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Global existence of solutions for an m-component reaction–diffusion system with a tridiagonal 2-Toeplitz diffusion matrix and polynomially growing reaction terms

Salem Abdelmalek[a], Samir Bendoukha[b,*]

[a]*Department of mathematics, University of Tebessa 12002 Algeria.*
[b]*Electrical Engineering Department, College of Engineering at Yanbu, Taibah University, Saudi Arabia.*

Abstract

This paper is concerned with the local and global existence of solutions for a generalized m-component reaction–diffusion system with a tridiagonal 2–Toeplitz diffusion matrix and polynomial growth. We derive the eigenvalues and eigenvectors and determine the parabolicity conditions in order to diagonalize the proposed system. We, then, determine the invariant regions and utilize a Lyapunov functional to establish the global existence of solutions for the proposed system. A numerical example is used to illustrate and confirm the findings of the study. ⓒ2017 All rights reserved.

1. Introduction

In this study, we consider the generalized m-component reaction–diffusion system with $m \geq 2$:

$$\frac{\partial U}{\partial t} - A\Delta U = F\left(U\right), \tag{1.1}$$

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*Corresponding author
Email addresses:* `sallllm@gmail.com` (Salem Abdelmalek), `sbendoukha@taibahu.edu.sa` (Samir Bendoukha)

in $\Omega \times (0, +\infty)$, where Ω is an open bounded domain of class C^1 in $\mathbb{R}^M$ with boundary $\partial\Omega$. The diffusion matrix A is assumed to be of the form

$$
A = \begin{pmatrix}
\alpha_1 & \gamma_1 & 0 & \cdots & \cdots & 0 \\
\beta_1 & \alpha_2 & \gamma_2 & \ddots & & \vdots \\
0 & \beta_2 & \alpha_1 & \gamma_1 & \ddots & \vdots \\
\vdots & \ddots & \beta_1 & \alpha_2 & \gamma_2 & 0 \\
\vdots & & \ddots & \beta_2 & \ddots & \ddots \\
0 & \cdots & \cdots & 0 & \ddots & \ddots
\end{pmatrix}_{m \times m}, \tag{1.2}
$$

with $\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2 > 0$ being positive real numbers representing the self and cross–diffusion constants and satisfying the inequality

$$
\frac{\sqrt{\alpha_1 \alpha_2}}{\max\{\beta_1 + \gamma_1, \beta_2 + \gamma_2\}} > \cos\left(\frac{\pi}{m+1}\right). \tag{1.3}
$$

The Laplacian operator $\Delta = \sum_{i=1}^{M} \frac{\partial^2}{\partial x_i^2}$ has a spatial dimension of M and $F(U)$ is a polynomially growing functional representing the reaction terms of the system.

The boundary conditions and initial data for the proposed system are assumed to satisfy

$$
\alpha U + (1 - \alpha) \partial_\eta U = B \quad \text{on } \partial\Omega \times (0, +\infty), \tag{1.4}
$$

or

$$
\alpha U + (1 - \alpha) A \partial_\eta U = B \quad \text{on } \partial\Omega \times (0, +\infty) \tag{1.5}
$$

and

$$
U(x, 0) = U_0(x) \quad \text{on } \Omega, \tag{1.6}
$$

respectively. For generality, we will consider three types of boundary conditions in this paper:

(i) Nonhomogeneous Robin boundary conditions, corresponding to

$$
0 < \alpha < 1, \ B \in \mathbb{R}^m;
$$

(ii) Homogeneous Neumann boundary conditions, corresponding to

$$
\alpha = 0 \text{ and } B \equiv 0;
$$

(iii) Homogeneous Dirichlet boundary conditions, corresponding to

$$
1 - \alpha = 0 \text{ and } B \equiv 0.
$$

Note that $\frac{\partial}{\partial\eta}$ denotes the outward normal derivative on $\partial\Omega$ and the vectors U, F, and B are defined as

$$
\begin{aligned}
U &:= (u_1, ..., u_m)^T, \\
F &:= (f_1, ..., f_m)^T, \\
B &:= (\beta_1, ..., \beta_m)^T.
\end{aligned}
$$

The initial data is assumed to be in the region given by

$$
\Sigma_{\mathfrak{L}, \emptyset} = \{U_0 \in \mathbb{R}^m : \langle V_\ell, U_0 \rangle \geq 0, \ \ell \in \mathfrak{L}\}, \tag{1.7}
$$

subject to

$$\langle V_\ell, B \rangle \geq 0, \ell \in \mathfrak{L}. \tag{1.8}$$

The study at hand builds upon numerous previous works found in the literature. Among the most relevant studies is that of Abdelmalek in [1] where he considered an m-component tridiagonal matrix of the form

$$A = \begin{pmatrix} \alpha & \gamma & 0 & \cdots & 0 \\ \beta & \alpha & \gamma & \ddots & \vdots \\ 0 & \beta & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \gamma \\ 0 & \cdots & 0 & \beta & \alpha \end{pmatrix}_{m \times m}$$

and proved the global existence of solutions subject to the parabolicity condition

$$\frac{\alpha}{\beta + \gamma} > \cos \frac{\pi}{m+1},$$

which can be easily shown to fall under the general condition in (1.3) with $\alpha = \alpha_1 = \alpha_2$, $\beta = \beta_1 = \beta_2$, and $\gamma = \gamma_1 = \gamma_2$.

