)$ and $a\in B(x,r^-)\cap B(y,s^-)$. Then

$$\begin{aligned}G(y,w,w)&\le\max\{G(x,w,w),G(y,x,x)\}\\&\le\max\{G(x,w,w),G(y,a,a),G(a,x,x)\}\\&\le\max\{G(x,w,w),G(y,a,a),G(a,a,x),G(x,a,a)\}\\&\le\max\{r,s\}=s,\end{aligned}$$

which implies that $B(x,r^-)\subseteq B(y,s^-)$.

(c) Let $B(a,r^-)$ be a G-ball in G-ultrametric space X. Then for all x,y and z in X,

$$\begin{aligned}G(x,y,z)&\le\max\{G(x,a,a),G(a,y,z)\}\\&\le\max\{G(x,a,a),G(y,a,a),G(z,a,a)\}\\&<r,\end{aligned}$$

which implies that

$$Diam(B(a,r^-)) = \sup\{G(x,y,z) \mid x,y,z \in B(a,r^-)\} \leq r.$$

□

Proposition 3. *Let (X,G) be a G-ultrametric space. Then the following statements hold.*

(a) If $x \in S(x_0,r)$, then $B(x,r^-) \subseteq S(x_0,r)$ and

$$S(x_0,r) = \cup_{x\in S(x_0,r)} B(x,r^-),$$

which $S(x_0,r) = \{y \in X : G(x_0,y,y) = r\}$.

(b) The spheres $S(x_0,r)(r > 0)$ are open and closed.

(c) The dressed balls of positive radius are open and the stripped balls are closed.

Proof. (a) Let w be an arbitrary point in $B(x,r^-)$. Then

$$\begin{aligned} r = G(x_0,x,x) &\leq G(x_0,w,x) \\ &\leq \max\{G(w,w,x), G(x_0,w,w)\} \\ &= G(x_0,w,w) \leq r, \end{aligned}$$

it follows that $B(x,r^-) \subseteq S(x_0,r)$ and $S(x_0,r) = \cup_{x\in S_r(x_0)} B(x,r^-)$.

(b) This is an immediate consequence of (a).

(c) If $r > 0$, then $B(a,r) = B(a,r^-) \cup S(a,r)$ is open. Also, if $r > 0$, then $B(a,r^-) = B(a,r) - S(a,r)$ is closed. If $r = 0$, then $B(a,r^-) = \emptyset$ is closed. □

Consequently, the G-ultrametric topology $\tau(G)$ is zero-dimensional and coincides with the ultrametric topology arising from d_G. Thus, while isometrically distinct, every G-ultrametric space is topologically equivalent to an ultrametrics space. This allows us to transport many concepts and results from ultrametric spaces into the G-ultrametric space setting.

3. The Main Theorem

Theorem 3.1. *Let (X,G) be a spherically complete G-ultrametric space. If f and $T : X \to X$ are self maps on X satisfying*

$$T(X) \subseteq f(X)$$

and for every tree distinct points x,y,z in X,

$$\begin{aligned} G(Tx,Ty,Tz) < \max\{&G(fx,fy,fz), G(fx,Tx,Tx), \\ &G(fy,Ty,Ty), G(fz,Tz,Tz)\}, \end{aligned}$$

then, there exists $z \in X$ such that $fz = Tz$. Further, if f and T are coincidentally commuting at z, then z is the unique common fixed point of f and T.

Proof. Let $B_a = B(fa, G(fa,Ta,Ta))$ denotes the closed spheres centered at fa with the radius $G(fa,Ta,Ta)$ and let $\mathbb{A}$ be the collection of these spheres for all $a \in X$.

