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# Randomness of Lacunary statistical acceleration convergence of $\chi^3$ over $p$-metric spaces defined by Orlicz functions

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**Abstract**

In this article, the notion of Randomness of Lacunary statistical acceleration convergence of $\chi^3$ over $p$-metric spaces defined by sequence of Orlicz has been introduced and some theorems related to that concept have been established using four dimensional matrix transformations. Author's construction with new definitions and also new statement of theorems of proofs are formulated. ©2017 All rights reserved.

*Keywords:* Analytic sequence, double sequences, $\chi^3$ space, Musielak-Orlicz function, random $p$-metric space, lacunary sequence, statistical convergence, converging faster, converging at the same rate, acceleration field, triple natural density.
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## 1. Introduction

The faster convergence of sequences particularly the acceleration of convergence of sequence of partial sums of series via linear and nonlinear transformations are widely used in finding solutions of mathematical as well as different scientific and engineering problems. The problem of acceleration convergence often occurs in numerical analysis. To accelerate the convergence, the standard interpolation and extrapolation methods of numerical mathematics are quite helpful. It is useful to study about the acceleration of convergence methods, which transform a slowly converging sequence into a new sequence, converging to the same limit faster than the original sequence. The speed of convergence of sequences is of the central importance in the theory of sequence transformation.

The concept of statistical convergence plays a vital role not only in pure mathematics but also in other branches of science involving mathematics, especially in information theory, computer science, biological

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science, dynamical systems, geo-graphic information systems, population modeling, and motion planning in robotics.

The notion of statistical convergence was introduced by Fast [6] and Schoenberg [9] independently. Over the years and under different names statistical convergence has been discussed in the theory of fourier analysis, ergodic theory, and number theory. In the recent years, generalizations of statistical convergence have appeared in the study of strong integral summability and the structure of ideals of bounded continuous functions on Stone-Čech compactification of the natural numbers. Moreover statistical convergence is closely related to the concept of convergence in probability.

The notion of statistical convergence depends on the density of subsets of $\mathbb{N}$. A subset of $\mathbb{N} \times \mathbb{N} \times \mathbb{N}$ is said to have density $\delta(E)$ if

$$\delta(E) = \lim_{r,s,t \to \infty} \frac{1}{rst} \sum_{m=1}^{r} \sum_{n=1}^{s} \sum_{k=1}^{t} \chi^3 E(mnk) = 0.$$

Throughout this paper, $w$, $\Gamma$, and $\Lambda$ denote the classes of all, entire, and analytic scalar valued single sequences, respectively.

We write $w^3$ for the set of all complex sequences $(x_{mnk})$, where $m, n, k \in \mathbb{N}$, the set of positive integers. Then, $w^3$ is a linear space under the coordinate wise addition and scalar multiplication.

Let $(x_{mnk})$ be a triple sequence of real or complex numbers. Then the series $\sum_{m,n,k=1}^{\infty} x_{mnk}$ is called a triple series. The triple series $\sum_{m,n,k=1}^{\infty} x_{mnk}$ is said to be convergent if and only if the triple sequence $(S_{mnk})$ is convergent, where

$$S_{mnk} = \sum_{i,j,q=1}^{m,n,k} x_{ijq} (m, n, k = 1, 2, 3, \ldots).$$

A triple sequence (real or complex) can be defined as a function $x : \mathbb{N} \times \mathbb{N} \times \mathbb{N} \to \mathbb{R}(\mathbb{C})$, where $\mathbb{N}, \mathbb{R}$ and $\mathbb{C}$ denote the set of natural numbers, real numbers and complex numbers respectively. The different types of notions of triple sequence was introduced and investigated at the initial by Sahiner et al. [7, 8], Esi et al. [3–5], Datta et al. [1], Subramanian et al. [10], Debnath et al. [2] and many others.
A sequence $x = (x_{mnk})$ is said to be triple analytic if

$$\sup_{m,n,k} |x_{mnk}|^{\frac{1}{m+n+k}} < \infty.$$

The vector space of all triple analytic sequences is usually denoted by $\Lambda^3$. A sequence $x = (x_{mnk})$ is called triple chi sequence if

$$((m+n+k)! \, |x_{mnk}|)^{\frac{1}{m+n+k}} \to 0 \text{ as } m, n, k \to \infty.$$

The vector space of all triple chi sequences is usually denoted by $\chi^3$. The space $\chi^3$ is a metric space with the metric

$$d(x, y) = sup_{m,n,k} \left\{ ((m+n+k)! \, |x_{mnk} - y_{mnk}|)^{\frac{1}{m+n+k}} : m, n, k : 1, 2, 3, \ldots \right\}$$

for all $x = \{x_{mnk}\}$ and $y = \{y_{mnk}\}$ in $\chi^3$. Let $\phi = \{\text{finite sequences}\}$.