Another important study is that of Kouachi and Rebiai in [7] where the authors established the global existence of solutions for a 3×3 tridiagonal 2–Toeplitz matrix of the form

$$A = \begin{pmatrix} \alpha_1 & \gamma_1 & 0 \\ \beta_1 & \alpha_2 & \gamma_2 \\ 0 & \beta_2 & \alpha_1 \end{pmatrix},$$

subject to the parabolicity condition

$$2\sqrt{\alpha_1 \alpha_2} > \sqrt{(\beta_1 + \gamma_1)^2 + (\beta_2 + \gamma_2)^2}.$$

Note that this condition is weaker than

$$\sqrt{2\alpha_1 \alpha_2} > \max\{\beta_1 + \gamma_1, \beta_2 + \gamma_2\},$$

which is obtained from (1.3) for $m = 3$. Although the work carried out in [7] is important to us here, it is necessary to note that the authors failed to identify all the invariant regions of the proposed system and settled for only 4 of them.

This paper will build upon the work of these two studies by assuming the diffusion matrix to be m–component tridiagonal 2–Toeplitz and determining all the possible invariant regions for the system. A Lyapunov functional will be used to establish the global existence of solutions in these regions.

The remainder of this paper is organized as follows: Section 2 uses the three point Chebyshev recurrence relationhip of polynomials to derive the eigenvalues and eigenvectors of the transposed dffusion matrix for the odd and even dimension cases, respectively. Section 3 derives the parabolicity conditions for the proposed system, which is essential for the diagonalization process, which follows in Section 4. Section 4 shows how the invariant regions of the equivalent digonalized system can be identified and proves the local and global existence of solutions. The last section of this paper will present a confirmation and validation of the findings through the use of numerical examples solved by means of the finite difference approximation method.

2. Eigenvalues and Eigenvectors

For reasons that will become apparent in the following section, we will first derive the eigenvalues and eigenvectors of matrix A^T with A being the proposed tridiagonal 2–Toeplitz diffusion matrix. We refer to the work of Gover in [4] where the characteristic polynomial of a tridiagonal 2–Toeplitz matrix was shown to be closely connected to polynomials that satisfy the three point Chebyshev recurrence relationship. First, we have

$$A^T = \begin{pmatrix} \alpha_1 & \beta_1 & 0 & \cdots & \cdots & 0 \\ \gamma_1 & \alpha_2 & \beta_2 & \ddots & & \vdots \\ 0 & \gamma_2 & \alpha_1 & \beta_1 & \ddots & \vdots \\ \vdots & \ddots & \gamma_1 & \alpha_2 & \beta_2 & 0 \\ \vdots & & \ddots & \gamma_2 & \ddots & \ddots \\ 0 & \cdots & \cdots & 0 & \ddots & \ddots \end{pmatrix}_{m \times m}. \tag{2.1}$$

The exact shape and characteristics of A^T differ for odd and even values of the dimension m. Hence, we will consider the two cases separately. Before we present the main findings of [4], let us define the constants

$$\beta = \sqrt{\frac{\beta_2 \gamma_2}{\beta_1 \gamma_1}} \text{ and } s = \sqrt{\frac{\gamma_1 \gamma_2}{\beta_1 \beta_2}}. \tag{2.2}$$

We also define the polynomials

$$\begin{cases} q_0(\mu) = 1, \ q_1(\mu) = \mu + \beta, \\ q_{n+1}(\mu) = \mu q_n(\mu) - q_{n-1}(\mu) \end{cases} \tag{2.3}$$

and

$$\begin{cases} p_0(\mu) = 1, \ p_1(\mu) = \mu, \\ p_{n+1}(\mu) = \mu p_n(\mu) - p_{n-1}(\mu), \end{cases} \tag{2.4}$$

whose zeros are denoted by Q_r and P_r, respectively, for $r = 1, ..., n$. We note that $p_n(\mu)$ is a Chebyshev polynomial of the first kind, whereas $q_n(\mu)$ is not. As shown in [4], the zeros of $p_n(\mu)$ can be given by

$$P_r = 2\cos\frac{r\pi}{n+1},$$

whereas for Q_r no explicit form was found.

Let us now summarize the eigenvalues and eigenvectors for the odd and even cases separately. First, for $m = 2n + 1$, we obtain the following results:

Theorem 2.1. *The eigenvalues of the matrix A^T of order $m = 2n + 1$ given in (2.1) are α_1 along with the solutions of the quadratic equations*

$$\frac{(\alpha_1 - \lambda)(\alpha_2 - \lambda)}{\sqrt{\beta_1 \beta_2 \gamma_1 \gamma_2}} - \frac{1}{\beta} - \beta = P_r, \tag{2.5}$$

for $r = 1, 2, ..., n$.

Note that for every P_r there exist two eigenvalues for matrix A^T, which along with α_1 yields $m = 2n+1$ eigenvalues. For notational purposes, let us define a duplicated set of zeros given by

$$P'_{2r} = P'_{2r-1} = P_r,$$

for $r = 1, 2, ..., n$.

Theorem 2.2. *The eigenvector of the matrix A^T of order $m = 2n + 1$ given in (2.1) associated with the eigenvalue λ_r, for $r = 1, ..., 2n$, is given by*

$$V_{\lambda_r} = (v_{1\lambda_r}, v_{2\lambda_r}, ..., v_{m\lambda_r})^T, \tag{2.6}$$

where

$$v_{\ell\lambda_r} = \begin{cases} s^{\frac{\ell-1}{2}} q_{\frac{\ell-1}{2}}\left(P_r'\right), & \ell \text{ is odd,} \\ -\frac{1}{\beta_1} s^{\frac{\ell}{2}-1} (\alpha_1 - \lambda_r) p_{\frac{\ell}{2}-1}\left(P_r'\right), & \ell \text{ is even,} \end{cases} \tag{2.7}$$

for $\ell = 1, ..., m$. The eigenvector associated with the eigenvalue α_1 is

$$V_{\alpha_1} = (v_{1\alpha_1}, v_{2\alpha_1}, ..., v_{m\alpha_1})^T, \tag{2.8}$$

with

$$v_{\ell\alpha_1} = \begin{cases} \left(-\frac{\gamma_1}{\beta_2}\right)^{\frac{\ell-1}{2}}, & \ell \text{ is odd} \\ 0, & \ell \text{ is even,} \end{cases} \tag{2.9}$$

for $\ell = 1, ..., m$.

