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Fixed point and coupled fixed point theorems for multi-valued contractions with respect to the excess functional

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Abstract

In this paper, we will consider the coupled fixed point problem for a multi-valued operator satisfying a contraction condition with respect to the excess functional in b-metric spaces. The approach is based on a fixed point theorem for a multi-valued operator in the setting of a b-metric space. On one hand, we will consider the problem of the existence of the solutions and on the other hand, data dependence, well-posedness, Ulam-Hyers stability and limit shadowing property of the coupled fixed point problem are discussed. Some applications to a system of integral inclusions and to a multi-valued periodic boundary value problem are also given.

Keywords: multi-valued operator, ordered metric space, coupled fixed point, data dependence, well-posedness, Ulam-Hyers stability, limit shadowing, system of integral inclusions, multi-valued periodic boundary value problem.
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1. Introduction

Nadler's contraction principle [15] is an extension to the multi-valued case of the classical Banach's contraction principle. There are many applications of these results, mainly in the theory of operator equations and inclusions, see [3, 4, 6, 10, 20]. On the other hand, several theorems were given for the so-called coupled fixed point problem ([5, 9, 14]). In the multi-valued setting, this problem is as follows:

Let (X, d) be a metric space and $P(X)$ be the family of all nonempty subsets of X. If $G : X \times X \to P(X)$ is a multi-valued operator, then by definition, a coupled fixed point problem for G means to find a pair $(x^*, y^*) \in X \times X$ satisfying,

$$\begin{cases} x^* \in G(x^*, y^*), \\ y^* \in G(y^*, x^*). \end{cases} \tag{1.1}$$

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The purpose of this paper is to give new fixed point results and new coupled fixed point theorems for multi-valued operators satisfying a contraction type condition with respect to the excess functional. We will consider here the context of a b-metric space. Our results are new even for the case of metric spaces and they extend some theorems given in [8, 18, 19] and other papers in the literature. Some applications to integral and differential inclusions are also given.

2. Preliminaries

We will recall first the definition of a b-metric space.

Definition 2.1 (Bakhtin [1], Czerwik [7])**.** Let X be a nonempty set and let $s \geq 1$ be a given real number. A functional $d : X \times X \to \mathbb{R}_+$ is said to be a b-metric with constant s, if all the axioms of the metric take place with the following modification of the triangle inequality property:

$$d(x, z) \leq s[d(x, y) + d(y, z)],\ for\ all\ x, y, z \in X.$$

Under the above hypotheses, the pair (X, d) is called a b-metric space with constant s.

In the framework of a b-metric space, a set $Y \subset X$ is said to be closed, if for any sequence $(x_n)_{n\in\mathbb{N}}$ in Y which converges (with respect to d) to an element x, we have $x \in Y$. The notions of bounded or compact sets are defined in a similar way to the case of usual metric spaces. For example, the set $Y \subset X$ is said to be bounded, if its diameter,

$$\delta(Y) := \sup_{x,y\in Y} d(x, y),$$

is finite, while Y is compact, if every sequence sequence $(x_n)_{n\in\mathbb{N}} \subset Y$ has a convergent subsequence in Y.

Let (X, d) be a b-metric space and $\mathcal{P}(X)$ be the set of all subsets of X. We denote $P(X) := \{Y \in \mathcal{P}(X) \mid Y \neq \emptyset\}$, $P_{cl}(X) := \{Y \in \mathcal{P}(X) \mid Y \text{ is closed }\}$, $P_{b,cl}(X) := \{Y \in \mathcal{P}(X) \mid Y \text{ is bounded and closed }\}$, $P_{cp}(X) := \{Y \in \mathcal{P}(X) \mid Y \text{ is compact }\}$.

Let (X, d) be a b-metric space. If $T : X \to P(X)$ is a multi-valued operator, then $x \in X$ is called fixed point for T, if and only if $x \in T(x)$. We denote by $Fix(T)$ the fixed point set of T and by $SFix(T)$ the set of all strict fixed points of T, i.e., the elements $x \in X$ such that $T(x) = \{x\}$.

If X, Y are two nonempty sets and $T : X \to P(Y)$, then we will denote by,

$$Graph(T) := \{(x, y) \in X \times Y : y \in T(x)\},$$

the graph of T. In the context of a b-metric space (X, d) the following generalized functionals are used in the main sections of the paper.

1. The gap functional generated by d:

$$D_d : P(X) \times P(X) \to \mathbb{R}_+, D_d(A, B) = inf\{d(a, b) \mid a \in A, b \in B\}.$$

2. The excess generalized functional:

$$\rho_d : P(X) \times P(X) \to \mathbb{R}_+ \cup \{+\infty\}, \rho_d(A, B) = sup\{D_d(a, B)|a \in A\}.$$

3. The Hausdorff-Pompeiu generalized functional :

$$H_d : P(X) \times P(X) \to \mathbb{R}_+ \cup \{+\infty\}, H_d(A, B) = \max\{\rho_d(A, B), \rho_d(B, A)\}.$$

4. The Pompeiu generalized functional:

$$H_d^+ : P(X) \times P(X) \to \mathbb{R}_+ \cup \{+\infty\},\ H_d^+(A, B) := \frac{1}{2}\{\rho_d(A, B) + \rho_d(B, A)\}.$$

Definition 2.2 ([18]). Let X be a nonempty set, let " $\leq$ " be a partial order on X and d be a b-metric on X with constant $s \geq 1$. Then the triple $(X, \leq, d)$ is called an ordered b-metric space if,

i. " $\leq$ " is a partial order on X.
ii. d is a b-metric on X with constant $s \geq 1$.
iii. If $(x_n)_{n\in\mathbb{N}}$ is a monotone increasing sequence in X and $x_n \to x^* \in X$, then $x_n \leq x^*$, for all $n \in \mathbb{N}$.
iv. If $(y_n)_{n\in\mathbb{N}}$ is a monotone decreasing sequence in X, and $y_n \to y^* \in X$, then $y_n \geq y^*$, for all $n \in \mathbb{N}$.

Definition 2.3 ([18]). Let $(X, \leq)$ be a partially ordered set. Then, the partial order $\leq$ induces on the product space $X \times X$ the following partial order,

$$\text{for } (x, y), (u, v) \in X \times X, (x, y) \leq_p (u, v) \Leftrightarrow x \leq, y \geq v.$$

Definition 2.4 ([19]). Let (X, d) be a b-metric space and $G : X \times X \to P(X)$ be a multi-valued operator. Then by definition, a coupled fixed point for G is a pair $(x, y) \in X \times X$ satisfying,

$$(P1) \begin{cases} x \in G(x, y), \\ y \in G(y, x). \end{cases} \tag{2.1}$$

We will denote by $CFix(G) = \{(x, y) \in X \times X | x \in G(x, y) \text{ and } y \in G(y, x)\}$ the coupled fixed point set of G.

Lemma 2.5 ([19]). *Let (X, d) be a b-metric space with constant $s \geq 1$, $A, B \in P(X)$ and $q > 1$. Then, for every $a \in A$, there exists $b \in B$ such that $\rho_d(A, B) \leq qd(a, b)$.*

3. Fixed point theorems

In this part, we will present some fixed point theorems for a multi-valued operator in b-metric spaces. See also [2, 11, 12, 17] for related results.

Definition 3.1. Let (X,d) be a b-metric space with constant $s \geq 1$. A multi-valued operator $T : X \to P_{b,cl}(X)$ is called H^+-contraction with constant k if,

1. there exists a fixed real number $0 < k < 1$ such that, for every $x, y \in X$, we have,

$$H_d^+(T(x), T(y)) \leq kd(x, y),$$

2. for every $x \in X$ and $y \in T(x)$ we have,

$$D_d(y, T(y)) \leq H^+(T(x), T(y)).$$

For the case of metric spaces see [16].

Definition 3.2. Let $(X, \preceq)$ be a partially ordered set and $A, B \in P(X)$. We will denote:
a) $A \leq_{st} B \Leftrightarrow \forall a \in A, \forall b \in B$ we have $a \preceq b$,
b) $A \leq_{wk} B \Leftrightarrow \forall a \in A, \exists b \in B$ such that $a \preceq b$.

Remark 3.3. Notice that, if $A, B, C \in P(X)$, then $A \leq_{st} B$ and $B \leq_{st} C$ implies $A \leq_{st} C$. The same property also holds for $\leq_{wk}$.

Definition 3.4. Let $(X, \preceq)$ be a partially ordered set and $T : X \to P(X)$ be a multi-valued operator. We say that T is strong increasing (respectively strong decreasing) on X, if for every $x, y \in X$ with $x \preceq y$, we have that $T(x) \leq_{st} T(y)$ (respectively $T(x) \geq_{st} T(y)$).

The first result of this section is a fixed point theorem in an ordered b-metric space.

Theorem 3.5. *Let $(X, \preceq, d)$ be a complete ordered b-metric space with constant $s \geq 1$. Let $T : X \to P_{cl}(X)$ be a multi-valued operator strong increasing with respect to "$\preceq$". Suppose that,*

i) there exists $k \in \left(0, \frac{1}{s}\right)$ such that $\rho_d(T(x), T(y)) \leq kd(x, y)$, for all $x, y \in X$ with $x \preceq y$,
ii) there exists an element $x_0 \in X$ such that $x_0 \leq_{wk} T(x_0)$.

