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On entropy of action of amenable groups

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Abstract

In this paper, we assign a linear operator to the action of an amenable group on a compact metric space. Then we extract the entropy of the action in terms of the eigenvalues of the operator. In this way we present a spectral representation of the entropy of action of amenable groups.

Keywords: Amenable group, entropy, Følner sequence, entropy kernel operator.
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1. Introduction

In the classical ergodic theory, the concept of entropy is defined for measure-preserving $\mathbb{Z}$-actions. The definition of entropy is stated via different approaches, but with the same origin [1, 3, 4, 5, 10, 11, 12, 13, 16, 17, 19, 21].

Entropy of $\mathbb{Z}$-actions is generalized to actions of general amenable groups. To have a nice entropy theory for actions of amenable groups, the concept of Følner sequence is applied. A Følner sequence for an action, is a sequence of finite sets which exhaust the space and do not move too much when acted on by any group element. Many classical results for $\mathbb{Z}$-actions, such as Shannon-McMillan-Brieman theorem [2, 7, 15], Ergodic theorems [22, 24, 25, 26] and Rokhlin-Sinai results [14], are generalized for actions of general amenable groups. Traditionally, entropy of an action is a nonnegative extended real number which is invariant under isomorphism. For $\mathbb{Z}$-actions it is replaced by linear operators on Banach spaces [10, 11].

In this paper, we consider an operator theory approach to the concept of entropy of action of amenable groups. In this approach, we consider the entropy of the action as a linear operator on a Hilbert space, rather than a nonnegative extended real number. In case of actions with finite entropy, the entropy of the action of an amenable group is represented in terms of the eigenvalues of a compact positive operator on a Hilbert space. This approach results in a spectral representation of the entropy of the action of an amenable group.

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2. Preliminaries

In this section, we present some preliminary facts which will be used in the remaining of the paper.

2.1. Invariant measures, ergodic measures and ergodic decomposition

Definition 2.1. Suppose that G is a topological group acting on a probability space $(X, \mathcal{B}, \mu)$ such that the action $G \times X \to X$ is measurable. The measure μ is called G-invariant if $\mu(gA) = \mu(A)$ for any $g \in G$.

Definition 2.2. An invariant measure μ is called ergodic, if for any measurable set A we have,

$$\forall g \in A, \quad gA = A \Longrightarrow \mu(A) = 0 \quad \text{or} \quad \mu(A) = 1.$$

The collection of all probability measures on $\mathcal{B}$ is denoted by $M(X)$ and the collection of all G-invariant measures on $\mathcal{B}$ is denoted by $M(G, X)$. We also write $E(G, X)$ for the collection of all ergodic measures. It is known that $M(X)$, equipped by the weak* topology, is a compact meatrizable space [23]. The proof of the following theorem is similar to Theorem 6.10 of [23] .

Theorem 2.3. *Suppose that G acts on a metric space X and μ is a G-invariant measure on $\mathfrak{B}_X$ –the σ-algebra of Borel sets of X– then,*

1. *$M(G, X)$ is a compact subset of $M(X)$.*
2. *$M(G, X)$ is convex.*
3. $\mathrm{ext}(M(G, X)) = E(G, X)$, *i.e., the collection of ergodic measures equals to the extreme points of the collection of G-invariant measures.*

In the following, we recall the Choquet's representation Theorem.

Theorem 2.4 (Phelps [9])**.** *Suppose that Y is a compact convex metrizable subset of a locally convex space E, and that $x_0 \in Y$. Then there exists a probability measure τ on Y which represents x_0 and is supported by the extreme points of Y, i.e., $\Psi(x_0) = \int_Y \Psi d\tau$ for every continuous linear functional Ψ on E and $\tau(ext(Y)) = 1$.*

Let $\mu \in M(G, X)$ and $f : X \to \mathbb{R}$ be a bounded measurable function. Since $E(G, X)$ agrees with the set of extreme points of $M(X, \phi)$, by applying Choquet's representation Theorem for $Y = M(G, X)$ and $\Psi(\mu) = \int_X f d\mu$, we will have the following corollary.