Consider a triple sequence $x = (x_{mnk})$. The $(m, n, k)^{th}$ section $x^{[m,n,k]}$ of the sequence is defined by $x^{[m,n,k]} = \sum_{i,j,q=0}^{m,n,k} x_{ijq} \delta_{ijq}$ for all $m, n, k \in \mathbb{N}$, where $\delta_{mnk}$ is a three dimensional matrix with 1 in the $(m, n, k)^{th}$ position and zero otherwise.

Consider a triple sequence $x = (x_{mnk})$. The $(m, n, k)^{th}$ section $x^{[m,n,k]}$ of the sequence is defined by $x^{[m,n,k]} = \sum_{i,j,q=0}^{m,n,k} x_{ijq} \Im_{ijq}$ for all $m, n, k \in \mathbb{N}$; where $\Im_{ijq}$ denotes the triple sequence whose only non zero term is a 1 in the $(i, j, k)^{th}$ place for each $i, j, q \in \mathbb{N}$.

An Orlicz function is a function $f : [0, \infty) \to [0, \infty)$ which is continuous, non-decreasing, and convex with $f(0) = 0$, $f(x) > 0$ for $x > 0$ and $f(x) \to \infty$ as $x \to \infty$. If convexity of Orlicz function $f$ is replaced by $f(x+y) \leq f(x) + f(y)$, then this function is called modulus function. An Orlicz function $f$ is said to satisfy $\Delta^2$-condition for all values $u$, if there exists $K > 0$ such that $f(2u) \leq Kf(u), u \geq 0$.

**Lemma 1.1.** *Let $f$ be an Orlicz function which satisfies $\Delta_2$-condition and let $0 < \delta < 1$. Then for each $t \geq \delta$, we have $f(t) < K\delta^{-1} f(2)$ for some constant $K > 0$.*

*A sequence $f = (f_{mnk})$ of Orlicz function is called a Musielak-Orlicz function. A sequence $g = (g_{mnk})$ defined by*

$$g_{mnk}(v) = \sup\{|v| u - (f_{mnk})(u) : u \geq 0\}, m, n, k = 1, 2, \ldots$$

*is called the complementary function of a sequence of Musielak-Orlicz $f$. For a given sequence of Musielak-Orlicz function $f$, the Musielak-Orlicz sequence space $t_f$ is defined as follows*

$$t_f = \left\{ x \in w^3 : I_f(|x_{mnk}|)^{1/m+n+k} \to 0 \ as\ m, n, k \to \infty \right\},$$

*where $I_f$ is a convex modular defined by*

$$I_f(x) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} f_{mnk}(|x_{mnk}|)^{1/m+n+k}, x = (x_{mnk}) \in t_f.$$

## 2. Definition and preliminaries

Let $n \in \mathbb{N}$ and $X$ be a real vector space of dimension $w$, where $n \leq w$. Let real valued function $d_p(x_1, \ldots, x_n) = \|(d_1(x_1, 0), \ldots, d_n(x_n, 0))\|_p$ on $X$ satisfying the following five conditions:

(i) $\|(d_1(x_1, 0), \ldots, d_n(x_n, 0))\|_p = 0$ if and and only if $d_1(x_1, 0), \ldots, d_n(x_n, 0)$ are linearly dependent;

(ii) $\|(d_1(x_1, 0), \ldots, d_n(x_n, 0))\|_p$ is invariant under permutation;

(iii) $\|(\alpha d_1(x_1, 0), \ldots, d_n(x_n, 0))\|_p = |\alpha| \, \|(d_1(x_1, 0), \ldots, d_n(x_n, 0))\|_p, \alpha \in \mathbb{R}$;