The relation $B_a \leq B_b$ holds, if and only if $B_b \subseteq B_a$ is a partial order on $\mathbb{A}$. Now, consider a totally ordered subfamily $\mathbb{A}_1$ of $\mathbb{A}$. Since spherically complete X, we have

$$\bigcap_{B_a\in\mathbb{A}_1} B_a = B \neq \emptyset.$$

Let $b \in B$ and $B_a \in \mathbb{A}_1$. Since $fb \in B_b \cap B_a$, we have $G(fa, fb, fb) \leq G(fa, Ta, Ta)$. Now, we claim that $Tb \in B_a$. By assumption, we have

$$\begin{aligned} G(fa, Tb, Tb) &\leq \max\{G(Ta, Tb, Tb), G(fa, Ta, Ta)\} \\ &\leq \max\{G(fa, fb, fb), G(fa, Ta, Ta), G(fb, Tb, Tb)\} \\ &= G(fa, Ta, Ta). \end{aligned}$$

Therefore, $Tb \in B_a$. It follows from Proposition 2 that $G(fb, Tb, Tb) \leq G(fa, Ta, Ta)$. So we have proved that $B_a \leq B_b$. Thus B_b is the upper bound for the family $\mathbb{A}_1$. By Zorn's lemma, $\mathbb{A}$ has a maximal element, say $B_z, z \in X$. We are going to prove that $fz = Tz$. Suppose, for contradiction, that $fz \neq Tz$. Since $Tz \in T(X) \subseteq f(X)$, there exists $w \in X$ such that $Tz = fw$. Clearly $z \neq w$. Now we have

$$\begin{aligned} G(fw, Tw, Tw) &= G(Tz, Tw, Tw) \\ &< \max\{G(fz, fw, fw), G(fz, Tz, Tz), G(fw, Tw, Tw)\} \\ &= G(fz, fw, fw). \end{aligned}$$

Thus $fz \notin B_w$. Hence $B_z \nsubseteq B_w$. It is a contradiction to the maximality of B_z. Hence $fz = Tz$. Further assume that f and T are coincidentally commuting at z . Then $f^z = f(fz) = fTz = Tfz = T(Tz) = T^z$. Suppose $fz \neq z$. Now, we have

$$G(Tfz, Tz, Tz) < \max\{G(f^2z, fz, fz), G(f^2z, Tfz, Tfz), G(fz, Tz, Tz), G(Tfz, Tz, Tz)\}.$$

Hence $fz = z$. Thus $fz = z$. Let u be a different fixed point of f and T. For $u \neq z$ we have that

$$\begin{aligned} G(u, z, z) &= G(Tu, Tz, Tz) \\ &< \max\{G(fu, fz, fz), G(fu, Tu, Tu), G(fz, Tz, Tz)\} \\ &= G(fu, fz, fz) = G(u, z, z) \end{aligned}$$

which is a contradiction. □

Theorem 3.2. *Let (X, G) be a G-ultrametric space, $f, S, T : X \to X$ satisfying:*

(1) $f(X)$ is spherically complete.

(2) For every tree distinct points $x, y, z \in X$,

$$G(Sx, Ty, Tz) < \max\{G(fx, fy, fz), G(fx, Sx, Sx), G(fy, Ty, Ty), G(fz, Tz, Tz)\}.$$

(3) $fS = Sf$, $fT = Tf$, $ST = TS$.

(4) $S(X) \subseteq f(X)$, $T(X) \subseteq f(X)$.

Then either $fw = Sw$ or $fw = Tw$ for some $w \in X$.

Proof. For $a \in X$, let $B_a = B(fa, \max\{G(fa, Sa, Sa), G(fa, Ta, Ta)\})$ denote the closed sphere centered at fa with the radius $\max\{G(fa, Sa, Sa), G(fa, Ta, Ta)\}$. Let $\mathbb{A}$ be the collection of all the spheres for all $a \in f(X)$. Then the relation $B_a \leq B_b$ iff $B_b \subseteq B_a$ is a partial order on $\mathbb{A}$. Now, consider a totally ordered subfamily $\mathbb{A}_1$ of $\mathbb{A}$. Since $f(X)$ is spherically complete, we have $\cap_{B_a \in \mathbb{A}_1} B_a = B \neq \emptyset$. Let $fb \in B$ where $b \in f(X)$ and $B_a \in \mathbb{A}_1$. Then $fb \in B_a$. Hence, $G(fb, fa, fa) \leq \; max\{G(fa, Sa, Sa), G(fa, Ta, Ta)\}$ If $a = b$ then $B_a = B_b$. Assume that $B_a \neq B_b$. Let $x \in B_b$. Then