Corollary 2.5. *Suppose that G is a topological group acting continuously on the compact metric space X. Then for each $\mu \in M(G, X)$, there is a unique measure $\tau = \tau_\mu$ on the Borel subsets of the compact metrizable space $M(G, X)$ such that $\tau_\mu(E(G, X)) = 1$ and*

$$\int_X f(x) d\mu(x) = \int_{E(G,X)} \left(\int_X f(x) dm(x) \right) d\tau_\mu(m),$$

for every bounded measurable function $f : X \to \mathbb{R}$.

Under the assumptions of Corollary 2.5 we write $\mu = \int_{E(G,X)} m d\tau_\mu(m)$ and it is called the ergodic decomposition of μ.

2.2. Amenability, Følner sequences and entropy

Suppose that G is a countable and discrete group. There are many equivalent formulations for the concept of amenability. In discrete case, one of the convenient definitions of amenability for discrete groups is as follows.

Definition 2.6. A discrete group G is amenable, if for any finite set $K \subset G$ and $\delta > 0$, there is a finite set $F \subset G$ such that,

$$\forall k \in K \quad |F \Delta kF| < \delta |F|.$$

Such a set F is called (K, δ)-invariant. A sequence $\{F_n\}_{n\geq 1}$ of finite subsets of G is called a Følner sequence, if for any K, $\delta > 0$ and for all large enough n, F_n is (K, δ)-invariant. Without loss of generality, we may assume that $|F_n| \geq n$.

Assume that G acts from the left on a measure space $(X, \mathcal{B}, \mu)$ with $\mu(X) = 1$. Let also μ preserves the action of G on X. We have the following mean ergodic theorem for amenable groups. It may easily be proved by the same method applied for $\mathbb{Z}$-actions.

Theorem 2.7. *If G is amenable and acts ergodically on $(X, \mathcal{B}, \mu)$, then for any $f \in L^1(\mu)$ and Følner sequence $\{F_n\}_{n\geq 1}$,*

$$A(F_n, f)(x)_{n\to\infty} \longrightarrow \int_X f d\mu \quad in \quad L^1(\mu),$$

where

$$A(F, f)(x) := \frac{1}{|F|} \sum_{g\in F} f(gx).$$

The pointwise version of Theorem 2.7 does not necessarily hold for any given Følner sequence [6].

Definition 2.8 (A. Shulman [18])**.** A sequence of sets $\{F_n\}_{n\geq 1}$ is said to be tempered, if for some $c > 0$ and all $n \in \mathbb{N}$,

$$|\bigcup_{k\leq n} {F_k}^{-1} F_{n+1}| \leq c|F_{n+1}|.$$

A version of maximal ergodic theorem was proved for tempered sequences [20]. We also have the following theorem for tempered Følner sequences [6].

Theorem 2.9 (Pointwise ergodic theorem)**.** *Let G be an amenable group acting on a measure space $(X, \mathcal{B}, \mu)$, and let $\{F_n\}_{n\geq 1}$ be a tempered Følner sequence. Then for any $f \in L^1(\mu)$,*

$$\lim_{n\to\infty} A(F_n, f)(x) = \int_X f d\mu \quad a.e.$$

A space $(X, \mathcal{B}, \mu)$ on which acts, together with a partition $\mathcal{P}$ of X, is called a process. If $x \in X$ and $\mathcal{P}$ is a partition, then we denote the unique element of $\mathcal{P}$ containing x by $\mathcal{P}(x)$. If also $F \subset G$ we set,

$$\mathcal{P}^F := \bigvee_{g\in F} g^{-1}\mathcal{P},$$

where $\bigvee$ denotes the joint operation on the set of finite partitions. We recall the definition of the entropy of a process.