The second case is where the matrix A^T (2.1) has an even dimension $m = 2n$. The following holds:

Theorem 2.3. *The eigenvalues of the matrix A^T of order $m = 2n + 1$ given in (2.1) denoted by λ_r are the solutions of the quadratic equations*

$$\frac{(\alpha_1 - \lambda)(\alpha_2 - \lambda)}{\sqrt{\beta_1 \beta_2 \gamma_1 \gamma_2}} - \frac{1}{\beta} - \beta = Q_r, \tag{2.10}$$

for $r = 1, 2, ..., n$, where Q_r are the zeros of $q_n(\mu)$.

Similar to P_r, there exist two eigenvalues for matrix A^T associated with every value of Q_r, which yields m eigenvalues. In order to simplify the notation, we define the duplicated set of zeros given by

$$Q_{2r}' = Q_{2r-1}' = Q_r,$$

for $r = 1, 2, ..., n$.

Theorem 2.4. *The eigenvector of the matrix A^T of order $m = 2n + 1$ given in (2.1) associated with the eigenvalue λ_r is given by*

$$V_r = (v_{1\lambda_r}, v_{2\lambda_r}, ..., v_{m\lambda_r})^T, \tag{2.11}$$

with

$$v_{\ell\lambda_r} = \begin{cases} s^{\frac{\ell-1}{2}} q_{\frac{\ell-1}{2}}\left(Q_r'\right), & \ell \text{ is odd} \\ -\frac{1}{\beta_1} s^{\frac{\ell}{2}-1} (\alpha_1 - \lambda_r) p_{\frac{\ell}{2}-1}\left(Q_r'\right), & \ell \text{ is even,} \end{cases} \tag{2.12}$$

for $\ell = 1, ..., m$.

3. Parabolicity

In this section, we will derive the parabolicity condition for the proposed system. Parabolicity is crucial to the diagonalization process, which we will be discussed later on in Section 4. In order to ensure the parabolicity of the system, we examine the positive definiteness of the proposed diffusion matrix. Generally speaking, a matrix is said to be positive definite if and only if its top-left corner principal minors are all positive. To this end, Andelic and da Fonesca [2] and others examined the parabolicity condition for a tridiagonal symmetric matrix. The following theorem holds.

Proposition 3.1. *Let T be the tridiagonal matrix defined as*

$$T = \begin{pmatrix} a_1 & b_1 & 0 & \cdots & & 0 \\ b_1 & a_2 & b_2 & & & \vdots \\ 0 & b_2 & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & b_{m-1} & \\ 0 & \cdots & \cdots & b_{m-1} & a_m \end{pmatrix},$$

with positive diagonal entries. If

$$a_i a_{i+1} > 4b_i^2 \cos^2 \left(\frac{\pi}{m+1} \right) \tag{3.1}$$

for $i = 1, ..., m-1$, then T is positive definite.

Since the diffusion matrix considered here is not symmetric, Proposition 3.1 does not apply directly to it. However, we know that if a matrix is not symmetric, its quadratic form $Q = \langle X, AX \rangle = X^T AX$, with X being an arbitrary column vector, is said to be positive definite if and only if the principal minors in the top–left corner of $\frac{1}{2}\left(A + A^T\right)$ are all positive. In order to derive sufficient conditions for matrix A in (1.2), we apply Proposition 3.1 to produce the following theorem.

Theorem 3.2. *Let A be the tridiagonal 2–Toeplitz matrix defined in (1.2). The quadratic form of A is positive definite iff condition (1.3) is satisfied. It follows that subject to (1.3), the reaction diffusion system (1.1) satisfies the parabolicity condition.*

Proof. Condition (3.1) can be rearranged to the form

$$\sqrt{a_i a_{i+1}} > 2 |b_i| \cos \left(\frac{\pi}{m+1} \right). \tag{3.2}$$

The symmteric counterpart of A as defined in (1.2) can be given by

$$\frac{1}{2}\left(A + A^T\right) = \begin{pmatrix} \alpha_1 & \frac{\beta_1+\gamma_1}{2} & 0 & \cdots & \cdots & 0 \\ \frac{\beta_1+\gamma_1}{2} & \alpha_2 & \frac{\beta_2+\gamma_2}{2} & \ddots & & \vdots \\ 0 & \frac{\beta_2+\gamma_2}{2} & \alpha_1 & \frac{\beta_1+\gamma_1}{2} & \ddots & \vdots \\ \vdots & \ddots & \frac{\beta_1+\gamma_1}{2} & \alpha_2 & \frac{\beta_2+\gamma_2}{2} & 0 \\ \vdots & & \ddots & \frac{\beta_2+\gamma_2}{2} & \ddots & \ddots \\ 0 & \cdots & \cdots & 0 & \ddots & \ddots \end{pmatrix}. \tag{3.3}$$

Now, substituting (3.3) in (3.2) yields the set of $m-1$ conditions

$$\begin{cases} \text{for } i = 1: & \sqrt{\alpha_1 \alpha_2} > (\beta_1 + \gamma_1) \cos \left(\frac{\pi}{m+1} \right) \\ \text{for } i = 2: & \sqrt{\alpha_1 \alpha_2} > (\beta_2 + \gamma_2) \cos \left(\frac{\pi}{m+1} \right) \\ \text{for } i = 3: & \sqrt{\alpha_1 \alpha_2} > (\beta_1 + \gamma_1) \cos \left(\frac{\pi}{m+1} \right) \\ \quad \vdots & \qquad \vdots \\ \text{for } i = m-1: & \begin{cases} \sqrt{\alpha_1 \alpha_2} > (\beta_2 + \gamma_2) \cos \left(\frac{\pi}{m+1} \right), & \text{if } m \text{ is odd,} \\ \sqrt{\alpha_1 \alpha_2} > (\beta_1 + \gamma_1) \cos \left(\frac{\pi}{m+1} \right), & \text{if } m \text{ is even.} \end{cases} \end{cases}$$

However, we notice that the $m-1$ conditons reduce to only 2, which can be combined to form condition (1.3). $\square$

4. Existence of Solutions

This section shows how the proposed system can be diagonalized using the eigenvectors derived in Section 2 above. We start by examining the invariant regions of the system and then move to diagonalize the system and establish the local and global existence of solutions given the initial data lies within the invariant regions.