(iv) $d_p((x_1, y_1), (x_2, y_2) \cdots (x_n, y_n)) = (d_X(x_1, x_2, \cdots x_n)^p + d_Y(y_1, y_2, \cdots y_n)^p)^{1/p}$ for $1 \leq p < \infty$; (or)

(v) $d((x_1, y_1), (x_2, y_2), \cdots (x_n, y_n)) := \sup\{d_X(x_1, x_2, \cdots x_n), d_Y(y_1, y_2, \cdots y_n)\}$

for $x_1, x_2, \cdots x_n \in X, y_1, y_2, \cdots y_n \in Y$ (is called the $p$-product metric of the Cartesian product of $n$-metric spaces) is the $p$-norm of the $n$-vector of the norms of the $n$-sub spaces.

A trivial example of $p$-product metric of $n$-metric space the $p$-norm space is $X = \mathbb{R}$ equipped with the following Euclidean metric in the product space the $p$-norm:

$$\begin{aligned} \|(d_1(x_1, 0), \ldots, d_n(x_n, 0))\|_E &= \sup\left(|\det(d_{mn}(x_{mn}, 0))|\right) \\ &= \sup\left( \begin{vmatrix} d_{11}(x_{11}, 0) & d_{12}(x_{12}, 0) & \cdots & d_{1n}(x_{1n}, 0) \\ d_{21}(x_{21}, 0) & d_{22}(x_{22}, 0) & \cdots & d_{2n}(x_{1n}, 0) \\ \vdots & \vdots & \ddots & \\ d_{n1}(x_{n1}, 0) & d_{n2}(x_{n2}, 0) & \cdots & d_{nn}(x_{nn}, 0) \end{vmatrix} \right), \end{aligned}$$

where $x_i = (x_{i1}, \cdots x_{in}) \in \mathbb{R}^n$ for each $i = 1, 2, \ldots, n$.

If every Cauchy sequence in $X$ converges to some $L \in X$, then $X$ is said to be complete with respect to the $p$-metric. Any complete $p$-metric space is said to be $p$-Banach metric space.

**Definition 2.1.** A sequence space $E_F$ of fuzzy numbers is said to be (i) solid (or normal) if $(Y_{mnk}) \in E_F$ whenever $(X_{mnk}) \in E_F$ and $\bar{d}(Y_{mnk}, \bar{0}) \leq \bar{d}(X_{mnk}, \bar{0})$ for all $m, n, k \in \mathbb{N}$; (ii) symmetric if $(X_{mnk}) \in E_F$ implies $(X_{\pi(mnk)}) \in E_F$ where $\pi$ is a permutation of $\mathbb{N} \times \mathbb{N} \times \mathbb{N}$.

Let $K = \{k_1 < k_2 < \cdots\} \subseteq \mathbb{N}$ and $E$ be a sequence space. A $K$-step space of $E$ is a sequence space

$$\lambda^E_{mnk} = \left\{ \left( X_{m_p n_p k_p} \right) \in w^3 : (m_p n_p k_p) \in E \right\}.$$

A canonical preimage of a sequence $\left\{ \left( x_{m_p n_p k_p} \right) \right\} \in \lambda^E_K$ is a sequence $\{y_{mnk}\} \in w^3$ defined as

$$y_{mnk} = \begin{cases} x_{mnk}, & \text{if } m, n, k \in E, \\ 0, & \text{otherwise.} \end{cases}$$

A canonical preimage of a step space $\lambda^E_K$ is a set of canonical preimages of all elements in $\lambda^E_K$, i.e., $y$ is canonical preimage of $\lambda^E_K$ if and only if $y$ is canonical preimage of some $x \in \lambda^E_K$.

**Definition 2.2.** A sequence space $E_F$ is said to be monotone if $E_F$ contains the canonical pre-images of all its step spaces.

By the convergence of a triple sequence we mean the convergence on the Pringsheim sense that is, a triple sequence $x = (x_{mnk})$ has Pringsheim limit $L$ (denoted by $P - \lim x = L$) provided that for given $\epsilon > 0$ there exists $q \in \mathbb{N}$ such that $|x_{mnk} - L| < \epsilon$ whenever $m, n, k > q$. We shall write more briefly as $P$-convergent.