Definition 2.10. For any $F \subset G$ and $\epsilon > 0$, we set,

$$b(F, \epsilon, \mathcal{P}) := \min\{|\mathcal{C}| : \mathcal{C} \subset \mathcal{P}^F, \mu(\cup\mathcal{C}) > 1 - \epsilon\},$$

then the entropy $h_\mu(\mathcal{P})$ is defined as,

$$h_\mu(\mathcal{P}) := \lim_{\epsilon\to\infty} \liminf_{n\to\infty} \frac{\log b(F_n, \epsilon, \mathcal{P})}{|F_n|},$$

where $\{F_n\}_{n\geq 1}$ is a Følner sequence for G.

The following theorem is a generalized version of Shannon-McMillan-Breiman theorem [6].

Theorem 2.11. *Let $\mathcal{P}$ be a finite partition and assume that G is an amenable group acting ergodically on a measure space $(X, \mathcal{B}, \mu)$. Let $h_\mu(\mathcal{P})$ denote the entropy of this process. Assume that $\{F_n\}_{n\geq 1}$ is a tempered sequence of Følner sets. Then for almost every x,*

$$\frac{-\log(\mu(\mathcal{P}^{F_n}(x)))}{|F_n|} \longrightarrow h_\mu(\mathcal{P}) \quad as \quad n \to \infty.$$

3. Entropy operator of action of amenable groups

In the rest of the paper, let X be a metric space and $\mathfrak{B}_X$ be the σ-algebra of all Borel partitions. Let G be an amenable group acting on the space $(X, \mathfrak{B}_X)$, $\mu \in M(G, X)$ and $\mathcal{P}$ be a measurable partition of X. Let also $\{F_n\}_{n\geq 1}$ be a tempered Følner sequence of G, such that $|F_{n+1}| \geq |F_n|$.

Definition 3.1. For $x, y \in X$ and $n \in \mathbb{N}$, we set,

$$\gamma_n(x, y; \mathcal{P}) := \limsup_{m\to\infty} \frac{1}{|F_m|} \mathrm{card}(\{g \in F_m : y \in g^{-1}\mathcal{P}^{F_n}(x)\})$$

and

$$\gamma_n^*(x, y; \mathcal{P}) := \begin{cases} -\frac{1}{|F_n|} \log \gamma_n(x, y; \mathcal{P}) : \gamma_n(x, y; \mathcal{P}) \neq 0, \\ 0 \qquad\qquad\qquad\qquad\quad : \gamma_n(x, y; \mathcal{P}) = 0. \end{cases}$$

Lemma 3.2. *For $x, y \in X$ and a partition $\mathcal{P}$, the sequence $\{\gamma_n^*(x, y; \mathcal{P})\}_{n\geq 1}$ is increasing.*

Proof. Let $k \leq n$. The partition $\mathcal{P}^{F_n}$ is finer than $\mathcal{P}^{F_k}$ therefore, if $x \in X$ then $\mathcal{P}^{F_n}(x) \subset \mathcal{P}^{F_k}(x)$ and consequently $g^{-1}\mathcal{P}^{F_n}(x) \subset g^{-1}\mathcal{P}^{F_k}(x)$ for any $g \in G$. Now, for $m \in \mathbb{N}$ we have,

$$\{g \in F_m : y \in g^{-1}\mathcal{P}^{F_n}\} \subset \{g \in F_m : y \in g^{-1}\mathcal{P}^{F_k}\},$$

which easily results in $\gamma_n(x, y; \mathcal{P}) \leq \gamma_k(x, y; \mathcal{P})$, therefore $\gamma_k^*(x, y; \mathcal{P}) \leq \gamma_n^*(x, y; \mathcal{P})$. □

By Lemma 3.2, $\lim_{n\to\infty} \gamma_n^*(x, y; \mathcal{P})$ exists as an extended real non-negative number. So, we may have the following definition.