4.1. Invariant Regions

Let us denote the positive and descendingly ordered eigenvalues of matrix A^T by λ_ℓ, with $\ell = 1, ..., m$, and the corresponding eigenvectors by $V_\ell = (v_{1\ell}, ..., v_{m\ell})^T$, where $\lambda_1 > \lambda_2 > ... > \lambda_m$. Assuming the proposed system satisfies the parabolicity condition (1.3), matrix A^T is guaranteed to have strictly positive eigenvalues, and thus is unitarily diagonalizable. Generally, the diagonalizing matrix can be formed containing as its columns the normalized eigenvectors of A. Recalling that for every eigenvalue there exist two eigenvectors with unit norm and opposite directions, we can define the diagonalizing matrix as

$$P = \left((-1)^{i_1} V_1 \mid (-1)^{i_2} V_2 \mid ... \mid (-1)^{i_m} V_m \right), \tag{4.1}$$

where each power i_ℓ is either equal to 1 or 2. In order to simplify the notation, let us consider the two disjoint sets

$$\mathfrak{Z} = \{\ell | i_\ell = 1\}$$

and

$$\mathfrak{L} = \{\ell | i_\ell = 2\},$$

which satisfy the properties

$$\mathfrak{L} \cap \mathfrak{Z} = \phi \text{ and } \mathfrak{L} \cup \mathfrak{Z} = \{1, 2, ..., m\}. \tag{4.2}$$

Each permutation of $\mathfrak{Z}$ and $\mathfrak{L}$ satisfying (4.2) yields a valid diagonalizing matrix. The total number of possible permutations is thus 2^m, which is also the number of invariant regions $\Sigma_{\mathfrak{L},\mathfrak{Z}}$ for the proposed system. These regions may be written as

$$\Sigma_{\mathfrak{L},\mathfrak{Z}} := \{U_0 \in \mathbb{R}^m : \langle V_z, U_0 \rangle \le 0 \le \langle V_\ell, U_0 \rangle, \ \ell \in \mathfrak{L}, \ z \in \mathfrak{Z}\}, \tag{4.3}$$

subject to

$$\langle V_z, B \rangle \le 0 \le \langle V_\ell, B \rangle, \ \ell \in \mathfrak{L}, \ z \in \mathfrak{Z}. \tag{4.4}$$

For simplicity, we will only consider one of the invariant regions which corresponds to the sets $\mathfrak{L} = \{1, 2, ..., m\}$ and $\mathfrak{Z} = \emptyset$ and is defined in (1.7) and (1.8). This yields the diagonalizing matrix

$$P = (V_1 \mid V_2 \mid ... \mid V_m). \tag{4.5}$$

Note that the work carried out in the following subsections can be trivially extended to the remaining $2^m - 1$ regions.

4.2. Diagonalization and Local Existence of Solutions

In order to establish the local existence of solutions for the proposed system (1.1), we start by diagonalizing the system by means of the diagonalizing matrix defined in (4.5). We follow the same work performed in [1] to obtain the equivalent diagonal system. First, let

$$W = (w_1, w_2, ..., w_m)^T = P^T U, \tag{4.6}$$

where

$$w_\ell := \langle V_\ell, U \rangle$$
$$= \begin{cases} \langle V_\ell, U \rangle, & \ell \in \mathfrak{L} \\ \langle (-1) V_\ell, U \rangle, & \ell \in \mathfrak{Z}. \end{cases}$$

Let us also define the functional

$$\mathcal{F}(W) = (\mathcal{F}_1, \mathcal{F}_2, ..., \mathcal{F}_m)^T = P^T F(U), \tag{4.7}$$

with each function

$$\mathcal{F}_\ell := \langle V_\ell, F \rangle$$

fulfilling the following conditions:

(A1) Must be continuously differentiable on $\mathbb{R}_+^m$ for all $\ell = 1, ..., m$, satisfying $\mathcal{F}_\ell(w_1, ..., w_{\ell-1}, 0, w_{\ell+1}, ..., w_m) \geq 0$, for all $w_\ell \geq 0$; $\ell = 1, ..., m$.

(A2) Must be of polynomial growth (see the work of Hollis and Morgan [6]), which means that for all $\ell = 1, ..., m$:

$$|\mathcal{F}_\ell(W)| \leq C_1 (1 + \langle W, 1 \rangle)^N, N \in \mathbb{N}, \text{on } (0, +\infty)^m. \tag{4.8}$$

(A3) Must satisfy the inequality:

$$\langle D, \mathcal{F}(W) \rangle \leq C_2 (1 + \langle W, 1 \rangle), \tag{4.9}$$

where

$$D := (D_1, D_2, ..., D_{m-1}, 1)^T,$$

for all $w_\ell \geq 0$, $\ell = 1, ..., m$,. All the constants D_ℓ satisfy $D_\ell \geq \overline{D_\ell}$, $\ell = 1, ..., m$ where $\overline{D_\ell}$, $\ell = 1, ..., m$, are sufficiently large positive constants.

Note that C_1 and C_2 are uniformly bounded positive functions defined on $\mathbb{R}_+^m$.

