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An existence result for a class of nonlinear Volterra functional integral equations

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Abstract

Recently [N. Khodabakhshi, S. M. Vaezpour, Fixed Point Theory, to appear] provides sufficient conditions for the existence of common fixed point for two commuting operators using the technique associated to an abstract measure of non-compactness in Banach spaces. In this paper, we develope their work with further applicative investigation. More precisely, we give suitable assumptions in order to obtain the existence of solutions for a nonlinear integral equation.

Keywords: Coincidence point, common fixed point, measure of noncompactness, integral equation.
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1. Introduction

Measure of non-compactness technique is one of the fruitful tools in nonlinear analysis which is extensively applied to investigate the existence of solutions to nonlinear integral equations via Darbo's fixed point theorem (see, for example, [3, 4, 6, 8, 13, 15]).

In our previous work [14], we provided sufficient conditions for the existence of common fixed point for two commuting operators using the technique associated to an abstract measure of non-compactness in Banach spaces, which generalize some recent papers [2, 11, 12]. In this paper, in order to confirm the applicability of our main result, we consider sufficient conditions for the existence of solutions of nonlinear integral equation.

This paper is organized as follows: in Section 2, some facts and results about measure of non-compactness and related theorems are given. We will also recall our main result which was proved in [14]. Finally, in Section 3, we will give sufficient conditions for the existence of solutions of nonlinear integral equation.

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2. Preliminaries

In this section, we present some definitions and results which will be needed later.
Let $(E, \|.\|)$ be a Banach space. We write $B(x, r)$ to denote the closed ball centered at x with radius r. Moreover, let $\mathfrak{M}_E$ indicate the family of all nonempty and bounded subsets of E and $\mathfrak{N}_E$ indicate its subfamily consisting of all relatively compact sets.

We mention the following definition of an abstract measure of non-compactness, given in [7].

Definition 2.1. A mapping $\mu : \mathfrak{M}_E \to \mathbb{R}$ is said to be a measure of noncompactness in E if it satisfies the following conditions:

(1) The family $\ker \mu = \{X \in \mathfrak{M}_E : \mu(X) = 0\}$ is nonempty and $\ker \mu \subseteq \mathfrak{N}_E$.
(2) $X \subseteq Y \Rightarrow \mu(X) \leq \mu(Y)$.
(3) $\mu(\overline{X}) = \mu(X)$.
(4) $\mu(convX) = \mu(X)$.
(5) $\mu(\lambda X + (1-\lambda)Y) \leq \lambda\mu(X) + (1-\lambda)\mu(Y)$, for $\lambda \in [0, 1]$.
(6) If (X_n) is a sequence of closed sets from $\mathfrak{M}_E$ such that $X_{n+1} \subset X_n$ $(n = 1, 2, ...)$ and if $\lim_{n\to\infty} \mu(X_n) = 0$, then the intersection set $X_\infty = \bigcap_{n=1}^{\infty} X_n$ is nonempty and $X_\infty \in \ker \mu$.

Now, we mention the following two theorems stated in [1, 7].

Theorem 2.2 (Schauder's fixed point theorem)**.** *Let Ω be a nonempty, bounded, closed and convex subset of a Banach space E. Then each continuous and compact map $T : \Omega \to \Omega$ has at least one fixed point in the set Ω.*

Theorem 2.3 (Darbo's fixed point theorem)**.** *Let Ω be a nonempty, bounded, closed and convex subset of a Banach space E and let $T : \Omega \to \Omega$ be a continuous operator such that $\mu(TX) \leq k\mu(X)$ for all nonempty subset X of Ω, where $k \in [0, 1)$ is a constant. Then T has a fixed point in the set Ω.*

Definition 2.4 ([9])**.** A mapping T of a convex set M is said to be affine if it satisfies the identity

$$T(kx + (1-k)y) = kTx + (1-k)Ty,$$

whenever $0 < k < 1$, and $x, y \in M$.