Definition 3.3. For $x, y \in X$ and the partition $\mathcal{P}$ of X, set

$$\Gamma_G(x, y) := \sqrt{\lim_{n\to\infty} \gamma_n^*(x, y; \mathcal{P})}.$$

The function $\Gamma_G : X \times X \to [0, +\infty]$ is called the entropy kernel of G-action on X.

Definition 3.4. Let $\mathcal{A}(G)$ be the set of all measurable functions $f : X \to \mathbb{R}$ such that the integral

$$\int_X \Gamma_G(x, y) f(y) d\mu(y),$$

exists for almost every $x \in X$.

For $f \in \mathcal{A}(G)$ set

$$\Phi_G^* f(x) := \int_X \Gamma_G(x, y) f(y) d\mu(y). \tag{3.1}$$

Before we mention our first main result, we need to note that, when G is an amenable countably infinite discrete group, for any finite measurable partition $\mathcal{P}$ of X and any $\mu \in M(G, X)$, one has the equality

$$h_\mu(\mathcal{P}) = \int_{E(G,X)} h_m(\mathcal{P}) d\tau_\mu(m), \tag{3.2}$$

where $\mu = \int_{E(G,X)} m d\tau_\mu(m)$ is the ergodic decomposition of μ. One can deduce (3.2) from Proposition 5.3.2 and 5.3.5 of [8] and the proof in the case $G = \mathbb{Z}$ like Theorem 8.4.(i) of [23] .

Theorem 3.5. *$h_\mu(\mathcal{P}) < +\infty$, if and only if $\Gamma_G \in L^2(X \times X, \mu \times \mu)$. Moreover, under the previous condition we have,*

$$||\Gamma_G||_{L^2(X\times X,\mu\times\mu)} = \sqrt{h_\mu(\mathcal{P})}.$$

Proof. First, let $m \in E(G,X)$, $x, y \in X$ and $n \in \mathbb{N}$. For almost all $y \in X$, by pointwise ergodic theorem, we have,

$$\begin{aligned}\gamma_n(x,y;\mathcal{P}) =& \limsup_{k\to+\infty} \frac{1}{|F_k|} \mathrm{card}(\{g \in F_k : y \in g^{-1}\mathcal{P}^{F_n(x)}\})\\ =& \limsup_{k\to+\infty} \frac{1}{|F_k|} \sum_{g\in F_k} \chi_{g^{-1}\mathcal{P}^{F_n}(x)}(y)\\ =& \limsup_{k\to+\infty} \frac{1}{|F_k|} \sum_{g\in F_k} \chi_{\mathcal{P}^{F_n}(x)}(gy)\\ =& \limsup_{k\to+\infty} A(F_k, \chi_{\mathcal{P}^{F_n}(x)})(y)\\ =& \int_X \chi_{\mathcal{P}^{F_n}(x)} dm(y)\\ =& m(\mathcal{P}^{F_n}(x)),\end{aligned}$$

so, for almost all $y \in X$,

$$\gamma_n^*(x,y;\mathcal{P}) = -\frac{\log m(\mathcal{P}^{F_n}(x))}{|F_n|}.$$

By Theorem 2.11,

$$\lim_{n\to+\infty} \gamma_n^*(x,y;\mathcal{P}) = h_m(\mathcal{P}),$$

for almost all $x, y \in X$, so

$$\Gamma_G(x,y) = h_m(\mathcal{P}),$$

for almost all $x, y \in X$. This easily results in

$$||\Gamma_G||_{L^2(X\times X, m\times m)} = \sqrt{h_m(\mathcal{P})}.$$

Now, let in general $\mu \in M(G,X)$, then $\mu\times\mu \in M(G\times G, X\times X)$. Let τ_μ and $\tau_{\mu\times\mu}$ be the probability measures in Corollary 2.5, corresponding to μ and $\mu\times\mu$ respectively.