Finally, let

$$\Lambda = P^T B.$$

Now, by observing the similarity transformation

$$P^T A (P^T)^{-1} = (P^{-1} A^T P)^T$$
$$= \text{diag}(\lambda_1, \lambda_2, ..., \lambda_m), \tag{4.10}$$

we can propose the following:

Proposition 4.1. *Diagonalizing system* (1.1) *by means of* P^T *yields*

$$W_t - \text{diag}(\lambda_1, \lambda_2, ..., \lambda_m)\Delta W = \mathcal{F}(W) \quad \text{in } \Omega \times (0, +\infty) \tag{4.11}$$

with the boundary condition

$$\alpha W + (1 - \alpha) \partial_n W = \Lambda \quad \text{on } \partial\Omega \times (0, +\infty) \tag{4.12}$$

or

$$\alpha W + (1 - \alpha) \text{diag}(\lambda_1, \lambda_2, ..., \lambda_m)\partial_n W = \Lambda, \tag{4.13}$$

and the initial data

$$W(x, 0) = W_0 \quad \text{on } \Omega. \tag{4.14}$$

The proof of Proposition 4.1 is trivial and can be looked up in [1]. The diagonal system in (4.11) is equivalent to (1.1) in the invariant region given in (1.7) and (1.8).

By considering the equivalent diagonal system in (4.11), we can now establish the local existence and uniqueness of solutions for the original system (1.1) with initial data in $C(\overline{\Omega})$ or $L^p(\Omega)$, $p \in (1, +\infty)$ using the basic existence theory for abstract semilinear differential equations (Friedman [3], Henry [5] and Pazy [8]). It simply follows that the solutions are classical on $(0, T_{\max})$, with $T_{\max}$ denoting the eventual blow up time in $L^\infty(\Omega)$. The local solution is continued globally by *apriori* estimates.

4.3. Global Existence of Solutions

The aim here is to establish the global existence of solutions for the equivalent system (4.11) and consequently the original system (1.1) subject to the parabolicity condition (1.3) through the use of an appropriate Lyapunov functional. The results obtained here are similar to those of [1]. Hence, no detailed proofs will be given here.

Let us define

$$K_l^r = K_{r-1}^{r-1} K_l^{r-1} - \left[H_l^{r-1}\right]^2, \ r = 3, ..., l, \tag{4.15}$$

where

$$H_l^r = \det_{\substack{1 \le \ell, \kappa \le l}} \left((a_{\ell,\kappa})_{\substack{\ell \ne l,...r+1 \\ \kappa \ne l-1,..r}} \right) \prod_{k=1}^{k=r-2} (\det [k])^{2^{(r-k-2)}}, \ r = 3, ..., l-1,$$

$$K_l^2 = \lambda_1 \lambda_l \underbrace{\prod_{k=1}^{l-1} \theta_k^{2(p_k+1)^2} \prod_{k=l}^{m-1} \theta_k^{2(p_k+2)^2}}_{\text{positive value}} \left[\prod_{k=1}^{l-1} \theta_k^2 - A_{1l}^2 \right],$$

and

$$H_l^2 = \lambda_1 \sqrt{\lambda_2 \lambda_l} \underbrace{\theta_1^{2(p_1+1)^2} \prod_{k=2}^{l-1} \theta_k^{(p_k+2)^2+(p_k+1)^2} \prod_{k=l}^{m-1} \theta_k^{2(p_k+2)^2}}_{\text{positive value}} \left[\theta_1^2 A_{2l} - A_{12} A_{1l} \right].$$

The term $\det_{\substack{1 \le \ell, \kappa \le l}} \left((a_{\ell,\kappa})_{\substack{\ell \ne l,...r+1 \\ \kappa \ne l-1,..r}} \right)$ denotes the determinant of the r square symmetric matrix obtained from $(a_{\ell,\kappa})_{1 \le \ell, \kappa \le m}$ by removing the $(r+1)^{th}, (r+2)^{th}, ..., l^{th}$ rows and the $r^{th}, (r+1)^{th}, ..., (l-1)^{th}$ columns. where $\det [1], ..., \det [m]$ are the minors of the matrix $(a_{\ell,\kappa})_{1 \le \ell, \kappa \le m}$. The elements of the matrix are:

$$a_{\ell\kappa} = \frac{\lambda_\ell + \lambda_\kappa}{2} \theta_1^{p_1^2} ... \theta_{(\ell-1)}^{p_{(\ell-1)}^2} \theta_\ell^{(p_\ell+1)^2} ... \theta_{\kappa-1}^{(p_{(\kappa-1)}+1)^2} \theta_\kappa^{(p_\kappa+2)^2} ... \theta_{(m-1)}^{(p_{(m-1)}+2)^2}. \tag{4.16}$$

where λ_ℓ in (2.4)-(2.2). Note that $A_{\ell\kappa} = \frac{\lambda_\ell + \lambda_\kappa}{2\sqrt{\lambda_\ell \lambda_\kappa}}$ for all $\ell, \kappa = 1, ..., m$, and $\theta_\ell; \ell = 1, ..., (m-1)$ are positive constants.

Theorem 4.2. *Suppose that the functions $F_\ell; \ell = 1, ..., m$ are of polynomial growth and satisfy condition (4.9) for some positive constants $D_\ell; \ell = 1, ..., m$ sufficiently large. Let $(w_1(t, .), w_2(t, .), ..., w_m(t, .))$ be a solution of (4.11) and*

$$L(t) = \int_\Omega H_{p_m}(w_1(t, x), w_2(t, x), ..., w_m(t, x)) dx, \tag{4.17}$$

where

$$H_{p_m}(w_1, ..., w_m) = \sum_{p_{m-1}=0}^{p_m} ... \sum_{p_1=0}^{p_2} C_{p_m}^{p_{m-1}} ... C_{p_2}^{p_1} \theta_1^{p_1^2} ... \theta_{(m-1)}^{p_{(m-1)}^2} w_1^{p_1} w_2^{p_2-p_1} ... w_m^{p_m-p_{m-1}},$$

with p_m a positive integer and $C_{p_\kappa}^{p_\ell} = \frac{p_\kappa!}{p_\ell!(p_\kappa - p_\ell)!}$.
Also suppose that the following condition is satisfied

$$K_l^l > 0; \ l = 2, ..., m, \tag{4.18}$$

It follows from these conditions that the functional L is uniformly bounded on the interval $[0, T^*], T^* < T_{\max}$.