Theorem 2.5 ([14])**.** *Let E be a Banach space, Ω be a nonempty, closed, convex and bounded subset of E and T, S be two continuous operators from Ω into Ω such that,*

a) $T \circ S = S \circ T$.

b) For any $M \subset \Omega$,
$$T(\overline{conv}(M)) \subset \overline{conv}(T(M)).$$

c) For any $M \subset \Omega$,
$$\psi(\mu(S(M))) \leq \psi(\mu(T(M))) - \phi(\mu(T(M))),$$

where $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous, monotone non-decreasing mapping and $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ is lower semicontinuous and monotone decreasing mapping such that $\phi(0) = 0$ and $\phi(t) > 0$ for $t > 0$. Then,

1) The set $\{x \in \Omega : S(x) = x\}$ is nonempty and compact.

2) T has a fixed point and the set $\{x \in \Omega : T(x) = x\}$ is closed and invariant by S.

3) If T is affine then T and S have a common fixed point and the set $\{x \in \Omega : T(x) = S(x) = x\}$ is compact.

Theorem 2.6 ([14])**.** *Let E be a Banach space, Ω be a nonempty, closed, convex and bounded subset of E and T, S be two continuous operators from Ω into Ω such that,*

a) $T \circ S = S \circ T$.

b) For any $M \subset \Omega$,
$$T(\overline{conv}(M)) \subset \overline{conv}(T(M)).$$

c) For any $M \subset \Omega$,
$$\mu(S(M))) \leq \psi(\mu(T(M))),$$

where $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ is monotone nondecreasing mapping such that $\lim_{n\to\infty} \psi^n(t) = 0$, $t > 0$. Then,

1) The set $\{x \in \Omega : S(x) = x\}$ is nonempty and compact.

2) T has a fixed point and the set $\{x \in \Omega : T(x) = x\}$ is closed and invariant by S.

3) If T is affine then T and S have a common fixed point and the set $\{x \in \Omega : T(x) = S(x) = x\}$ is compact.

Theorem 2.7 ([14])**.** *Let E be a Banach space, Ω be a nonempty, closed, convex and bounded subset of E and T, S be two continuous operators from Ω into Ω such that:*

a) $T \circ S = S \circ T$.

b) T is affine.

c) For any $M \subset \Omega$,
$$\psi(\mu(ST(M))) \leq \psi(\mu(T(M))) - \phi(\mu(T(M))),$$

where $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous, monotone non-decreasing mapping and $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ is a lower semicontinuous and monotone decreasing mapping such that $\phi(0) = 0$, $\phi(t) > 0$ for $t > 0$. Then T and S have a common fixed point.

3. Main Result

In this section we present sufficient conditions for the existence of solutions of the following nonlinear integral equation in $BC(\mathbb{R}^+)$ —the space of all bounded, continuous functions $x : \mathbb{R}^+ \to \mathbb{R}$.

$$x(t) = \lambda f(t, \int_0^t g(t, s, x(\alpha(s)))ds) + (1 - \lambda) \int_0^t g(t, s, x(s))ds, \quad t \geq 0, \lambda \in (0, 1). \tag{3.1}$$

For any nonempty bounded subset X of $BC(\mathbb{R}^+)$; $x \in X$; $T > 0$ and $\epsilon \geq 0$ let

$$\begin{aligned}
&\omega^T(x, \varepsilon) = \sup\{|x(t) - x(s)| : s, t \in [0, T], \ \ |t - s| \leq \varepsilon\},\\
&\omega^T(X, \varepsilon) = \sup\{\omega^T(x, \varepsilon) : \ x \in X\}, \qquad \omega_0^T(X) = \lim_{\varepsilon\to 0} \omega^T(X, \varepsilon),\\
&\omega_0(X) = \lim_{T\to\infty} \omega_0^T(X), \qquad X(t) = \{x(t) : x \in X\},\\
&\mathrm{diam}X(t) = \sup\{|x(t) - y(t)| : x, y \in X\},\\
&\mu(X) = \omega_0(X) + \limsup_{t\to\infty} \mathrm{diam}X(t).
\end{aligned} \tag{3.2}$$

Banaś has shown in [5] that the function μ is a measure of noncompactness in the space $BC(\mathbb{R}^+)$. Our main result is the following existence theorem.

Theorem 3.1. *The nonlinear integral equation* (3.1) *has at least one solution in the space* $BC(\mathbb{R}^+)$*, if the following conditions are satisfied:*

(A_0) The function $\alpha : \mathbb{R}^+ \to \mathbb{R}^+$ *is continuous,* $\alpha(t) \overset{t\to\infty}{\longrightarrow} \infty$.