**Definition 2.3.** The triple sequence $\theta_{i,\ell,j} = \{(m_i, n_\ell, k_j)\}$ is called triple lacunary if there exist three increasing sequences of integers such that

$$\begin{aligned} m_0 &= 0, h_i = m_i - m_{r-1} \to \infty \text{ as } i \to \infty \text{ and} \\ n_0 &= 0, \overline{h_\ell} = n_\ell - n_{\ell-1} \to \infty \text{ as } \ell \to \infty, \\ k_0 &= 0, \overline{h_j} = k_j - k_{j-1} \to \infty \text{ as } j \to \infty. \end{aligned}$$

Let $m_{i,\ell,j} = m_i n_\ell k_j, h_{i,\ell,j} = h_i \overline{h_\ell h_j}$, and $\theta_{i,\ell,j}$ be determined by

$$\begin{aligned} I_{i,\ell,j} &= \{(m, n, k) : m_{i-1} < m < m_i \text{ and } n_{\ell-1} < n \leq n_\ell \text{ and } k_{j-1} < k \leq k_j\}, \\ q_k &= \frac{m_k}{m_{k-1}}, \overline{q_\ell} = \frac{n_\ell}{n_{\ell-1}}, \overline{q_j} = \frac{k_j}{k_{j-1}}. \end{aligned}$$

## 3. Notion of $\lambda_{mnk}-$ triple $\chi^3$ and triple $\Lambda^3$ sequences

Let $\lambda = (\lambda_{mnk})$ be a strictly increasing triple sequence of positive reals tending to infinity, that is $\lambda_{m+1,n,k} \leq \lambda_{mnk} + 1, \lambda_{m,n+1,k} \leq \lambda_{mnk} + 1, \lambda_{m,n,k+1} \leq \lambda_{mnk} + 1, \lambda_{mnk} - \lambda_{m+1,n,k} - \lambda_{m,n+1,k} - \lambda_{m,n,k+1} \leq \lambda_{m,n,k} + 1 - \lambda_{m+1,n+1,k+1}, \lambda_{111} = 1$, we say a sequence $x = (x_{mnk}) \in w^3$ is a triple $\lambda-$ convergent to the number $L \in \mathbb{N}$, called as the triple $\lambda-$ limit of $x$, if $\Lambda_{abc}(x) \to L$ as $a, b, c \to \infty$, where $\Lambda_{abc} = \frac{1}{\lambda_{abc}} \sum_{m,n,k=0}^{abc} |\lambda_{mnk} - \lambda_{m+1,n,k} - \lambda_{m,n+1,k} - \lambda_{m,n,k+1}| \, x_{mnk}, a, b, c \in \mathbb{N}$.
The generalized de la Vallee-Poussin means is defined by

$$t_{mnk}(x) = \lambda^{-1}_{mnk} \sum_{m,n,k \in I_{mnk}} x_{mnk},$$

where $I_{mnk} = [mnk - \lambda_{mnk} + 1, mnk]$. A sequence $x = (x_{mnk})$ is said to $(V, \lambda)-$ summable to a number $L$ if $t_{mnk}(x) \to L$, as $m, n, k \to \infty$.
The notion of $\lambda-$ triple gai and triple analytic sequences as follows: Let $\lambda = (\lambda_{mnk})^\infty_{m,n,k=0}$ be a strictly increasing sequences of positive real numbers tending to infinity, that is

$$0 < \lambda_{000} < \lambda_{111} < \cdots \, and \, \lambda_{mnk} \to \infty \, as \, m, n, k \to \infty$$

and said that a sequence $x = (x_{mnk}) \in w^3$ is $\lambda-$ convergent to 0, called a the $\lambda-$ limit of $x$, if $B^{\mu}_{\eta}(x) \to 0\, as\, m, n, k \to \infty$, where

$$B^{\mu}_{\eta}(x) = \frac{1}{\varphi_{rst}} \sum_{m \in I_{rst}} \sum_{n \in I_{rst}} \sum_{k \in I_{rst}} \lambda_{mnk} - \lambda_{m,n+1,k} - \lambda_{m,n,k+1} + \lambda_{m,n+1,k+1} - \lambda_{m+1,n,k} + \lambda_{m+1,n+1,k} + \lambda_{m+1,n,k+1} - \lambda_{m+1,n+1,k+1} \left((m+n+k)!\, |\Delta^m x_{mn}|\right)^{1/m+n+k},$$