Corollary 4.3. *Under the assumptions of Theorem 4.2, all solutions of (4.11) with positive initial data in $L^\infty(\Omega)$ are in $L^\infty(0, T^*; L^p(\Omega))$ for some $p \ge 1$.*

Proposition 4.4. *Under the assumptions of Theorem 4.2 and given that the condition (1.3) is satisfied, all solutions of (4.11) with positive initial data in $L^\infty(\Omega)$ are global for some $p > \frac{MN}{2}$.*

5. Numerical Example

In order to put the findings of this study to the test, let us consider the following 5-component system

$$\frac{\partial U}{\partial t} - A\Delta U = F(U), \tag{5.1}$$

where the transposed diffusion matrix is given by

$$A^T = \begin{pmatrix} 1 & 0.5 & 0 & 0 & 0 \\ 0.3 & 1.5 & 0.7 & 0 & 0 \\ 0 & 0.25 & 1 & 0.5 & 0 \\ 0 & 0 & 0.3 & 1.5 & 0.7 \\ 0 & 0 & 0 & 0.25 & 1 \end{pmatrix} \tag{5.2}$$

and the reaction functional $F(U)$ is of the form

$$F(U) = \begin{pmatrix} F_1 & F_2 & F_3 & F_4 & F_5 \end{pmatrix}^T,$$

with

$$F_j(U) = U^T \Upsilon_j U + \sigma_j^T U, \quad j = 1, ..., 5.$$

For the purpose of this example, let Υ_j be the symmetric matrices given by

$$\Upsilon_1 = \begin{pmatrix} 0.0146 & -0.0257 & 0.0073 & -0.0088 & 0 \\ -0.0257 & 0.0202 & -0.004 & 0 & 0.0044 \\ 0.0073 & -0.004 & 0.0005 & 0.0011 & -0.0015 \\ -0.0088 & 0 & 0.0011 & -0.0043 & 0.0027 \\ 0 & 0.0044 & -0.0015 & 0.0027 & -0.0007 \end{pmatrix},$$

$$\Upsilon_2 = \begin{pmatrix} 0.1142 & 0.228 & 0.0571 & 0.1293 & 0 \\ 0.228 & -0.2281 & -0.0153 & 0 & -0.0646 \\ 0.0571 & -0.0153 & 0.0041 & 0.0158 & -0.0122 \\ 0.1293 & 0 & 0.0158 & 0.0489 & -0.0244 \\ 0 & -0.0646 & -0.0122 & -0.0244 & -0.0061 \end{pmatrix},$$

$$\Upsilon_3 = \begin{pmatrix} 0.3702 & -0.1245 & 0.1851 & 0.0194 & 0 \\ -0.1245 & 0.0371 & -0.0817 & 0 & -0.0097 \\ 0.1851 & -0.0817 & 0.0132 & 0.0364 & -0.0397 \\ 0.0194 & 0 & 0.0364 & -0.0079 & 0.0133 \\ 0 & -0.0097 & -0.0397 & 0.0133 & -0.0198 \end{pmatrix},$$

$$\Upsilon_4 = \begin{pmatrix} -0.1316 & -0.1013 & -0.0658 & -0.0743 & 0 \\ -0.1013 & 0.1177 & 0.0236 & 0 & 0.0371 \\ -0.0658 & 0.0236 & -0.0047 & -0.0154 & 0.0141 \\ -0.0743 & 0 & -0.0154 & -0.0252 & 0.0108 \\ 0 & 0.0371 & 0.0141 & 0.0108 & 0.0070 \end{pmatrix},$$

and

$$\Upsilon_5 = \begin{pmatrix} -0.1651 & 0.5295 & -0.0825 & 0.2108 & 0 \\ 0.5295 & -0.4429 & 0.0539 & 0 & -0.1054 \\ -0.0825 & 0.0539 & -0.0059 & -0.0081 & 0.0177 \\ 0.2108 & 0 & -0.0081 & 0.0949 & -0.0567 \\ 0 & -0.1054 & 0.0177 & -0.0567 & 0.0088 \end{pmatrix}.$$

Also, suppose that

$$\begin{cases} \sigma_1 = \begin{pmatrix} 0.0795 & 0.0303 & -0.0243 & -0.014 & 0.0059 \end{pmatrix}^T, \\ \sigma_2 = \begin{pmatrix} -0.6466 & -0.6144 & 0.0798 & 0.2844 & 0.0572 \end{pmatrix}^T, \\ \sigma_3 = \begin{pmatrix} 0.4549 & -0.2791 & -0.2846 & 0.1292 & 0.1635 \end{pmatrix}^T, \\ \sigma_4 = \begin{pmatrix} 0.2682 & 0.3879 & 0.0097 & -0.1796 & -0.0618 \end{pmatrix}^T, \\ \sigma_5 = \begin{pmatrix} -1.6033 & -0.8159 & 0.4251 & 0.3777 & -0.0608 \end{pmatrix}^T. \end{cases}$$

The system clearly satisfies the parabolicity condition (1.3) as

$$\frac{\sqrt{\alpha_1 \alpha_2}}{\max\{\beta_1 + \gamma_1, \beta_2 + \gamma_2\}} = \frac{\sqrt{1.5}}{0.95} = 1.2892 > \cos\left(\frac{\pi}{5}\right) = 0.8090.$$