Then the following conclusions hold,

a) $Fix(T) \neq \emptyset$ and there exists a sequence $(x_n)_{n\in\mathbb{N}} \subset X$ of successive approximation of T starting from $x_0 \in X$ which converges to a fixed point of T.
b) In particular, if d is continuous b-metric on X, then

$$d(x_n, x^*) \leq \frac{sk^n}{1-sk} d(x_0, x_1), \forall n \in \mathbb{N}^* \text{ and } x_1 \in T(x_0).$$

Proof. a) Let $x_0 \in X$ such that $x_0 \leq_{wk} T(x_0)$. By *ii)* we get that there exists $x_1 \in T(x_0)$ such that $x_0 \preceq x_1$. Using Lemma 2.5 we obtain that, for any $q > 1$, there exists $x_2 \in T(x_1)$ such that,

$$d(x_1, x_2) \leq q\rho(T(x_0), T(x_1)) \leq qkd(x_0, x_1).$$

Since $x_1 \in T(x_0)$, $x_2 \in T(x_1)$, $x_0 \preceq x_1$ and T is a strong increasing multi-valued operator, we obtain that $x_1 \preceq x_2$. By Lemma 2.5 we get that there exists $x_3 \in T(x_2)$ such that,

$$d(x_2, x_3) \leq q\rho(T(x_1), T(x_2)) \leq qkd(x_1, x_2) \leq (qk)^2 d(x_0, x_1).$$

By induction, we obtain a sequence $(x_n)_{n\in\mathbb{N}}$ in X with the following properties:

1) $x_{n+1} \in T(x_n)$, for all $n \in \mathbb{N}$,
2) $x_n \preceq x_{n+1}$, for all $n \in \mathbb{N}$,
3) $d(x_n, x_{n+1}) \leq (qk)^n d(x_0, x_1)$, for all $n \in \mathbb{N}$.

Let $1 < q < \frac{1}{sk}$, we show that $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence. Indeed, we have,

$$\begin{aligned} d(x_n, x_{n+p}) &\leq sd(x_n, x_{n+1}) + s^2 d(x_{n+1}, x_{n+2}) + ... + s^{p-1} d(x_{n+p-1}, x_{n+p}) \\ &\leq s(qk)^n d(x_0, x_1) + s^2 (qk)^{n+1} d(x_0, x_1) + ... + s^{p-1} (qk)^{n+p-1} d(x_0, x_1) \\ &= s(qk)^n \frac{1-(sqk)^p}{1-sqk} d(x_0, x_1) \leq s(qk)^n \frac{1}{1-sqk} d(x_0, x_1). \end{aligned}$$

Hence,

$$d(x_n, x_{n+p}) \leq \frac{s(qk)^n}{1-sqk} d(x_0, x_1) \to 0, \text{ as } n \to \infty, \ \forall p \in \mathbb{N}. \tag{3.1}$$

This implies that the sequence $(x_n)_{n\in\mathbb{N}}$ is Cauchy in X. Since (X, d) is a complete b-metric space, there exists $x^* \in X$ such that $x_n \to x^*$. Moreover $x_n \preceq x^*$, for all $n \in \mathbb{N}$. Then, we have,

$$\begin{aligned} D(x^*, T(x^*)) &\leq s[d(x^*, x_{n+1}) + D_d(x_{n+1}, T(x^*))] \\ &\leq s[d(x^*, x_{n+1}) + \rho_d(T(x_n), T(x^*))] \\ &\leq s[d(x^*, x_{n+1}) + kd(x_n, x^*)]. \end{aligned}$$

Hence $D(x^*, T(x^*)) \to 0$ as $n \to \infty$. Since T has closed values, we obtain that $x^* \in T(x^*)$.
(b) By (3.1), letting $p \to \infty$ and $q \searrow 1$, we get that,

$$d(x_n, x^*) \leq \frac{sk^n}{1-sk} d(x_0, x_1), \forall n \in \mathbb{N}^*.$$

□

Remark 3.6. If we only assume that X is a nonempty set, $\preceq$ is a partially ordering on X and d is a complete b-metric with constant $s \geq 1$, then the conclusions of the above theorem hold, if additionally, the graph of T is closed in $X \times X$.

The second result is a fixed point theorem in a b-metric space under a contraction type condition on the whole space. In this case, we do not need a monotonicity condition on the multi-valued operator.

Theorem 3.7. *Let (X, d) be a complete b-metric space with constant $s \geq 1$ and $T : X \to P_{cl}(X)$ a multi-valued operator for which there exists $k \in \left(0, \frac{1}{s}\right)$ such that,*

$$\rho_d(T(x), T(y)) \leq kd(x, y), \text{ for all } x, y \in X.$$

Then:

a) Fix(T) $\neq \emptyset$ and there exists a sequence $(x_n)_{n\in\mathbb{N}} \subset X$ of successive approximations of T starting from any $(x_0, x_1) \in Graph(T)$ which converges to a fixed point x^* of T.
b) In particular, if d is a continuous b-metric, then

$$d(x_n, x^*) \leq \frac{sk^n}{1 - sk} d(x_0, x_1), \forall n \in \mathbb{N}^*.$$

c) If additionally $SFix(T) \neq \emptyset$, then $Fix(T) = SFix(T) = \{x^*\}$.

Proof. a) Let $x_0 \in X$ and $x_1 \in T(x_0)$ be arbitrary chosen. By Lemma 2.5 we have that for any $q > 1$, there exists $x_2 \in T(x_1)$ such that,

$$d(x_1, x_2) \leq q\rho(T(x_0), T(x_1)) \leq qkd(x_0, x_1).$$

Repeating the same argument $n-$times, we get a sequence of successive approximations for T starting from $(x_0, x_1) \in Graph(T)$ such that, for each $n \in \mathbb{N}$, we have,

$$d(x_n, x_{n+1}) \leq (qk)^n d(x_0, x_1), \text{ for all } n \in \mathbb{N}.$$

Now, we will prove that $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence on X. Indeed, for $1 < q < \frac{1}{ks}$, we have,

$$d(x_n, x_{n+p}) \leq s(qk)^n \frac{1 - (sqk)^p}{1 - sqk} d(x_0, x_1) \to 0, \text{ as } n \to \infty.$$

This implies that the sequence $(x_n)_{n\in\mathbb{N}}$ is Cauchy and hence it is convergent in (X, d) to some $x^* \in X$. Then,

$$\begin{aligned} D(x^*, T(x^*)) &\leq s[d(x^*, x_{n+1}) + D_d(x_{n+1}, T(x^*))] \\ &\leq s[d(x^*, x_{n+1}) + \rho(T(x_n), T(x^*))] \\ &\leq s[d(x^*, x_{n+1}) + kd(x_n, x^*)] \to 0, \quad as \quad n \to \infty. \end{aligned}$$

Thus, $Fix(T) \neq \emptyset$.
b) If the b-metric d is continuous, then by,

$$d(x_n, x_{n+p}) \leq s(qk)^n \frac{1 - (sqk)^p}{1 - sqk} d(x_0, x_1),$$

letting $p \to \infty$ and $q \searrow 1$, we obtain the following desired estimation,

$$d(x_n, x^*) \leq \frac{sk^n}{1 - sk} d(x_0, x_1), \forall n \in \mathbb{N}^*.$$

c) Let $y^* \in SFix(T)$. We consider another $y \in SFix(T)$ such that $y \neq y^*$. Then,

$$d(y, y^*) = \rho(T(y), T(y^*)) \leq kd(y, y^*).$$

Thus, since $k < \frac{1}{s} < 1$, we obtain that $d(y, y^*) = 0$ and so $SFix(T) = \{y^*\}$. Let $x^* \in Fix(T)$ with $x^* \neq y^*$. We have,

$$d(x^*, y^*) = D_d(x^*, T(y^*)) \leq \rho_d(T(x^*), T(y^*)) \leq kd(x^*, y^*).$$

Since $k < \frac{1}{s} < 1$, we obtain that $d(y^*, x^*) = 0$. Thus $y^* = x^*$ and $Fix(T) = \{x^*\}$. Therefore, $SFix(T) = Fix(T) = \{x^*\}$. □

Remark 3.8. If we assume that,

$$\rho_d(T(x), T(y)) \leq kd(x, y), \text{ for all } x, y \in X,$$

then by reversing the values of x and y we get that the following condition also holds,

$$\rho_d(T(y), T(x)) \leq kd(x, y), \text{ for all } x, y \in X.$$

Thus, by adding the above two relations we get that,

$$H_d^+(T(x), T(y)) \leq kd(x, y), \text{ for all } x, y \in X.$$

Thus, we get the assumption (1) in Definition (3.1). Of course, the reverse implication do not takes place.

Remark 3.9. The above results generalize some theorems given in Wang [21] for the case of usual metric spaces. Moreover, some extensions of the above results for different types of multivalued generalized contractions (Reich type, Ćirić type, ., see [17]) can be given by applying the same method.

4. Coupled fixed point theorems

We will start this section by recalling some useful notion and results.

Definition 4.1 ([18]). Let $(X, \preceq)$ a partially ordered set and $G : X \times X \to P(X)$. We say that G has the strict mixed monotone property with respect to the partial order " $\preceq$ ", if the following implications holds:

1. $x_0 \preceq x_1 \Rightarrow G(x_0, y) \leq_{st} G(x_1, y)$, $\forall y \in X$.
2. $y_0 \succeq y_1 \Rightarrow G(x, y_0) \leq_{st} G(x, y_1)$, $\forall x \in X$.