Set $\Delta := \{m\times m : m \in E(G,X)\}$, then $\Delta \subset E(G\times G, X\times X)$. If $\psi : E(G,X) \to \Delta$ is the bijection $\psi(m) = m\times m$ then $\tau_{\mu\times\mu} = \tau_\mu\psi^{-1}$ on the Borel subsets of Δ. Therefore,

$$\begin{aligned}\tau_{\mu\times\mu}(E(G\times G, X\times X)\setminus\Delta) =& 1-\tau_{\mu\times\mu}(\Delta)\\ =& 1-\tau_\mu\psi^{-1}(\Delta)\\ =& 1-\tau_\mu(E(G,X))\\ =& 0.\end{aligned}$$

For $n \geq 1$, let $g_n := \min\{\Gamma_G^2, n\}$. Then $\{g_n\}_{n\geq1}$ is an increasing sequence of non-negative bounded measurable functions on $X\times X$ such that $g_n \uparrow \Gamma_G^2$. By Monotone Convergence Theorem and Corollary 2.5 we have,

$$\begin{aligned}
||\Gamma_G||^2_{L^2(X\times X,\mu\times\mu)} &= \int_{X\times X} \Gamma_G^2 d\mu\times\mu \\
&= \lim_{n\to+\infty} \int_{X\times X} g_n^2 d\mu\times\mu \\
&= \lim_{n\to+\infty} \int_{E(G\times G,X\times X)} \left(\int_{X\times X} g_n d\nu\right) d\tau_{\mu\times\mu}(\nu) \\
&= \int_{E(G\times G,X\times X)} \left(\int_{X\times X} \Gamma_G^2 d\nu\right) d\tau_{\mu\times\mu}(\nu) \\
&= \int_{\Delta} \left(\int_{X\times X} \Gamma_G^2 d\nu\right) d\tau_\mu\psi^{-1}(\nu) \\
&= \int_{E(G,X)} \left(\int_{X\times X} \Gamma_G^2 dm\times m\right) d\tau_\mu(m) \\
&= \int_{E(G,X)} h_m(\mathcal{P}) d\tau_\mu(m) \\
&= h_\mu(\mathcal{P}).
\end{aligned}$$

This completes the proof. □

Corollary 3.6. *If $h_\mu(\mathcal{P}) < +\infty$, then,*

1. $L^2(X,\mu) \subset \mathcal{A}(G)$.
2. $L^2(X,\mu)$ *is* Φ_G^**-invariant, i.e.,* $\Phi_G^*(L^2(X,\mu)) \subset L^2(X,\mu)$.

Proof. To prove part 1, let $f \in L^2(X,\mu)$, then,

$$\begin{aligned}
\left|\int_X \Gamma_G(x,y)f(y)d\mu(y)\right| &\le \int_X \Gamma_G(x,y)|f(y)|d\mu(y) \\
&\le \left(\int_X \Gamma_G(x,y)^2 d\mu(y)\right)^{\frac{1}{2}} ||f||_{L^2(X,\mu)}.
\end{aligned} \tag{3.3}$$

Set $g(x) := \int_X \Gamma_G(x,y)^2 d\mu(y)$. Since $h_\mu(\mathcal{P}) < +\infty$, by Theorem 3.5 $\Gamma_G \in L^2(X\times X,\mu\times\mu)$, therefore,

$$\int_X g(x)d\mu(x) = \int_X\int_X \Gamma_G(x,y)^2 d\mu(y)d\mu(x) = ||\Gamma_G||^2_{L^2(X\times X,\mu\times\mu)} < +\infty.$$

So $g(x) = \int_X \Gamma_G(x,y)^2 d\mu(y)$ is finite for almost all $x \in X$, therefore by (3.3), $\int_X \Gamma_G(x,y)f(y)d\mu(y)$ exists for almost all $x \in X$, which means $f \in \mathcal{A}(G)$.