We have from (2.2)

$$\beta = \sqrt{\frac{0.7 \times 0.25}{0.5 \times 0.3}} = 1.0801. \tag{5.3}$$

Hence, we can form the polynomial $p_n(\mu)$ as

$$\begin{cases} p_0(\mu) = 1, \ p_1(\mu) = \mu \\ p_2(\mu) = \mu(\mu) - 1 = \mu^2 - 1, \end{cases}$$

with solutions

$$P_1 = 1 \text{ and } P_2 = -1. \tag{5.4}$$

Now, the eigenvalues are α_1 along with the solutions of the following two equations derived from (2.5)

$$\begin{cases} \frac{(1-\lambda)(1.5-\lambda)}{\sqrt{0.5 \times 0.7 \times 0.3 \times 0.25}} - \frac{1}{1.0801} - 1.0801 = 1, \\ \frac{(1-\lambda)(1.5-\lambda)}{\sqrt{0.5 \times 0.7 \times 0.3 \times 0.25}} - \frac{1}{1.0801} - 1.0801 = -1, \end{cases}$$

which can be simplified to

$$\begin{cases} 6.1721(\lambda - 1)(\lambda - 1.5) - 3.0059 = 0, \\ 6.1721(\lambda - 1)(\lambda - 1.5) - 1.0059 = 0. \end{cases}$$

Solving the two quadratic equations in λ yields the four eigenvalues of A, which in descending order can be given by

$$\begin{cases} \lambda_1 = 1.9913, \\ \lambda_2 = 1.7248, \\ \lambda_3 = 1, \\ \lambda_4 = 0.77516, \\ \lambda_5 = 0.50871. \end{cases} \tag{5.5}$$

Hence,

$$D = \text{diag}(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5).$$

Similarly, formula (2.11)-(2.12) can be used to derive the eigenvectors of A^T, which are arranged according to the corresponding eigenvalues to form the diagonalizing matrix

$$P = \begin{pmatrix} 0.3848 & -0.5265 & -0.5632 & -0.9063 & -0.8769 \\ 0.7629 & -0.7633 & 0.5534 & 0.0000 & 0.3943 \\ 0.3705 & -0.0195 & -0.5423 & 0.3884 & -0.0325 \\ 0.3531 & 0.3534 & 0.2562 & 0.0000 & -0.1825 \\ \underbrace{0.0891}_{V_1} & \underbrace{0.1219}_{V_2} & \underbrace{-0.1303}_{V_3} & \underbrace{-0.1665}_{V_4} & \underbrace{0.2030}_{V_5} \end{pmatrix}. \tag{5.6}$$
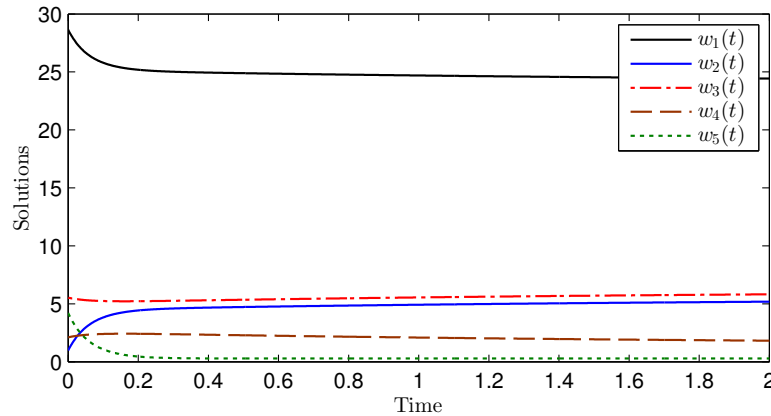
Figure 1: The solutions of the equivalent diagonal system described in (5.7) in the diffusion-free case with the initial data given in (5.9).

Matrix P^T is used to diagonalize the system yields the equivalent system

$$\begin{cases} \frac{\partial w_1}{\partial t} - 1.9913\Delta w_1 = -0.5w_1w_5 + 0.65w_2, \\ \frac{\partial w_2}{\partial t} - 1.7248\Delta w_2 = 0.5w_1w_5 - 0.65w_2, \\ \frac{\partial w_3}{\partial t} - \Delta w_3 = -0.32w_3w_5 + 0.41w_4, \\ \frac{\partial w_4}{\partial t} - 0.77516\Delta w_4 = 0.32w_3w_5 - 0.41w_4, \\ \frac{\partial w_5}{\partial t} - 0.50871\Delta w_5 = -0.5w_1w_5 + 0.65w_2 - 0.32w_3w_5 + 0.41w_4. \end{cases} \tag{5.7}$$

Note that for simplicity, we have neglected small terms and rounded the polynomial coefficients to four decimal points. The resulting reaction terms clearly satisfy conditions (A1) through (A3) as discussed in Section 4.2 above.