where,

$$\begin{aligned}((m+n+k)!\, |\Delta^m x_{mnk}|)^{1/m+n+k} &= (m+n+k)!^{1/m+n+k} \Delta^{m-1} \lambda_{mnk} x_{mnk} - \Delta^{m-1} \lambda_{m,n+1,k} x_{m,n+1,k} \\ &- \Delta^{m-1} \lambda_{m,n,k+1} x_{m,n,k+1} + \Delta^{m-1} \lambda_{m,n+1,k+1} x_{m,n+1,k+1} \\ &- \Delta^{m-1} \lambda_{m+1,n,k} x_{m+1,n,k} + \Delta^{m-1} \lambda_{m+1,n+1,k} x_{m+1,n+1,k} \\ &+ \Delta^{m-1} \lambda_{m+1,n,k+1} x_{m+1,n,k+1} - \Delta^{m-1} \lambda_{m+1,n+1,k+1} x^{1/m+n+k}_{m+1,n+1,k+1}.\end{aligned}$$

In particular, we say that $x$ is a $\lambda_{mnk}-$ triple gai sequence if $B^{\mu}_{\eta}(x) \to 0$ as $m, n, k \to \infty$. Further we say that $x$ is $\lambda_{mnk}-$ triple analytic sequence if $sup_{mn} |B^{\mu}_{\eta}(x)| < \infty$. We have

$$\lim_{m,n,k \to \infty} \left|B^{\mu}_{\eta}(x) - a\right| = \lim_{m,n,k \to \infty} \frac{1}{\varphi_{rst}} \sum_{m \in I_{rst}} \sum_{n \in I_{rst}} \sum_{k \in I_{rst}} \lambda_{mnk} - \lambda_{m,n+1,k} - \lambda_{m,n,k+1} + \lambda_{m,n+1,k+1} - \lambda_{m+1,n,k} + \lambda_{m+1,n+1,k} + \lambda_{m+1,n,k+1} - \lambda_{m+1,n+1,k+1} \left((m+n)!\, |\Delta^m x_{mn}|\right)^{1/m+n} = 0.$$

So we can say that $lim_{m,n,k \to \infty} |B^{\mu}_{\eta}(x)| = a$. Hence $x$ is $\lambda_{mnk}-$ convergent to $a$. Let $I^3-$ be an admissible ideal of $2^{\mathbb{N} \times \mathbb{N} \times \mathbb{N}}, \theta_{rst}$ be a double lacunary sequence, $f = (f_{mnk})$ be a sequence of Musielak-Orlicz functions, and $\left(X, \|(d(x_1, 0), d(x_2, 0), \cdots, d(x_{n-1}, 0))\|_p\right)$ be a $p$-metric space, $q = (q_{mnk})$ be triple analytic sequence of positive real numbers. By $w^3(p - X)$ we denote the space of all sequences defined over $\left(X, \|(d(x_1, 0), d(x_2, 0), \ldots, d(x_{n-1}, 0))\|_p\right)$. The following inequality will be used throughout the paper. If $0 \le q_{mnk} \le \sup q_{mnk} = H, K = \max\left(1, 2^{H-1}\right)$, then

$$|a_{mnk} + b_{mnk}|^{q_{mnk}} \le K \left\{|a_{mnk}|^{q_{mnk}} + |b_{mnk}|^{q_{mnk}}\right\}$$

for all $m, n, k$ and $a_{mnk}, b_{mnk} \in \mathbb{C}$. Also $|a|^{q_{mnk}} \le \max\left(1, |a|^H\right)$ for all $a \in \mathbb{C}$.

## 4. Acceleration convergence of multiple sequences of fuzzy numbers

In this section some more definitions related to triple sequences of fuzzy numbers have been defined and some interesting theorems regarding acceleration convergence of triple sequences of fuzzy numbers have been discussed.