Since for all $f \in L^2(X,\mu)$ we have,

$$||\Phi_G^* f||_{L^2(X,\mu)} \le ||\Gamma_G||_{L^2(X\times X,\mu\times\mu)} ||f||_{L^2(X,\mu)}, \tag{3.4}$$

Part 2 also holds.

If $h_\mu(\mathcal{P}) < +\infty$, we set $\Phi_G := \Phi_G^*|_{L^2(X,\mu)}$ which by Corollary 3.6, is a linear operator on the Hilbert space $H = L^2(X,\mu)$. The linear operator $\Phi_G : L^2(X,\mu) \to L^2(X,\mu)$ is called the entropy operator of the action of G on X. In this case, we have even more about Φ_G.

Theorem 3.7. *If $h_\mu(\mathcal{P}) < +\infty$, then,*

1. Φ_G *is a compact bounded linear operator on* $L^2(X,\mu)$ *such that* $||\Phi_G||_{op} \le ||\Gamma_G||_{L^2(X\times X,\mu\times\mu)}$.

2. *If* $\{\lambda_n\}_{n\geq 1}$ *is the sequence of eigenvalues of* Φ_G, *then,*

$$h_\mu(\mathcal{P}) = \sum_{n=1}^{+\infty} \lambda_n^2 \dim(E_n),$$

where E_n *is the eigenspace corresponding to* λ_n.

Proof. Part 1 is a direct result of (3.4).

To prove part 2, consider an orthonormal basis $\mathcal{B} = \bigcup_{n=0}^{+\infty} \mathcal{B}_n$ for $L^2(X,\mu)$ where $\mathcal{B}_0 = \{f_k^0\}_{k=1}^{d_0}$ is an orthonormal basis for $\ker \Phi_G$ and $\mathcal{B}_n = \{f_k^n\}_{k=1}^{d_n}$ $(n \geq 1)$ is an orthonormal basis for E_n. Then we have,

$$\begin{aligned}
\sum_{n=1}^{\infty} \lambda_n^2 \dim(E_n) &= \sum_{n=1}^{\infty}\sum_{k=1}^{d_n} ||\Phi_G f_k^n||^2_{L^2(\mu)} \\
&= \sum_{n=0}^{\infty}\sum_{k=1}^{d_n} ||\Phi_G f_k^n||^2_{L^2(\mu)} \\
&= \sum_{n=0}^{\infty}\sum_{k=1}^{d_n} \int_X |(\Phi_G f_k^n)(x)|^2 \, d\mu(x) \\
&= \sum_{n=0}^{\infty}\sum_{k=1}^{d_n} \int_X \left| \int_X \Gamma_G(x,y) f_k^n(y) d\mu(y) \right|^2 d\mu(x) \\
&= \sum_{n=0}^{\infty}\sum_{k=1}^{d_n} \int_X d\mu(x) \, |< \Gamma_G(x,\cdot), f_k^n >|^2 \\
&= \int_X d\mu(x) \sum_{n=0}^{\infty}\sum_{k=1}^{d_n} |< \Gamma_G(x,\cdot), f_k^n >|^2 \\
&= \int_X d\mu(x) ||\Gamma_G(x,\cdot)||^2_{L^2(\mu)} \\
&= \int_X d\mu(x) \left(\int_X \Gamma_G(x,y)^2 d\mu(y) \right) \\
&= \int_X \int_X \Gamma_G(x,y)^2 d\mu(y) d\mu(x) \\
&= ||\Gamma_G||^2_{L^2(X\times X, \mu\times\mu)} \\
&= h_\mu(\mathcal{P}). \quad \square
\end{aligned}$$

4. Conclusion:

motivated by [10, 11], in this paper, we consider the entropy of action of amenable groups as a linear operator instead of a non-negative number. In case of finite entropy, a Hilbert-Schmidt operator on a Hilbert space is assigned to the action of an amenable group such that the entropy of the action is expressed in terms of the spectrum of the operator. So, we have a spectral representation of the entropy of the action of an amenable group on a compact metric space.

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