$$\begin{aligned} G(x, fb, fb) &\leq \max\{G(fb, Sb, Sb), G(fb, Tb, Tb)\} \\ &\leq \max\{G(fb, fa, fa), G(fa, Ta, Ta), G(Ta, Sb, Sb), G(fb, fa, fa), \\ &\qquad G(fa, Sa, Sa), G(Sa, Tb, Tb)\} \\ &< \max\{G(fb, fa, fa), G(fa, Ta, Ta), G(fa, Sa, Sa), \max\{G(fb, fa, fa), \\ &\qquad G(fb, Sb, Sb), G(fa, Ta, Ta)\}, \max\{G(fa, fb, fb), G(fa, Sa, Sa), \\ &\qquad G(fb, Tb, Tb)\}\} \\ &= \max\{G(fa, Sa, Sa), G(fa, Ta, Ta)\}. \end{aligned}$$

Thus

$$G(x, fb, fb) < \max\{G(fa, Sa, Sa), G(fa, Ta, Ta)\}.$$

Now,

$$\begin{aligned} G(x, fa, fa) &\le \max\{G(x, fb, fb), G(fb, fa, fa)\} \\ &\le \max\{G(fa, Sa, Sa), G(fa, Ta, Ta)\}. \end{aligned}$$

Thus $x \in B_a$. Hence $B_b \subseteq B_a$ for any $B_a \in \mathbb{A}_1$. Thus B_b is an upper bound in $\mathbb{A}$ for the family $\mathbb{A}_1$ and hence by Zorn's Lemma, there is a maximal element in $\mathbb{A}$, say B_z, $z \in f(X)$. There exists $w \in X$ such that $z = fw$. Suppose $fw \neq Sw$ and $fw \neq Tw$.

$$\begin{aligned} G(Sfw, TSw, TSw) &< \max\{G(f^2w, fSw, fSw), G(f^2w, Sfw, Sfw), G(fSw, TSw, TSw)\} \\ &= G(f^2w, fSw, fSw) \end{aligned}$$

since $fS = Sf$.

$$\begin{aligned} G(STw, Tfw, Tfw) &< \max\{G(fTw, f^2w, f^2w), G(fTw, STw, STw), G(f^2w, Tfw, Tfw)\} \\ &= G(f^2w, fTw, fTw) \end{aligned}$$

since $fT = Tf$.

$$\begin{aligned} G(Sfw, S^2w, S^2w) &\le \max\{G(Sfw, TSw, TSw), G(TSw, Tfw, Tfw), G(Tfw, S^2w, S^2w)\} \\ &< \max\{G(f^2w, fSw, fSw), G(f^2w, fTw, fTw), \max\{G(fSw, f^2w, f^2w), \\ &\qquad G(fSw, S^2w, S^2w), G(f^2w, Tfw, Tfw)\}\} \\ &= \max\{G(f^2w, fSw, fSw), G(f^2w, fTw, fTw)\}. \end{aligned}$$

We have

$$\max\{G(STw, Tfw, Tfw), G(Tfw, T^2w, T^2w)\} < max\{G(f^2w, fTw, fTw), G(f^2w, fSw, fSw)\}.$$

If

$$\max\{G(f^2w, fTw, fTw), G(f^2w, fSw, fSw)\} = G(f^2w, fSw, fSw).$$

Then

$$\max\{G(Sfw, TSw, TSw), G(Sfw, S^2w, S^2w)\} < G(f^2w, fSw, fSw),$$

which gives $f^2w \notin B_{Sw}$. Hence $fz \notin B_{Sw}$. But $fz \in B_z$. Hence $B_z \nsubseteq B_{Sw}$. It is a contradiction to the maximality of B_z in $\mathbb{A}$, since $S_w \in S(X) \subseteq f(X)$. If

$$\max\{G(f^2w, fTw, fTw), G(f^2w, fSw, fSw)\} = G(f^2w, fTw, fTw),$$

then $\max\{G(STw, Tfw, Tfw), G(Tfw, T^2w, T^2w)\} < G(f^2w, fTw, fTw)$, which gives $f^2w \notin B_Tw$. Hence $fz \notin B_{Tw}$, but $fz \in Bz$, so $B_z \nsubseteq B_{Tw}$. It is a contradiction to the maximality of B_z in $\mathbb{A}$, since $Tw \in T(X) \subseteq f(X)$. Therefore, either$fw = Sw$ or $fw = Tw$. □

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