**Definition 4.1.** Let $x = (x_{mnk})$ and $y = (y_{mnk})$ be two triple sequences of real numbers. Then the sequence $x$ is said to converge $P-\chi^3$ faster than the sequence $y$, written as $x <^P y$, if $P-\lim_{mnk} \left|\frac{((m+n+k)! x_{mnk})^{1/m+n+k}}{((m+n+k) y_{mnk})^{1/m+n+k}}\right| = 0$.

**Definition 4.2.** The sequence $x = (x_{mnk})$ is said to converge at the same rate in Pringsheim's sense as the sequence $y = (y_{mnk})$, written as $x \approx^{P-\chi^3} y$, if

$$0 < P - \liminf \left|\frac{((m+n+k)! x_{mnk})^{1/m+n+k}}{((m+n+k)! y_{mnk})^{1/m+n+k}}\right| \le P - \liminf \left|\frac{((m+n+k)! x_{mnk})^{1/m+n+k}}{((m+n+k)! y_{mnk})^{1/m+n+k}}\right| < \infty.$$

**Definition 4.3.** The four dimensional matrix $A = (a_{k,\ell,m,n})$ is said to $P$-accelerate the convergence of the sequence $x = (x_{mnk})$ if $Ax <^P x$. We define the $P$-acceleration field of $A$ as the set

$$\left\{x = (x_{mnk}) \in w^3 : Ax <^P x\right\}.$$

Now we define the acceleration convergence of triple sequences of fuzzy numbers as follows.

**Definition 4.4.** Let $X = (X_{mnk})$ and $Y = (Y_{mnk})$ be two double sequences of fuzzy numbers with $X_{mnk} \to \overline{0}$ and $Y_{mnk} \to \overline{0}$. Then the sequence $X$ converges to $\overline{0}$, $P-\chi^3$ faster than the sequence $Y$ converges to $\overline{0}$, written as $X <^{P-\chi^3} Y$, if $P-\lim_{mnk} \left|\frac{\overline{d}\left(((m+n+k)!X_{mnk})^{1/m+n+k},\overline{0}\right)}{\overline{d}\left(((m+n+k)!Y_{mnk})^{1/m+n+k},\overline{0}\right)}\right| = 0$, provided $\overline{d}\left(((m+n+k)!Y_{mnk})^{1/m+n+k},\overline{0}\right) \neq 0$ for all $m,n,k \in \mathbb{N}$.

**Definition 4.5.** The triple sequence $X = (X_{mnk})$ converges to $\overline{0}$ at the same rate in Pringsheim's sense as the sequence $Y = (Y_{mnk})$ converges to $\overline{0}$, written as $X \approx^{P-\chi^3} Y$, if

$$0 < P - \liminf \left|\frac{\overline{d}\left(((m+n+k)!X_{mnk})^{1/m+n+k},\overline{0}\right)}{\overline{d}\left(((m+n+k)!Y_{mnk})^{1/m+n+k},\overline{0}\right)}\right| \leq P - \liminf \left|\frac{\overline{d}\left(((m+n+k)!x_{mnk})^{1/m+n+k},\overline{0}\right)}{\overline{d}\left(((m+n+k)!y_{mnk})^{1/m+n+k},\overline{0}\right)}\right| < \infty.$$

**Definition 4.6.** The four dimensional matrix $A = (a_{k,\ell,m,n})$ is said $P$-accelerate the convergence of the sequence $X = (X_{mnk})$ if $AX <^P X$. We define the $P$-acceleration field of $A$ as the set

$$\left\{X = (X_{mnk}) \in w^3 : AX <^P X\right\}.$$

**Definition 4.7.** A matrix transformation associated with the four-dimensional matrix $A$ is said to be an $\chi^{3F} - \chi^{3F}$ if $AX$ is in the set $\chi^{3F}$, whenever $X$ is in $\chi^{3F}$ and is analytic.