(A_1) The function $f : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ *is continuous and there exist a nondecreasing, continuous function* $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ *such that*

$$|f(t,x) - f(t,y)| \leq \psi(|x-y|).$$

Moreover for all $t, s \in \mathbb{R}^+$, $\psi(t) + \psi(s) \leq \psi(t+s),\ \psi(t) < t,\ \psi(0) = 0.$

(A_2)

$$L = \sup\{f(t,0) : t \in \mathbb{R}^+\} < \infty.$$

(A_3) The function $g : \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ *is continuous and linear with respect to third variable, there exists a continuous function* $b : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ *satisfying*

$$|g(t,s,x)| \leq b(t,s),$$

for all $t, s \in \mathbb{R}^+$ *and* $x \in \mathbb{R}$*, where*

$$\lim_{t\to\infty} \int_0^t b(t,s)ds = 0.$$

(A_4)

$$f(t, Kx(t)) = K(f(t, x(t))),$$

where $(Kx)(t) = \int_0^t g(t,s,x(s))ds$.

Proof. Take

$$(Fx)(t) = f(t, x(t)),$$

$$(Kx)(t) = \int_0^t g(t,s,x(s))ds.$$

Thus, equation (3.1) becomes

$$x(t) = Hx(t) := \lambda FKx(\alpha(t)) + (1-\lambda)(Kx)(t).$$

Let us define the operator $G : C(\mathbb{R}^+) \to C(\mathbb{R}^+)$ by

$$Gx(t) := \frac{Hx - (1-\lambda)(Kx)(t)}{\lambda} = FKx(\alpha(t))).$$

We show that G, K have a common fixed point.

First, taking into account our assumptions, function Gx is continuous on $\mathbb{R}^+$. Note that by hypothesis (A_3) there exists a constant $V > 0$ such that,

$$V = \sup_{t\geq 0} v(t) = \sup_{t\geq 0} \Big[\int_0^t b(t,s)ds \Big].$$

For arbitrary fixed $x \in BC(\mathbb{R}^+)$ and $t \in \mathbb{R}^+$ we get,

$$\begin{aligned} |(Gx)(t)| &\leq |f(t, \int_0^t g(t,s,x(\alpha(s)))ds) - f(t,0)| + |f(t,0)| \\ &\leq \psi(\int_0^t |g(t,s,x(\alpha(s)))|ds) + \sup\{|f(t,0)|, t \in \mathbb{R}^+\} \\ &\leq \psi(v(t)) + L \leq V + L. \end{aligned}$$

Hence $Gx \in BC(\mathbb{R}^+)$, also G transforms Q into itself, where Q is a subset of $BC(\mathbb{R}^+)$ defined by,

$$Q = \{x \in BC(\mathbb{R}^+) : ||x|| \leq r = L + V\}.$$

The set Q is nonempty, convex, bounded and closed in $BC(\mathbb{R}^+)$.

Similarly, we can show that K transforms Q into itself. Now, we show that G is continuous on Q. Let us fix arbitrarily $\epsilon > 0$ and take $x, y \in Q$ such that $||x - y|| \leq \epsilon$. Then,

$$\begin{aligned}
|(Gx)(t) - (Gy)(t)| &\leq |f(t, \int_0^t g(t,s,x(\alpha(s)))ds) - f(t, \int_0^t g(t,s,y(\alpha(s)))ds)| \\
&\leq \psi(|\int_0^t [g(t,s,x(\alpha(s))) - g(t,s,y(\alpha(s)))]ds|) \\
&\leq \psi(\int_0^t [|g(t,s,x(\alpha(s)))| + |g(t,s,y(\alpha(s)))|]ds) \\
&\leq \psi(\int_0^t [b(t,s)|ds + \int_0^t [b(t,s)ds]) \\
&\leq \psi(2\int_0^t b(t,s)ds) \leq \psi(2v(t)).
\end{aligned}$$