Remark 4.2. Let (X, d) be a b-metric space with constant $s \geq 1$ and $Z := X \times X$. Then the functional $\tilde{d} : Z \times Z \to \mathbb{R}_+$ defined by,

$$\tilde{d}((x, y), (u, v)) \leq d(x, u) + d(y, v), \text{ for all } (x, y), (u, v) \in Z,$$

is a b-metric on Z with the same constant $s \geq 1$ and if (X, d) is a complete b-metric space, then $(Z, \tilde{d})$ is a complete b-metric space, too.
Moreover, for $x, y \in X$, $A, B, U, V \in P(X)$, we have,

$$D_{\tilde{d}}((x, y), A \times B) = D_d(x, A) + D_d(y, B),$$

$$\rho_{\tilde{d}}(U \times V, A \times B) = \rho_d(U, A) + \rho_d(V, B),$$

$$H_{\tilde{d}}(U \times V, A \times B) \leq H_d(U, A) + H_d(V, B)$$

and

$$H_{\tilde{d}}^+(U \times V, A \times B) = H_d^+(U, A) + H_d^+(V, B).$$

Additionally, by the properties of the gap functional D_d, if $(x, y) \in X \times X$ and $A, B \in P_{cl}(X)$, then,

$$D_{\tilde{d}}((x, y), A \times B) = 0 \text{ if and only if } (x, y) \in A \times B.$$

The first main result of this section is the following theorem.

Theorem 4.3. *Let $(X, \preceq, d)$ be an ordered b-metric space with constant $s \geq 1$ such that, the b-metric d is complete. Let $G : X \times X \to P(X)$ be a multi-valued operator having the strict mixed monotone property with respect to "$\preceq$" and G has closed graph. Assume that:*

(i) there exists $k \in \left(0, \frac{1}{s}\right)$ such that,

$$\rho_d(G(x,y), G(u,v)) + \rho_d(G(y,x), G(v,u)) \leq k[d(x,u) + d(y,v)], \ \forall x \preceq u \text{ and } y \succeq v,$$

(ii) there exist $(x_0, y_0) \in X \times X$ and $(x_1, y_1) \in G(x_0, y_0) \times G(y_0, x_0)$ such that $x_0 \preceq x_1$ and $y_0 \succeq y_1$.

Then, the following conclusions hold:

(a) there exist $x^*, y^* \in X$ and there exist two sequences $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ in X with,

$$\begin{cases} x_{n+1} \in G(x_n, y_n), \\ y_{n+1} \in G(y_n, x_n), \end{cases}$$

for all $n \in \mathbb{N}$ such that $x_n \to x^*$, $y_n \to y^*$ as $n \to \infty$ and

$$\begin{cases} x^* \in G(x^*, y^*), \\ y^* \in G(y^*, x^*). \end{cases}$$

(b) If in addition, the b-metric d is continuous, then we have the following estimation holds,

$$d(x_n, x^*) + d(y_n, y^*) \leq \frac{sk^n}{1 - sk}[d(x_0, x_1) + d(y_0, y_1)], \text{ for all } n \in \mathbb{N}.$$

Proof. (a) We denote by $Z = X \times X$. Let $(x_0, y_0) \in Z$ and $(x_1, y_1) \in G(x_0, y_0) \times G(y_0, x_0)$ such that $x_0 \preceq x_1$ and $y_0 \succeq y_1$. Using Lemma 2.5 we obtain that for any $q > 1$, there exists $(x_2, y_2) \in G(x_1, y_1) \times G(y_1, x_1)$ such that,

$$\begin{cases} d(x_1, x_2) \leq q\rho(G(x_0, y_0), G(x_1, y_1)), \\ d(y_1, y_2) \leq q\rho(G(y_0, x_0), G(y_1, x_1)). \end{cases}$$

Then,

$$\begin{aligned} d(x_1, x_2) + d(y_1, y_2) &\leq q[\rho(G(x_0, y_0), G(x_1, y_1)) + \rho(G(y_0, x_0), G(y_1, x_1))] \\ &\leq qk[d(x_0, x_1) + d(y_0, y_1)]. \end{aligned}$$

From Lemma 2.5, we get that there exists $(x_3, y_3) \in G(x_2, y_2) \times G(y_2, x_2)$ such that,

$$\begin{cases} d(x_2, x_3) \leq q\rho(G(x_1, y_1), G(x_2, y_2)), \\ d(y_2, y_3) \leq q\rho(G(y_1, x_1), G(y_2, x_2)). \end{cases}$$

Now we want to prove that $(x_1, y_1) \leq_p (x_2, y_2)$ which is equivalent to show that $x_1 \preceq x_2$ and $y_1 \succeq y_2$. From $x_0 \preceq x_1$ we have that $G(x_0, y) \leq_{st} G(x_1, y)$, for all $y \in X$. If we take $y = y_0$, then we obtain,

$$G(x_0, y_0) \leq_{st} G(x_1, y_0). \tag{4.1}$$

From $y_0 \succeq y_1$ we have that $G(x, y_0) \leq_{st} G(x, y_1)$, for all $x \in X$. If we take $x = x_1$, then we obtain,

$$G(x_1, y_0) \leq_{st} G(x_1, y_1). \tag{4.2}$$

From (4.1) and (4.2) we obtain that $G(x_0, y_0) \leq_{st} G(x_1, y_1)$. Thus, $x_1 \preceq x_2$. From $x_0 \preceq x_1$, we have that $G(y, x_0) \geq_{st} G(y, x_1)$, for all $y \in X$. If we take $y = y_0$, then we obtain,

$$G(y_0, x_0) \geq_{st} G(y_0, x_1). \tag{4.3}$$

From $y_0 \succeq y_1$ we have that $G(y_0, x) \geq_{st} G(y_1, x)$, for all $x \in X$. If we take $x = x_0$, then we obtain,

$$G(y_0, x_1) \geq_{st} G(y_1, x_1). \tag{4.4}$$

From (4.3) and (4.4) we obtain that $G(y_0, x_0) \geq_{st} G(y_1, x_1)$. Thus, $y_1 \succeq y_2$. Then,

$$\begin{aligned} d(x_2, x_3) + d(y_2, y_3) &= q[\rho(G(x_1, y_1), G(x_2, y_2)) + \rho(G(y_1, x_1)) + G(y_2, x_2)] \\ &\leq qk[d(x_1, x_2) + d(y_1, y_2)] \\ &\leq (qk)^2[d(x_0, x_1) + d(y_0, y_1)]. \end{aligned}$$

By this procedure, we obtain two sequences $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ in X with the following properties:
(1) $x_n \preceq x_{n+1}$ and $y_n \geq y_{n+1}$, for all $n \geq 0$.
(2) $x_{n+1} \in G(x_n, y_n)$ and $y_{n+1} \in G(y_n, x_n)$, for all $n \geq 0$.
(3) $d(x_n, x_{n+1}) + d(y_n, y_{n+1}) \leq (qk)^n[d(x_0, x_1) + d(y_0, y_1)]$.
Let $\tilde{d} : Z \times Z \to \mathbb{R}_+$ be a functional defined by,

$$\tilde{d}((x, y), (u, v)) = d(x, u) + d(y, v), \ \textit{for all} \ (x, y), (u, v) \in Z.$$

Consider $z_0 = (x_0, y_0) \in Z$ and $z_n = (x_n, y_n) \in Z$, then the relation (3) is equivalent with,

$$\tilde{d}((x_n, y_n), (x_{n+1}, y_{n+1})) \leq (qk)^n \tilde{d}((x_0, y_0), (x_1, y_1)) \Leftrightarrow$$

$$\Leftrightarrow \tilde{d}(z_n, z_{n+1}) \leq (qk)^n d(z_0, z_1), \ \textit{for all} \ n \in \mathbb{N}.$$

We choose $1 < q < \dfrac{1}{ks}$ and we must to show that $z_n = (x_n, y_n) \in Z$ is a Cauchy sequence on $(Z, \tilde{d})$. We have,

$$\begin{aligned} \tilde{d}(z_n, z_{n+p}) &\leq s\tilde{d}(z_n, z_{n+1}) + s^2\tilde{d}(z_{n+1}, z_{n+2}) + ... + s^{p-1}\tilde{d}(z_{n+p-1}, z_{n+p}) \\ &\leq s(qk)^n\tilde{d}(z_0, z_1) + s^2(qk)^{n+1}\tilde{d}(z_0, z_1) + ... + s^{p-1}(qk)^{n+p-1}\tilde{d}(z_0, z_1) \\ &\leq s(qk)^n[1 + sqk + ... + (sqk)^{p-1}]\tilde{d}(z_0, z_1) \\ &= s(qk)^n\frac{1 - (sqk)^p}{1 - sqk}\tilde{d}(z_0, z_1) \leq \frac{s(qk)^n}{1 - sqk}\tilde{d}(z_0, z_1). \end{aligned}$$

Hence,

$$\tilde{d}(z_n, z_{n+p}) \leq \frac{s(qk)^n}{1 - sqk}\tilde{d}(z_0, z_1) \to 0 \ as \ n \to \infty, \ \forall p \in \mathbb{N}^*. \tag{4.5}$$

Thus $(z_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in Z. Since $(Z, \tilde{d})$ is a complete b-metric space, there exists $z^* = (x^*, y^*) \in Z$ such that $\lim_{n\to\infty} z_n = z^*$, therefore $\lim_{n\to\infty} x_n = x^*$ and $\lim_{n\to\infty} y_n = y^*$.
Now, we will prove that $(x^*, y^*) \in Z$ is a coupled fixed point of G i.e.,

$$\begin{cases} x^* \in G(x^*, y^*), \\ y^* \in G(y^*, x^*). \end{cases}$$

Consider $T : Z \to P_{cl}(Z)$ a multi-valued operator defined by,

$$T(x, y) = G(x, y) \times G(y, x), \ \forall (x, y) \in Z.$$

By definition of $\tilde{d}$ we have that,

$$\begin{aligned} D_{\tilde{d}}(z^*, T(z^*)) &= D_{\tilde{d}}((x^*, y^*), G(x^*, y^*) \times G(y^*, x^*)) \\ &= D_d(x^*, G(x^*, y^*)) + D_d(y^*, G(y^*, x^*)) \\ &\leq s[d(x^*, x_{n+1}) + D_d(x_{n+1}, G(x^*, y^*)] + s[d(y^*, y_{n+1}) + D_d(y_{n+1}, G(y^*, x^*)] \\ &\leq s[d(x^*, x_{n+1}) + d(y^*, y_{n+1})] + s[\rho(G(x_n, y_n), G(x^*, y^*)) + \rho(G(y_n, x_n), G(y^*, x^*))] \\ &\leq s[d(x^*, x_{n+1}) + d(y^*, y_{n+1})] + sk[d(x_n, x^*) + d(y_n, y^*)] \to 0 \ \ as \ \ n \to \infty. \end{aligned}$$