Observe that the proposed system has $2^5 = 32$ invariant regions where the resulting $w_{\ell 0}$ is guaranteed to be positive. We consider one of these regions corresponding to $w_{\ell 0} = \langle V_\ell, U_0 \rangle$ and given by

$$\Sigma_{\mathfrak{L}, \emptyset} = \{U_0 \in \mathbb{R}^m : \langle V_\ell, U_0 \rangle \geq 0, \ \ell = 1, ..., m\},$$

which yields five inequalities

$$\begin{cases} 0.3848u_{01} + 0.7629u_{02} + 0.3705u_{03} + 0.3531u_{04} + 0.0891u_{05} \geq 0, \\ -0.5265u_{01} - 0.7633u_{02} - 0.0195u_{03} + 0.3534u_{04} + 0.1219u_{05} \geq 0, \\ -0.5632u_{01} + 0.5534u_{02} - 0.5423u_{03} + 0.2562u_{04} - 0.1303u_{05} \geq 0, \\ -0.9063u_{01} + 0.3884u_{03} - 0.1665u_{05} \geq 0, \\ -0.8769u_{01} + 0.3943u_{02} - 0.0325u_{03} - 0.1825u_{04} + 0.2030u_{05} \geq 0, \end{cases} \tag{5.8}$$

with

$$U_0 = (u_{01}, u_{02}, u_{03}, u_{04}, u_{05})^T.$$

Solving this system of inequalities yields the first region where the initial data is assumed to lie. We will consider for instance the initial data

$$U_0 = (0, 15, 14, 29, 20)^T. \tag{5.9}$$

The equivalent diagonalized system (5.7) was solved numerically by means of the finite difference (FD) method. Figures 1 and 2 show the solutions to the diagonalized system (5.7) and the original system (5.1), respectively, in the diffusion free case. In the one dimensional case, a sinusoidal perturbation is added to the initial data to introduce spatial diversity into the model. The solutions are shown in Figures 3 and 4.
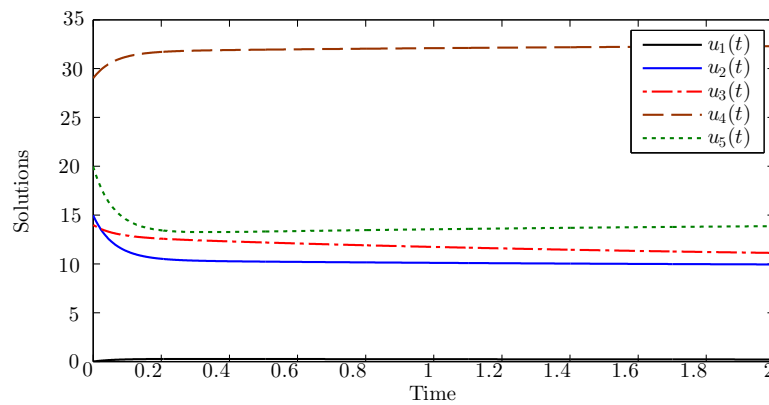
Figure 2: The solutions of the original system described in (5.1) in the diffusion-free case with the initial data given in (5.9).
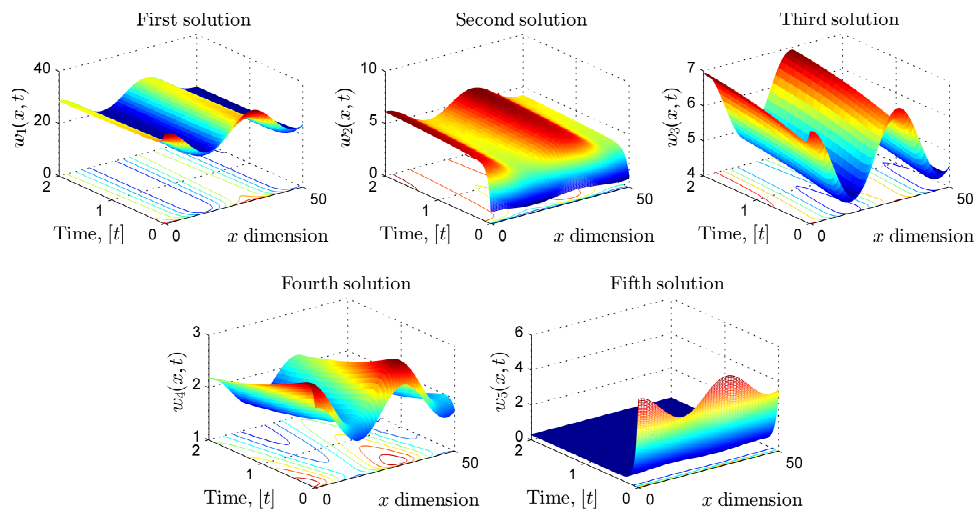

Figure 3: The solutions of the equivalent diagonal system described in (5.7) in the one-dimensional diffusion case.
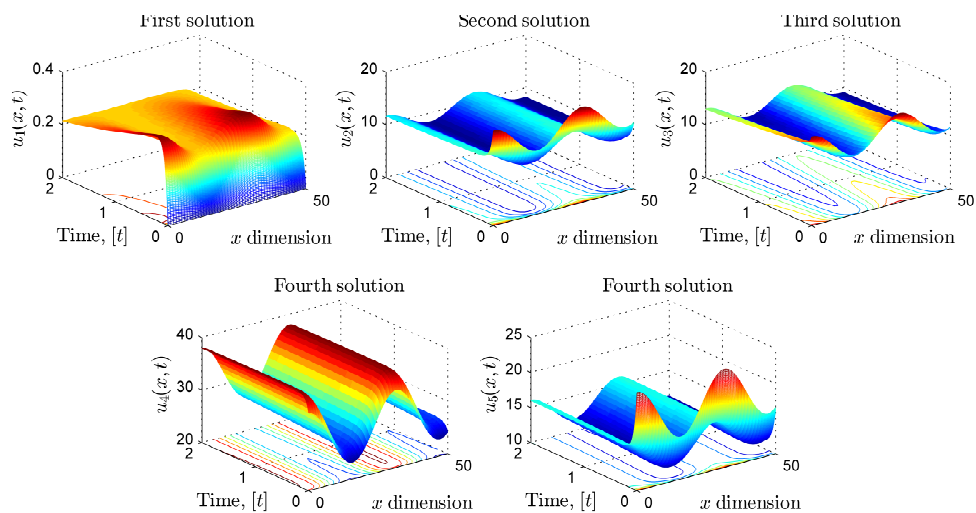

Figure 4: The solutions of the original system described in (5.1) in the one-dimensional diffusion case.

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