So, by conditions $(A_1), (A_3)$ there exists $T > 0$ such that for $t \geq T$ we have, $v(t) \leq \frac{\epsilon}{2}$ and thus we have

$$|(Gx)(t) - (Gy)(t)| \leq \psi(\epsilon) \leq \epsilon.$$

Now for $t \in [0, T]$,

$$\begin{aligned}
|(Gx)(t) - (Gy)(t)| &\leq |f(t, \int_0^t g(t,s,x(\alpha(s)))ds) - f(t, \int_0^t g(t,s,y(\alpha(s)))ds)| \\
&\leq \psi(|\int_0^t [g(t,s,x(\alpha(s))) - g(t,s,y(\alpha(s)))]ds|) \\
&\leq \psi(\int_0^t \omega_r^T(g,\epsilon)ds) \\
&\leq \psi(T\omega_r^T(g,\epsilon)),
\end{aligned}$$

where

$$\omega_r^T(g,\epsilon) = \sup\{|g(t,s,x) - g(t,s,y)| : t,s \in [0,T], x,y \in [-r,r], ||x-y|| \leq \epsilon\}.$$

By the continuity of g on $[0,T] \times [0,T] \times [-r,r]$ and by condition (A_1), we have $\psi(\omega_r^T(g,\epsilon)) \to 0$ as $\epsilon \to 0$. Hence, G is continuous on the set Q.

Similarly K is continuous. By hypotheses $(A_3), (A_4)$, the assumptions $(a), (b)$ of Theorem 2.6 are satisfied. Now, we show that condition (c) of Theorem 2.6 holds.

For any nonempty set $A \subset Q$, for fixed arbitrary $T > 0$ and $\epsilon > 0$, choose $x \in X$ and $t_1, t_2 \in [0,T]$ with $|t_2 - t_1| \leq \epsilon$. Then, we have,

$$\begin{aligned}
|(Gx)(t_2) - (Gx)(t_1)| = |f(t_2, &\int_0^{t_2} g(t_2,s,x(\alpha(s)))ds - f(t_1, \int_0^{t_2} g(t_2,s,x(\alpha(s)))ds \\
&+ f(t_1, \int_0^{t_2} g(t_2,s,x(\alpha(s)))ds - f(t_1, \int_0^{t_1} g(t_1,s,x(\alpha(s)))ds| \\
&\leq \omega_r^T(f,\epsilon) + \psi(|\int_0^{t_2} g(t_2,s,x(\alpha(s)))ds - \int_0^{t_1} g(t_1,s,x(\alpha(s)))ds|) \\
&\leq \omega_r^T(f,\epsilon) + \psi(|Kx(\alpha(t_2)) - Kx(\alpha(t_1))|) \\
&\leq \omega_r^T(f,\epsilon) + \psi(\omega^T(Kx, \omega^T(\alpha,\epsilon))),
\end{aligned}$$

where

$$\omega_r^T(f,\epsilon) = \sup\{|f(t_2,x) - f(t_1,x)| : t_1,t_2 \in [0,T], |t_2 - t_1| \le \epsilon \ , x \in [-r,r]\}.$$

By the uniform continuity of functions f on $[0,T] \times [-r,r]$ and α we have,

$$\lim_{\epsilon \to 0} \omega_r^T(f,\epsilon) = \lim_{\epsilon \to 0} \omega^T(\alpha,\epsilon) = 0.$$

Therefore,

$$\omega_0^T(GX) \le \psi(\omega_0^T(KX)),$$

and by letting $T \to \infty$ we have,

$$\omega_0(GX) \le \psi(\omega_0(KX)). \tag{3.3}$$

For arbitrary fixed $t \in \mathbb{R}^+$ and $x,y \in A$ we also obtain,

$$|(Gx)(t) - (Gy)(t)| \le \psi(|\int_0^t [g(t,s,x(\alpha(s))) - g(t,s,y(\alpha(s)))]ds|),$$

and so

$$diam(GX)(t) \le \psi(diam(KX(\alpha(t)))).$$

Therefore

$$\limsup_{t \to \infty} diam(GX) \le \psi(\limsup_{t \to \infty} diam(KX((t))). \tag{3.4}$$

So, by combining (3.3),(3.4) we obtain,

$$\begin{aligned}\mu(GX) &\le \psi(\omega_0(KX)) + \psi(\limsup_{t\to\infty} diam(KX((t))) \\ &\le \psi(\mu(KX)).\end{aligned}$$

Therefore, by Theorem 2.6, G,K have a common fixed point and thus H has a fixed point. As a consequence, the functional integral equation (3.1) has at least one solution in $BC(\mathbb{R}^+)$. □

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