Since G has closed values, we obtain that T has closed values too. As a consequence, $z^* \in T(z^*)$, i.e., $(x^*, y^*) \in G(x^*, y^*) \times G(y^*, x^*)$. This means that $x^* \in G(x^*, y^*)$ and $y^* \in G(y^*, x^*)$.
(b) Using (4.5) we have,

$$\begin{aligned}\tilde{d}(z_n, z_{n+p}) &= \tilde{d}((x_n, y_n), (x_{n+p}, y_{n+p})) = d(x_n, x_{n+p}) + d(y_n, y_{n+p}) \\ &\leq \frac{s(qk)^n}{1 - sqk}[d(x_0, x_1) + d(y_0, y_1)].\end{aligned}$$

Since d is a continuous b-metric and $1 < q < \dfrac{1}{sk}$ is arbitrary chosen, letting $p \to \infty$ and $q \searrow 1$, we get,

$$d(x_n, x^*) + d(y_n, y^*) \leq \frac{sk^n}{1 - sk}[d(x_0, x_1) + d(y_0, y_1)], \ \textit{for all } n \in \mathbb{N}.$$

□

Remark 4.4. Notice that the contraction condition (i) from Theorem 4.3 can be re-written, using the functional H^+, as follows:

$$H_d^+(T(x, y), T(u, v)) \leq 2k(d(x, u) + d(y, v)), \ \forall (x, y), (u, v) \in X \times X \text{ with } (x, y) \leq_p (u, v).$$

Remark 4.5. If instead of the hypothesis that the triple $(X, \preceq, d)$ is a complete ordered b-metric space, we only assume that X is a nonempty set, $\preceq$ is a partially ordering on X and d is a complete b-metric with constant $s \geq 1$, then the conclusions of the above theorem hold, if additionally, the graph of G is a closed set.

The following result is about the uniqueness of the coupled fixed point.

Theorem 4.6. *In addition to the hypotheses of Theorem* (4.3) *we suppose that,*

(i) there exists $(x^*, y^*) \in X \times X$ *such that,*

$$\begin{cases} G(x^*, y^*) = \{x^*\}, \\ G(y^*, x^*) = \{y^*\}, \end{cases}$$

(ii) for any solution $(\overline{x}, \overline{y})$ *of the coupled fixed point problem (P1) we have* $\overline{x} \leq x^*$ *and* $\overline{y} \geq y^*$ *or* $(\overline{x}, \overline{y}) \leq_p (x^*, y^*)$.

Then, we obtain that the coupled fixed point problem (P1) has an unique solution.

Proof. Let $(x^*, y^*) \in X \times X$ such that,

$$\begin{cases} G(x^*, y^*) = \{x^*\}, \\ G(y^*, x^*) = \{y^*\}. \end{cases}$$

We suppose that there exists $(\overline{x}, \overline{y}) \in X \times X$ such that,

$$\begin{cases} \overline{x} \in G(\overline{x}, \overline{y}), \\ \overline{y} \in G(\overline{y}, \overline{x}), \end{cases}$$

with $(\overline{x}, \overline{y}) \leq_p (x^*, y^*)$.
We denotes by $Z = X \times X$ and consider the functional $\tilde{d} : Z \times Z \to \mathbb{R}_+$ defined by,

$$\tilde{d}((x, y), (u, v)) = d(x, u) + d(y, v).$$

Then,

$$\begin{aligned}
\tilde{d}((\overline{x},\overline{y}),(x^*,y^*)) &= D_{\tilde{d}}((\overline{x},\overline{y}),G(x^*,y^*)\times G(y^*,x^*)) \\
&= D_d(\overline{x},G(x^*,y^*)) + D_d(\overline{y},G(y^*,x^*)) \\
&\le \rho(G(\overline{x},\overline{y}),G(x^*,y^*)) + \rho(G(\overline{y},\overline{x}),G(y^*,x^*)) \\
&\le k[d(\overline{x},x^*) + d(\overline{y},y^*)] \\
&= k\tilde{d}((\overline{x},\overline{y}),(x^*,y^*)).
\end{aligned}$$

Thus $(1-k)\tilde{d}((\overline{x},\overline{y}),(x^*,y^*)) \le 0$, which means that $\tilde{d}((\overline{x},\overline{y}),(x^*,y^*)) = 0$ and $(\overline{x},\overline{y}) = (x^*,y^*)$. □

Theorem 4.7. *We suppose that all the hypotheses of Theorem (4.6) take place and $x^* \le y^*$ or $y^* \le x^*$ where (x^*,y^*) is the unique coupled fixed point of G. Then $x^* = y^*$ i.e $G(x^*,x^*) = \{x^*\}$.*

Proof. From Theorem 4.6 we have that there exists $(x^*,y^*) \in X \times X$ such that,

$$\begin{cases} G(x^*,y^*) = \{x^*\}, \\ G(y^*,x^*) = \{y^*\}. \end{cases}$$

Let $x^* \le y^*$ then $(x^*,y^*) \le_p (y^*,x^*)$. We will prove now that G has a unique fixed point.

$$\begin{aligned}
d(x^*,y^*) &= \frac{1}{2}[d(x^*,y^*) + d(y^*,x^*)] \\
&= \frac{1}{2}[D_d(x^*,G(y^*,x^*)) + D_d(y^*,G(x^*,y^*))] \\
&\le \frac{1}{2}\rho(G(x^*,y^*),G(y^*,x^*)) + \frac{1}{2}\rho(G(y^*,x^*),G(x^*,y^*)) \\
&\le \frac{1}{2}k[d(x^*,y^*) + d(y^*,x^*)] = kd(x^*,y^*).
\end{aligned}$$

Thus $(1-k)d(x^*,y^*) \le 0$ and we get to the conclusion that $d(x^*,y^*) = 0$, i.e., $x^* = y^*$. □

If the contraction condition withe respect to the excess functional is assumed for every elements $x, y \in X$, then no monotonicity assumptions are needed for G. Hence, we get the following result.

Theorem 4.8. *Let (X,d) be a complete b-metric space with constant $s \ge 1$. Let $G : X \times X \to P_{cl}(X)$ be a multi-valued operator for which there exists $k \in (0,1)$ such that,*

$$\rho_d(G(x,y),G(u,v)) + \rho_d(G(y,x),G(v,u)) \le k[d(x,u) + d(y,v)]\ ,\forall (x,y),(u,v) \in X \times X.$$

Then, the following conclusions hold:

(a) $CFix(G) \ne \emptyset$ and for any initial point $(x_0,y_0) \in X \times X$, the sequences $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ defined by,

$$\begin{cases} x_{n+1} \in G(x_n,y_n), \\ y_{n+1} \in G(y_n,x_n), \end{cases}$$

converge to x^* and respectively to y^* as $n \to \infty$ where,

$$\begin{cases} x^* \in G(x^*,y^*), \\ y^* \in G(y^*,x^*). \end{cases}$$

(b) In particular, if the b-metric d is continuous, then we have the following estimation:

$$d(x_n,x^*) + d(y_n,y^*) \le \frac{sk^n}{1-sk}[d(x_0,x_1) + d(y_0,y_1)].$$

(c) *If additionally, there exists a pair* $(u^*, v^*) \in X \times X$ *such that,*

$$\begin{cases} G(u^*, v^*) = \{u^*\}, \\ G(v^*, u^*) = \{v^*\}, \end{cases}$$

then $CFix(G) = \{(x^*, y^*)\}$.

Proof. We denote by $Z = X \times X$ and consider the functional $\tilde{d} : Z \times Z \to \mathbb{R}_+$ defined by,

$$\tilde{d}((x, y), (u, v)) = d(x, u) + d(y, v).$$

Then, $\tilde{d}$ is a b-metric on Z with the same constant $s \geq 1$. Let $T : Z \to P(Z)$ defined by $T(x, y) = G(x, y) \times G(y, x)$ for all $(x, y) \in Z$. The multi-valued operator T is a contraction (with respect to $\rho_{\tilde{d}}$) on $(Z, \tilde{d})$. Let $z = (x, y)$, $w = (u, v) \in Z$. Then, we have,

$$\begin{aligned} \rho_{\tilde{d}}(T(z), T(w)) &= \rho_{\tilde{d}}(G(x, y) \times G(y, x), G(u, v) \times G(v, u)) \\ &= \rho_d(G(x, y), G(u, v) + \rho_d(G(y, x), G(v, u))) \\ &\leq k[d(x, u) + d(y, v)] = k\tilde{d}(z, w). \end{aligned}$$

Thus, by Theorem 3.7 we have that there exists $(x^*, y^*) \in Z$ such that $(x^*, y^*) \in T(x^*, y^*)$ i.e $(x^*, y^*) \in G(x^*, y^*) \times G(y^*, x^*)$. Thus $CFix(G) \neq \emptyset$. Using Theorem 3.7 we also obtain that there exists a sequence $(z_n)_{n\in\mathbb{N}} \subset Z$ of successive approximation for T starting from any point $(z_0, z_1) \in Graph(T)$, which converges to a fixed point (x^*, y^*) of T, i.e., $(x_n, y_n) \to (x^*, y^*)$ as $n \to \infty$. From Theorem 3.7, we have that,

$$d(x_n, x^*) + d(y_n, x^*) \leq \frac{sk^n}{1 - sk}[d(x_0, x_1) + d(y_0, y_1)], \forall n \in \mathbb{N},$$

or equivalently

$$\tilde{d}(x_n, y_n), (x^*, y^*) \leq \frac{sk^n}{1 - sk}\tilde{d}((x_0, y_0), (x_1, y_1)), \forall n \in \mathbb{N}.$$

(b) Let $(u^*, v^*) \in Z$ such that,

$$\begin{cases} G(u^*, v^*) = \{u^*\}, \\ G(v^*, u^*) = \{v^*\}. \end{cases}$$

This means that $\{(u^*, v^*)\} = G(u^*, v^*) \times G(v^*, u^*)$ and $\{(u^*, v^*)\} = F(u^*, v^*)$. Thus, $SFix(T) \neq \emptyset$. From Theorem 3.7 we have that $Fix(T) = SFix(T) = \{(x^*, y^*)\}$, which means that T has a unique fixed point in Z. In conclusion, G has also a unique coupled fixed point in Z. □

In the next part of this section, we will present some proprieties of the coupled fixed point problem (P1). The first one is the data dependence problem for the coupled fixed point problem (P1).

Theorem 4.9. *Let* (X, d) *be a complete b-metric space with constant* $s \geq 1$ *and let* $G : X \times X \to P_{cl}(X)$, $S : X \times X \to P(X)$ *be two multi-valued operators. We suppose that:*

(i) *there exists* $k \in \left(0, \frac{1}{s}\right)$ *such that,*

$$\rho_d(G(x, y), G(u, v)) + \rho_d(G(y, x), G(v, u)) \leq k[d(x, u) + d(y, v)], \ \forall (x, y), (u, v) \in X \times X,$$

(ii) *there exists* $(x^*, y^*) \in X \times X$ *such that,*

$$\begin{cases} G(u^*, v^*) = \{u^*\}, \\ G(v^*, u^*) = \{v^*\}, \end{cases}$$

(iii) there exists $(u^*, v^*) \in X \times X$ *such that,*

$$\begin{cases} u^* \in S(u^*, v^*), \\ v^* \in S(v^*, u^*), \end{cases}$$

(iv) there exists $\eta > 0$ *such that* $\rho_d(G(x,y), S(x,y)) \leq \eta$*, for all* $(x,y) \in X \times X$*.*

Then

$$\rho_{\tilde{d}}(CFix(S), CFix(G)) \leq \frac{2s\eta}{1-sk},$$

where $\tilde{d}((x,y),(u,v)) = d(x,u) + d(y,v),\ \forall (x,y),(u,v) \in X \times X$*.*

Proof. Let $(u^*, v^*) \in X \times X$ such that,

$$\begin{cases} u^* \in S(u^*, v^*), \\ v^* \in S(v^*, u^*), \end{cases}$$

and $(x^*, y^*) \in X \times X$ such that,

$$\begin{cases} G(x^*, y^*) = \{x^*\}, \\ G(y^*, x^*) = \{y^*\}. \end{cases}$$

We denote by $Z = X \times X$ and consider the functional $\tilde{d} : X \times X \to \mathbb{R}_+$ defined by,

$$\tilde{d}((x,y),(u,v)) = d(x,u) + d(y,v), \forall (x,y),(u,v) \in Z.$$

We have,

$$\begin{aligned} \tilde{d}((u^*,v^*),(x^*,y^*)) &= D_{\tilde{d}}((u^*,v^*), G(x^*,y^*) \times G(y^*,x^*)) \\ &= D_d(u^*, G(x^*,y^*)) + D_d(v^*, G(y^*,x^*)) \\ &= D_d(G(x^*,y^*), u^*) + D_d(G(y^*,x^*), v^*) \\ &\leq s[\rho_d(G(x^*,y^*), G(u^*,v^*)) + D_d(G(u^*,v^*), u^*)] \\ &+ s[\rho_d(G(y^*,x^*), G(v^*,u^*)) + D_d(G(v^*,u^*), v^*)] \\ &\leq s[\rho_d(G(x^*,y^*), G(u^*,v^*)) + \rho_d(G(y^*,x^*), G(v^*,u^*))] \\ &+ s[\rho_d(G(u^*,v^*), S(u^*,v^*)) + \rho_d(G(v^*,u^*), S(v^*,u^*))] \\ &\leq sk[d(x^*,u^*) + d(y^*,v^*)] + 2\eta s. \end{aligned}$$

Then,

$$\tilde{d}((x^*,y^*),(u^*,v^*)) \leq \frac{2\eta s}{1-sk}.$$

Thus, $D_{\tilde{d}}((u^*,v^*), CFix(G)) \leq \dfrac{2\eta s}{1-sk}$. Since $(u^*,v^*) \in CFix(S)$ are arbitrary chosen, we obtain that,

$$\rho_{\tilde{d}}(CFix(S), CFix(G)) \leq \frac{2s\eta}{1-sk}.$$

□

The second problem we intend to study is the well-posedness of the coupled fixed point problem.

Definition 4.10. We consider the coupled fixed point problem $(P1)$. By definition, $(P1)$ is well-posed for G with respect to D_d if:

(i) $CFix(G) = \{w^*\}$, where $w^* = (u^*, v^*) \in X \times X$,

(ii) if there exists a sequence $w_n = (u_n, v_n) \in X \times X$ with,

$$\begin{cases} D_d(u_n, G(u_n, v_n)) \to 0, \\ D_d(v_n, G(v_n, u_n)) \to 0, \end{cases}$$

then $d(u_n, u^*) + d(v_n, v^*) \to 0$ as $n \to \infty$.

Theorem 4.11. *We suppose that all the hypotheses of Theorem (4.8) take place. Then the coupled fixed point problem $(P1)$ is well-posed for G with respect to D_d.*

Proof. Since the multi-valued operator G satisfies the hypotheses of Theorem 4.8, we get that G has a unique coupled fixed point, i.e., $CFix(G) = \{w^*\}$ where $w^* = (u^*, v^*) \in X \times X$. Then we have,

$$\begin{cases} G(u^*, v^*) \in \{u^*\}, \\ G(v^*, u^*) \in \{v^*\}. \end{cases}$$

Let $w_n = (u_n, v_n) \subset X \times X$, with,

$$\begin{cases} D_d(u_n, G(u_n, v_n)) \to 0, \\ D_d(v_n, G(v_n, u_n)) \to 0, \end{cases}$$

as $n \to \infty$. Then we have,

$$\begin{aligned} d(u_n, x^*) + d(v_n, v^*) &= D_d(u_n, G(u^*, v^*)) + D_d(v_n, G(v^*, u^*)) \\ &\leq s[D_d(u_n, G(u_n, v_n)) + \rho_d(G(u_n, v_n), G(u^*, v^*))] \\ &+ s[D_d(v_n, G(v_n, u_n)) + \rho_d(G(v_n, u_n), G(v^*, u^*))] \\ &\leq sk[d(u_n, u^*) + d(v_n, v^*)] + s[D_d(u_n, G(u_n, v_n)) + D_d(v_n, G(v_n u_n))]. \end{aligned}$$

Hence,

$$d(u_n, u^*) + d(v_n, v^*) \leq \frac{s}{1 - sk}[D_d(u_n, G(u_n, v_n)) + D_d(v_n, G(v_n, u_n))] \to 0 \text{ as } n \to \infty.$$

□

Now we will present the Ulam-Hyers property of the coupled fixed point problem.

Definition 4.12. Let (X, d) be a b-metric space with constant $s \geq 1$ and let $G : X \times X \to P(X)$ be a multi-valued operator. Let $\tilde{d}$ be any b-metric on $X \times X$ generated by d. Let us consider the system of inclusions,

$$\begin{cases} x \in G(x, y), \\ y \in G(y, x), \end{cases} \tag{4.6}$$

and the inequality,

$$D_d(x, G(x, y)) + D_d(y, G(x, y)) \leq \epsilon, \tag{4.7}$$

where $\epsilon > 0$ and $(x, y) \in X \times X$.

By definition, the system of inclusions (4.6) is called Ulam-Hyers stable if and only if, there exists a function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ increasing, continuous in 0 with $\psi(0) = 0$, such that for each $\epsilon > 0$ and for each solution $(u^*, v^*) \in X \times X$ of the inequality (4.7), there exists a solution $(x^*, y^*) \in X \times X$ of (4.6) such that,

$$d(u^*, v^*) + d(v^*, y^*) \leq \psi(\epsilon) \Leftrightarrow \tilde{d}((x^*, y^*), (u^*, v^*)) \leq \psi(\epsilon).$$

Theorem 4.13. *Let $G : X \times X \to P_{cl}(X)$ be a multi-valued operator which verifies the hypotheses of Theorem (4.8). Then the system of inclusions (4.6) is Ulam-Hyers stable.*

Proof. From Theorem 4.8 we have that G has a unique coupled fixed point $(x^*, y^*) \in X \times X$ such that,

$$\begin{cases} G(x^*, y^*) \in \{x^*\}, \\ G(y^*, x^*) \in \{y^*\}. \end{cases}.$$

Let $\epsilon > 0$ and $(u^*, v^*) \in X \times X$ be a ϵ-solution of the coupled fixed point problem (4.6), i.e., a solution of the inequality (4.7). Then, we have $D_d(u^*, G(u^*, v^*)) + D_d(v^*, G(v^*, u^*)) \leq \epsilon$. Hence, we get,

$$\begin{aligned} d(u^*, x^*) + d(v^*, y^*) &= D_d(u^*, G(x^*, y^*)) + D_d(v6*, G(y^*, x^*)) \\ &\leq s[D_d(u^*, G(u^*, v^*)) + \rho_d(G(u^*, v^*), G(x^*, y^*))] \\ &+ s[D_d(v^*, G(v^*, u^*)) + \rho_d(G(v^*, u^*), G(y^*, x^*))] \\ &\leq sk[d(u^*, x^*) + d(v^*, y^*)] + s[D_d(u^*, G(u^*, v^*)) + D_d(v^*, G(v^*, u^*))]. \end{aligned}$$

Thus,

$$d(u^*, x^*) + d(v^*, y^*) \leq \frac{s}{1 - sk} \cdot \epsilon.$$

Therefore, if we consider $\psi : \mathbb{R}_+ \to \mathbb{R}_+$, given by $\psi(t) = ct$ (with $c = \dfrac{s}{1 - sk} > 0$), we can conclude that the system of inclusion (4.6) is Ulam-Hyers stable. □

For the next result, we need the following theorem (known as Cauchy's Lemma).

Lemma 4.14 (Cauchy's Lemma)**.** *Let $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ be two sequences of non-negative real numbers, such that $\sum_{p=0}^{\infty} a_p < +\infty$ and $\lim_{n\to\infty} b_n = 0$. Then, $\lim_{n\to\infty} \Big(\sum_{p=0}^{n} a_{n-p} b_p \Big) = 0$.*

Using the above result and the global existence result for the coupled fixed point problem (see Theorem 4.8), we can prove the limit shadowing property of the coupled fixed point problem (P1).

Definition 4.15. Let (X, d) be a b-metric space with constant $s \geq 1$ and $G : X \times X \to P(X)$ be a multi-valued operator. Let $\tilde{d}$ be any b-metric generated by d. By definition, the coupled fixed problem $(P1)$ has the limit shadowing property if, for any sequence $(x_n, y_n)_{n\in\mathbb{N}}$ in $X \times X$ for which,

$$D\tilde{d}((x_{n+1}, y_{n+1}), G(x_n, y_n) \times G(y_n, x_n)) \to 0 \text{ as } n \to \infty,$$

there exists a sequence $(u_n, v_n)_{n\in\mathbb{N}}$ in $X \times X$ such that,

$$\tilde{d}((x_n, y_n), (u_n, v_n)) \to 0, \text{ as } n \to \infty.$$

Theorem 4.16. *Let (X, d) be a b-metric space with constant $s \geq 1$ and $G : X \times X \to P_{cl}(X)$ be a multi-valued operator which verifies all the hypotheses of Theorem (4.8). Then, the coupled fixed point problem $(P1)$ has the limit shadowing property.*

Proof. By Theorem 4.8 we get that $CFix(G) = \{x^*, y^*\}$ and for any initial point $(u_0, v_0) \in X \times X$, there exist two sequence $(u_n)_{n\mathbb{N}}$ and $(v_n)_{n\mathbb{N}}$ in X with $u_{n+1} \in G(u_n, v_n)$ and $v_{n+1} \in G(v_n, u_n)$ for all $n \in \mathbb{N}$, such that $(u_n)_{n\in\mathbb{N}} \to x^*$, $(v_n)_{n\in\mathbb{N}} \to y^*$ as $n \to \infty$. We denote $Z = X \times X$ and consider the functional $\tilde{d} : Z \times Z \to \mathbb{R}_+$ defined by,

$$\tilde{d}((x, y), (u, v)) = d(x, u) + d(y, v).$$

We know that if d is a b-metric with constant $s \geq 1$ then $\tilde{d}$ is a b-metric with the same constant s. Let $(x_n, y_n)_{n\in\mathbb{N}}$ be a sequence in Z such that,

$$D_{\tilde{d}}((x_{n+1}, y_{n+1}), G(x_n, y_n) \times G(y_n, x_n)) \to 0 \text{ as } n \to \infty.$$

This also means that,

$$D_d((x_{n+1}, G(x_n, y_n)) + D_d(y_n, G(y_n, x_n)) \to 0, \text{ as } n \to \infty.$$

Thus,

$$\tilde{d}((x_{n+1}, y_{n+1}), (u_{n+1}, v_{n+1})) \leq s[\tilde{d}((x_{n+1}, y_{n+1}), (x^*, y^*)) + \tilde{d}((x^*, y^*), (u_{n+1}, v_{n+1}))].$$

Next, we can write,

$$\begin{aligned}
\tilde{d}((x_{n+1}, y_{n+1}), (x^*, y^*)) &= D_{\tilde{d}}((x_{n+1}, y_{n+1}), G(x^*, y^*) \times G(y^*, x^*)) \\
&= D_d(x_{n+1}, G(x^*, y^*)) + D_d(y_{n+1}, G(y^*, x^*)) \\
&\leq s[D_d(x_{n+1}, G(x_n, y_n)) + \rho_d(G(x_n, y_n), G(x^*, y^*))] \\
&+ s[D_d(y_{n+1}, G(y_n, x_n))] + \rho_d(G(y_n, x_n), G(y^*, x^*)) \\
&\leq s[D_d(x_{n+1}, G(x_n, y_n)) + D_d(y_{n+1}), G(y_n, x_n))] + sk[d(x_n, x^*) + d(y_n, y^*)] \\
&= s[D_d(x_{n+1}, G(x_n, y_n)) + D_d(y_{n+1}, G(y_n, x_n))] + sk\tilde{d}((x_n, y_n), (x^*, y^*)) \\
&\leq sk\{s[D_d(x_n, G(x_{n-1}, y_{n-1})) + D_d(y_n, G(y_{n-1}, x_{n-1}))] + sk\tilde{d}((x_{n-1}, y_{n-1}), (x^*, y^*))\} \\
&+ s[D_d(x_{n+1}, G(x_n, y_n)) + D_d(y_{n+1}), G(y_n, x_n))] \\
&\leq \dots \leq \\
&\leq s\sum_{p=0}^{n}(sk)^{n-p}[D_d(x_{p+1}, G(x_p, y_p)) + D_d(y_{p+1}, G(y_p, x_p))] + (sk)^{n+1}\tilde{d}((x_0, y_0), (x^*, y^*))
\end{aligned}$$

By Cauchy's Lemma we obtain that,

$$s\sum_{p=0}^{n}(sk)^{n-p}[D_d(x_{p+1}, G(x_p, y_p)) + D_d(y_{p+1}, G(y_p, x_p))] \to 0 \text{ as } n \to \infty.$$

Thus we obtain that,

$$\tilde{d}((x_{n+1}, y_{n+1}), (x^*, y^*)) \to 0 \text{ as } n \to \infty.$$

Then,

$$\tilde{d}((x_{n+1}, y_{n+1}), (u_{n+1}, v_{n+1})) \to 0 \text{ as } n \to \infty,$$

and so, we get the final conclusion,

$$d(x_n, u_n) + d(y_n, v_n) \to 0 \text{ as } n \to \infty.$$

□

5. Applications

The purpose of this section is to give some applications of the previous results.

Let us consider first the following system of integral inclusions,

$$\begin{cases} x(t) \in \int_0^t K(s, x(s), y(s))ds + g(t), \\ y(t) \in \int_0^t K(s, y(s), x(s))ds + g(t), \end{cases} for\ t \in [0, T], \tag{5.1}$$

where $g : [0, T] \to \mathbb{R}$ and $K : [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \to P(\mathbb{R}^n)$ are operators satisfying some appropriate conditions. A solution of the above system is a pair $(x, y) \in C([0, T], \mathbb{R}^n) \times C([0, T], \mathbb{R}^n)$ satisfying the above relations for all $t \in [0, T]$. We consider $X := C([0, T], \mathbb{R}^n)$ endowed with the partial order relation,

$$x \leq_C y \Leftrightarrow x(t) \leq y(t), \text{ for all } t \in [0, T],$$

where " $\leq$ " is the component wise ordering relation on $\mathbb{R}^n$. We will also denote by $|\cdot|$ a norm in $\mathbb{R}^n$. We also consider (for $p \in \mathbb{N}, p \geq 2$ an even number) the following functional,

$$d(x,y) := \max_{t\in[0,T]}[(x(t)-y(t))^p e^{-\tau t}].$$

Notice that d is a b-metric, for any $\tau > 0$ arbitrary chosen. Indeed, it is easy to check that the first conditions are satisfied for all $x, y \in X$. We show that the triangle's inequality is verified too. Using Hölder's inequality we obtain that,

$$\begin{aligned}(x(t)-y(t))^p &\leq 2^{\frac{p}{q}}[(x(t)-z(t))^p + (z(t)-y(t))^p]\cdot e^{-\tau t}\cdot e^{\tau t}\\ &\leq 2^{\frac{p}{q}}\max_{t\in[0,T]}((x(t)-z(t))^p e^{\tau t} + (z(t)-y(t))^p e^{\tau t})\cdot e^{\tau t}\\ &= 2^{\frac{p}{q}}(d(x,z)+d(y,z))\cdot e^{\tau t},\end{aligned}$$

with $\dfrac{1}{p}+\dfrac{1}{q}=1$.
After a multiplication with $e^{-\tau t}$ and taking then maximum over $t \in [0,T]$ we get that,

$$d(x,y) \leq 2^{p-1}(d(x,z)+d(z,y)).$$

Notice also that the triple $(X, \leq, d)$ is an ordered metric space.

Now, we can prove the following existence result.

Theorem 5.1. *Consider the integral system (5.1). We suppose that:*

(i) $g : [0,T] \to \mathbb{R}$ is a continuous function and $K : [0,T]\times\mathbb{R}^n\times\mathbb{R}^n \to \mathbb{R}^n$ is Lebesgue measurable in the first variable and jointly $H_{|\cdot|}$- continuous in the last two variables.

(ii) K is integrably bounded, i.e., there exists a mapping $r \in L^1[0,T]$ such that for each $(s,u,v) \in [0,T]\times\mathbb{R}^n\times\mathbb{R}^n$ and for any $w \in K(s,u,v)$, we have $|w| \leq r(t)$, a.e. $t \in [0,T]$.

(iii) $K(s,\cdot,\cdot)$ has the strict mixed monotone property with respect to the last two variables, for all $s \in [0,T]$.

(iv) There exist $\alpha, \beta \in L^1([0,T],\mathbb{R}_+)$ such that, for each $s \in [0,T]$, we have,

$$\rho_{|\cdot|}(K(s,x,y),K(s,u,v)) \leq \alpha(s)|x-u| + \beta(s)|y-v|,\ \forall x,y,u,v \in \mathbb{R}^n,$$

with $(x \leq u,\ y \geq v)$ or $(x \leq u,\ y \leq v)$.

(v) There exist $x_0, y_0 \in C[0,T]$ and two measurable selections $f_{x_0,y_0} : [0,T] \to \mathbb{R}^n$ of $K(\cdot,x_0(\cdot),y_0(\cdot))$ and $f_{y_0,x_0} : [0,T] \to \mathbb{R}^n$ of $K(\cdot,y_0(\cdot),x_0(\cdot))$, such that,

$$\begin{cases} x_0(t) \leq g(t) + \int\limits_0^t K(s,x_0(s),y_0(s))f_{x_0,y_0}(s)ds, \\ y_0(t) \geq g(t) + \int\limits_0^t K(s,y_0(s),x_0(s))f_{y_0,x_0}(s)ds, \end{cases} \tag{5.2}$$

or

$$\begin{cases} x_0(t) \geq g(t) + \int\limits_0^t K(s,x_0(s),y_0(s))f_{x_0,y_0}(s)ds, \\ y_0(t) \leq g(t) + \int\limits_0^t K(s,x_0(s),y_0(s))f_{y_0,x_0}(s)ds, \end{cases} \tag{5.3}$$

for all $t \in [0,T]$.

Then, there exists at least one solution (x^*, y^*) of the system (5.1).

Proof. We will work in the b- metric space (X, d), with $X = C([0, T], \mathbb{R}^n)$ and

$$d(x, y) := \max_{t\in[0,T]} [(x(t) - y(t))^p e^{-\tau t}] = ||(x - y)^p||_B.$$

We can prove that all the assumptions of Theorem 4.3 are satisfied. For this purpose, we define $S : X \times X \to P(X)$, $(x, y) \to S(x, y)$ given by,

$$S(x, y) := \{s \in X \ with \ s(t) := g(t) + \int\limits_0^t f_{x,y}(s)ds | f_{x,y} \in \mathcal{S}_{K(\cdot,x(\cdot),y(\cdot))}\},$$

where $\mathcal{S}_{K(\cdot,x(\cdot),y(\cdot))}$ denotes the set of all measurable selections of the multi-valued operator $s \to K(s, x(s), y(s))$. Notice that, by (i), for each measurable mappings $x, y : [0, T] \to \mathbb{R}$, the multi-valued operator$K(\cdot, x(\cdot), y(\cdot))$ is Lebesgue measurable (see, for example, [18]). Hence, by Kuratowski and Ryll-Nardzewski selection theorem, we obtain that $\mathcal{S}_{K(\cdot,x(\cdot),y(\cdot))}$ is nonempty for every $x, y \in X$ and by (ii), we have that $S_{K(\cdot,x(\cdot),y(\cdot))} \subseteq L^1([0, T], \mathbb{R}^n)$. Thus, S is well defined. Then, the system (5.1) can be written as a coupled fixed point problem for S as follows,

$$\begin{cases} x \in S(x, y), \\ y \in S(y, x). \end{cases} \tag{5.4}$$

We will show that S satisfies all the assumptions of Theorem 4.3. We will show first that S has closed values. Indeed, let $x, y \in X$ and let (s_n) be a sequence in $S(x, y)$ with $s_n \to s$ in (X, d). We have to show that $s \in S(x, y)$, i.e., there exists $f_{x,y} \in \mathcal{S}_{K(\cdot,x(\cdot),y(\cdot))}$ such that,

$$s(t) = g(t) + \int_0^t f_{x,y}(s)ds.$$

Since $s_n \in S(x, y)$, there exists $f^{(n)}_{x,y} \in \mathcal{S}_{K(\cdot,x(\cdot),y(\cdot))}$ such that,

$$s_n(t) = g(t) + \int_0^t f^{(n)}_{x,y}(s)ds, \quad \text{for } n \in \mathbb{N}, t \in [0, T].$$

Since (s_n) pointwise converges to s in X, for all t, we have $s_n(t) \to s(t)$. Since,

$$s_n(t) \in g(t) + \int_0^t K(s, x(s), y(s))ds, \ t \in [0, T].$$

(here the multi-valued integral is in the sense of Aumann [13]), using the fact that the set $g(t)+\int_0^t K(s, x(s), y(s))ds$ is closed (in fact it is compact, see [13]) we get that,

$$s(t) \in g(t) + \int_0^t K(s, x(s), y(s))ds, \ t \in [0, T].$$

Thus, there exists $f_{x,y} \in \mathcal{S}_{K(\cdot,x(\cdot),y(\cdot))}$ such that,

$$s(t) = g(t) + \int_0^t f_{x,y}(s)ds.$$

In the next step, we show that there exists $k \in \mathbb{R}^*_+$ with $k < \frac{1}{s}$, such that,

$$\rho(S(x, y), S(u, v)) + \rho(S(y, x), S(v, u)) \leq k[d(x, u) + d(y, v)],$$

for all ($x \leq u$ and $y \geq v$) or ($u \leq x$ and $v \geq y$).
For this purpose, let (for example) $x \leq u$ and $y \geq v$. First, we will prove that for each $w \in S(x, y)$, there exists $z \in S(u, v)$ such that,

$$d(w, z) \leq k(d(x, u) + d(y, v)).$$

Without the loss of generality, we can suppose that,

$$\rho_{|\cdot|}(K(s, x, y), K(s, u, v)) < \alpha(s)|x - u| + \beta(s)|y - v|, \quad for\ (x, y) \neq (u, v).$$

If $w \in S(x, y)$, then there exists $f_{x,y} \in S_{K(\cdot,x(\cdot),y(\cdot))}$ such that,

$$w(t) = g(t) + \int_0^t K(s, x(s), y(s)) f_{x,y}(s) ds,\ for\ a.e.\ t \in [0, T].$$

Since $f_{x,y}(t) \in K(t, x(t), y(t))$ for $t \in [0, T]$, we can find $r \in K(t, u(t), v(t))$ such that,

$$|f_{x,y}(t) - r| \leq \alpha(t)|x(t) - u(t)| + \beta(t)|y(t) - v(t)|,\ for\ t \in [0, T].$$

Thus, if we define the multi-valued operator $Q(t) := F(t, u(t), v(t)) \cap R(t)$, where,

$$R(t) := \{r || f_{x,y}(t) - r| \leq \alpha(t)|x(t) - u(t)| + \beta(t)|y(t) - v(t)|\}.$$

Then, $Q(t)$ is nonempty for $t \in [0, T]$ and Q is measurable (as an intersection of two measurable multi-valued operators). Thus, Q has measurable selections and let $f_{u,v}(t) \in Q(t)$, for $t \in [0, T]$ such that $f_{u,v} \in K(t, u(t), v(t))$ and

$$|f_{x,y}(t) - f_{u,v}(t)| < \alpha(t)|x(t) - u(t)| + \beta(t)|y(t) - v(t)|.$$

Define now,

$$z(t) := g(t) + \int_0^t K(s, x(s), y(s)) f_{u,v}(s) ds,\ \forall t \in [0, T].$$

Obviously, $z \in S(u,v)$ and the following estimations holds,

$$
\begin{aligned}
|w(t)-z(t)|^p &\leq [\int_0^t K(s,x(s),y(s))|f_{x,y}(s)-f_{u,v}(s)|]^p \\
&\leq [\int_0^t K(s,x(s),y(s))(\alpha(s)|x(s)-u(s)|e^{-\tau s}e^{\tau s}+\beta(s)|y(s)-v(s)|e^{-\tau s}e^{\tau s})ds]^p \\
&\leq [\int_0^t K(s,x(s),y(s))\alpha(s)\sqrt[p]{||(x-u)^p||_B}e^{\tau s}ds \\
&+\int_0^t K(s,x(s),y(s))\beta(s)\sqrt[p]{||(y-v)^p||_B}e^{\tau s}ds]^p \\
&\leq 2^{p-1}[(\int_0^t K(s,x(s),y(s))\alpha(s)\sqrt[p]{||(x-u)^p||_B}e^{\tau s}ds)^p \\
&+(\int_0^t K(s,x(s),y(s))\beta(s)\sqrt[p]{||(y-v)^p||_B}e^{\tau s}ds)^p] \\
&= 2^{p-1}[(\int_0^t K(s,x(s),y(s))\alpha(s)e^{\tau s}ds)^p||(x-u)^p||_B \\
&+(\int_0^t K(s,x(s),y(s))\beta(s)e^{\tau s}ds)^p||(y-v)^p||_B e^{\tau s}ds] \\
&\leq 2^{p-1}[(\int_0^t K(s,x(s),y(s))\alpha(s)ds)^p\cdot\int_0^t e^{p\tau s}ds||(x-u)^p||_B \\
&+(\int_0^t K(s,x(s),y(s))\beta(s)ds)^p\cdot\int_0^t e^{p\tau s}ds||(y-v)^p||_B] \\
&\leq 2^{p-1}\Big(\max_{t\in[0,T]}(\int_0^t K(s,x(s),y(s))\alpha(s)ds)^p\cdot||(x-u)^p||_B \\
&+\max_{t\in[0,T]}(\int_0^t (K(s,x(s),y(s)))\beta(s)ds)^p\cdot||(y-v)^p||_B\Big)\cdot\frac{1}{p\tau}e^{p\tau t}.
\end{aligned}
$$

After a multiplication with $e^{-p\tau t}$ and taking then maximum over $t\in[0,T]$ we get that,

$$||(w-z)^p||_B \leq \frac{2^{p-1}k}{p\tau}(||(x-u)^p||_B+||(y-v)^p||_B),$$

where

$$k := \max\{\max_{t\in[0,T]}(\int_0^t K(s,x(s),y(s))\alpha(s)ds)^p, \max_{t\in[0,T]}(\int_0^t K(s,x(s),y(s))\beta(s)ds)^p\}.$$

Hence, we get that,

$$d(w,z) \le \frac{2^{p-1}k}{p\tau}(d(x,u)+d(y,v)).$$

Now, from this proof and the analogous inequality obtained by interchanging the roles of x and y and respectively u and v, we can prove that for each $z \in S(y,x)$, there exists $w \in S(v,u)$ such that,

$$d(z,w) \le \frac{2^{p-1}k}{p\tau}(d(x,u)+d(y,v)).$$

Thus,

$$\rho(S(x,y),S(u,v)) + \rho(S(y,x),S(v,u)) \le \frac{2^p k}{p\tau}[d(x,u)+d(y,v)].$$

Since $\tau > 0$ is arbitrarily, we choose $\tau > \dfrac{2^p ks}{p}$. Thus $\dfrac{2^p k}{p\tau} < \dfrac{1}{s}$ and so all the assumption of Theorem 4.3 are satisfied. □

Now we can give an application of the previous result using the system of differential inclusions with periodic values. Lets consider the following system:

$$\begin{cases} x'(t) \in F(t,x(t),y(t)), \ x(0)=x^0, \\ y'(t) \in F(t,y(t),x(t)), \ y(0)=x^0. \end{cases} \tag{5.5}$$

where $x^0 \in \mathbb{R}^n$ and $F:[0,T]\times\mathbb{R}^n\times\mathbb{R}^n \to P(\mathbb{R}^n)$ is an operator satisfying some appropriate conditions.

We can observe that the system (5.5) is equivalent with the following system of integral inclusions:

$$\begin{cases} x(t) \in \int\limits_0^t F(s,x(s),y(s))ds + x^0, \ t\in[0,T], \\ y(t) \in \int\limits_0^t F(s,y(s),x(s))ds + x^0, \ t\in[0,T]. \end{cases} \tag{5.6}$$

By applying Theorem 5.1 to the above system, we immediately obtain the following existence result.

Theorem 5.2. *Consider the integral system (5.5). We suppose that:*

(i) $F:[0,T]\times\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}^n$ is upper semicontiunous and Lebesgue measurable in the first variable and jointly $H_{|\cdot|}$- continuous in the last two variables.

(ii) F is integrably bounded, i.e., there exists a mapping $r\in L^1[0,T]$ such that for each $(s,u,v)\in[0,T]\times\mathbb{R}^n\times\mathbb{R}^n$ and for any $w\in F(s,u,v)$, we have $|w|\le r(t)$, a.e. $t\in[0,T]$.

(iii) $F(s,\cdot,\cdot)$ has the strict mixed monotone property with respect to the last two variables, for all $s\in[0,T]$.

(iv) There exist $\alpha,\beta\in L^1([0,T],\mathbb{R}_+)$ such that, for each $s\in[0,T]$, we have,

$$\rho_{|\cdot|}(F(s,x,y),F(s,u,v)) \le \alpha(s)|x-u| + \beta(s)|y-v|, \ \forall x,y,u,v\in\mathbb{R}^n,$$

with ($x\le u$, $y\ge v$) or ($x\le u$, $y\le v$).

(v) There exist $x_0,y_0\in C[0,T]$ and two measurable selections $f_{x_0,y_0}:[0,T]\to\mathbb{R}^n$ of $F(\cdot,x_0(\cdot),y_0(\cdot))$ and $f_{y_0,x_0}:[0,T]\to\mathbb{R}^n$ of $F(\cdot,y_0(\cdot),x_0(\cdot))$, such that,

$$\begin{cases} x_0(t) \le x^0 + \int\limits_0^t F(s,x_0(s),y_0(s))f_{x_0,y_0}(s)ds, \\ y_0(t) \ge x^0 + \int\limits_0^t F(s,y_0(s),x_0(s))f_{y_0,x_0}(s)ds, \end{cases} \tag{5.7}$$

or

$$\begin{cases} x_0(t) \geq x^0 + \int\limits_0^t F(s, x_0(s), y_0(s)) f_{x_0,y_0}(s)ds, \ t \in [0,T], \\ y_0(t) \leq x^0 + \int\limits_0^t F(s, x_0(s), y_0(s)) f_{y_0,x_0}(s)ds, \ t \in [0,T], \end{cases} \tag{5.8}$$

Then, there exists at least one solution (x^*, y^*) *of the system* (5.6).

References

[1] I. A. Bakhtin, *The contraction mapping principle in quasimetric spaces*, Funct. Anal., Unianowsk Gos. Ped. Inst., **30** (1989), 26–37. 2.1

[2] V. Berinde, *Generalized contractions in quasimetric spaces*, Seminar on Fixed Point Theory, 1993, 3–9. 3

[3] V. Berinde, *Generalized coupled fixed point theorems for mixed monotone mappings in partially ordered metric spaces*, Nonlinear Anal., **74** (2011), 7347–7355. 1

[4] M. Boriceanu-Bota, A. Petruşel, I. A. Rus, *Fixed point theorems for some multi-valued generalized contractions in b-metric spaces*, International J. Math. Statistics, **6** (2010), 65–76. 1

[5] M. Boriceanu-Bota, A. Petruşel, G. Petruşel, B. Samet, *Coupled fixed point theorems for single-valued operators in b-metric spaces*, Fixed Point Theory Appl., **2015** (2015), 15 pages. 1

[6] H. Covitz and S. B. Nadler, *Multi-valued contraction mappings in generalized metric spaces*, Israel J. Math., **8** (1970), 5–11. 1

[7] S. Czerwik, *Nonlinear set-valued contraction mappings in B-metric spaces*, Atti Sem. Mat. Univ. Modena, **46** (1998), 263–276. 2.1

[8] T. Gnana Bhaskar, V. Lakshmikantham, *Fixed point theorems in partially ordered metric spaces and applications*, Nonlinear Anal., **65** (2006), 1379–1393. 1

[9] D. Guo, V. Lakshmikantham, *Coupled fixed points of nonlinear operators with applications*, Nonlinear Anal., **11** (1987), 623–632. 1

[10] C. J. Himmelberg, F. S. Van Vleck, *Lipschitzian generalized differential equations*, Rend. Sem. Mat. Univ. Padova, **48** (1972), 159–169. 1

[11] W. A. Kirk, N. Shahzad, *Fixed Point Theory in Distance Spaces*, Springer, Cham, (2014). 3

[12] W. A. Kirk, N. Shahzad, *Remarks on metric transforms and fixed-point theorems*, Fixed Point Theory Appl., **2013** (2013), 11 pages. 3

[13] M. Kisielewicz, *Differential Inclusions and Optimal Control*, Kluwer Academic Publishers, Dordrecht, (1991). 5

[14] V. Lakshmikantham, L. Ćirić, *Coupled fixed point theorems for nonlinear contractions in partially ordered metric spaces*, Nonlinear Anal., **70** (2009), 4341–4349. 1

[15] S. B. Nadler, *Multi-valued contraction mappings*, Pacific J. Math., **30** (1969), 475–488. 1

[16] H. K. Pathak, N. Shahzad, *A generalization of Nadlers fixed point theorem and its application to nonconvex integral inclusions*, Topol. Methods Nonlinear Anal., **41** (2013), 207–227. 3

[17] A. Petruşel, *Multi-valued weakly Picard operators and applications*, Sci. Math. Japon., **59** (2004), 169-202. 3, 3.9

[18] A. Petruşel, G. Petruşel, B. Samet, J.-C. Yao, *Coupled fixed point theorems for symmetric multi-valued contractions in b-metric space with applications to systems of integral inclusions*, to appear in J. Nonlinear Convex Anal. 1, 2.2, 2.3, 4.1, 5

[19] A. Petruşel, G. Petruşel, B. Samet, J.-C. Yao, *Coupled fixed point theorems for multi-valued operators in b-metric spaces with applications*, submitted. 1, 2.4, 2.5

[20] A. Razani, N. Karamikabir, *Proximal pairs and P-KKM mappings*, Commun. Nonlinear Anal., **1** (2016), 37–41. 1

[21] T. X. Wang, *Fixed point theorems and fixed point stability for multivalued mappings on metric spaces*, Nanjing Daxue Xuebao Shuxue Bannian Kan, **6** (1989), 16–23. 3.9