In the present paper we define the following sequence spaces:

$$\left[\chi^{3q}_{f\mu}, \|(d(X_1,0), d(X_2,0), \ldots, d(X_{n-1},0))\|_p^{\varphi}\right]^{I^3}_{\theta_{rst}}$$
$$= \left\{r,s,t \in I_{rst} : \left[f_{mnk}\left(\|\mu_{mnk}(X), (d(X_1,0), d(X_2,0), \ldots, d(X_{n-1},0))\|_p\right)\right]^{q_{mn}} \geq \epsilon\right\} \in I^3,$$
$$\left[\Lambda^{3q}_{f\mu}, \|(d(X_1,0), d(X_2,0), \ldots, d(X_{n-1},0))\|_p^{\varphi}\right]^{I^3}_{\theta_{rst}}$$
$$= \left\{r,s,t \in I_{rst} : \left[f_{mnk}\left(\|\mu_{mnk}(X), (d(X_1,0), d(X_2,0), \ldots, d(X_{n-1},0))\|_p\right)\right]^{q_{mnk}} \geq K\right\} \in I^3.$$

If we take $f_{mnk}(X) = X$, we get

$$\left[\chi^{3q}_{f\mu}, \|(d(X_1,0), d(X_2,0), \ldots, d(X_{n-1},0))\|_p^{\varphi}\right]^{I^3}_{\theta_{rst}}$$
$$= \left\{r,s,t \in I_{rst} : \left[\left(\|\mu_{mnk}(X), (d(X_1,0), d(X_2,0), \ldots, d(X_{n-1},0))\|_p\right)\right]^{q_{mnk}} \geq \epsilon\right\} \in I^3,$$
$$\left[\Lambda^{3q}_{f\mu}, \|(d(X_1,0), d(X_2,0), \ldots, d(X_{n-1},0))\|_p^{\varphi}\right]^{I^3}_{\theta_{rst}}$$
$$= \left\{r,s,t \in I_{rst} : \left[\left(\|\mu_{mnk}(X), (d(X_1,0), d(X_2,0), \ldots, d(X_{n-1},0))\|_p\right)\right]^{q_{mnk}} \geq K\right\} \in I^3.$$

If we take $q = (q_{mnk}) = 1$, we get

$$\left[\chi^{3}_{f\mu}, \|(d(X_1,0), d(X_2,0), \ldots, d(X_{n-1},0))\|_p^{\varphi}\right]^{I^3}_{\theta_{rst}}$$
$$= \left\{r,s,t \in I_{rst} : \left[f_{mnk}\left(\|\mu_{mnk}(X), (d(X_1,0), d(X_2,0), \ldots, d(X_{n-1},0))\|_p\right)\right] \geq \epsilon\right\} \in I^3,$$

$$\left[\Lambda^3_{f\mu}, \|(d(X_1,0), d(X_2,0), \ldots, d(X_{n-1},0))\|^{\varphi}_p\right]^{I^3}_{\theta_{rst}}$$
$$= \left\{r,s,t \in I_{rst} : \left[f_{mnk}\left(\|\mu_{mnk}(X), (d(X_1,0), d(X_2,0), \ldots, d(X_{n-1},0))\|_p\right)\right] \geq K\right\} \in I^3.$$

In the present paper we plan to study some topological properties and inclusion relation between the above defined sequence spaces,

$$\left[\chi^{3q}_{f\mu}, \|(d(X_1), d(X_2), \cdots, d(X_{n-1}))\|^{\varphi}_p\right]^{I^3}_{\theta_{rst}} \quad \text{and} \quad \left[\Lambda^{3q}_{f\mu}, \|(d(X_1), d(X_2), \cdots, d(X_{n-1}))\|^{\varphi}_p\right]^{I^3}_{\theta_{rst}},$$

which we shall discuss in this paper.

## 5. Main results

**Theorem 5.1.** *Let $f = (f_{mnk})$ be an Musielak-Orlicz function and $q = (q_{mnk})$ be a triple analytic of positive real numbers, then $\left[\chi^{3q}_{f\mu}, \|(d(X_1,0), d(X_2,0), \ldots, d(X_{n-1},0))\|^{\varphi}_p\right]^{I^3}_{\theta_{rst}}$ and $\left[\Lambda^{3q}_{f\mu}, \|\left(d(X_1,0), d(X_2,0), \ldots, d(X_{n-1},0)\right)\|^{\varphi}_p\right]^{I^3}_{\theta_{rst}}$ are linear spaces.*

*Proof.* It is routine verification. Therefore the proof is omitted. □

**Theorem 5.2.** *Let $f = (f_{mnk})$ be Musielak-Orlicz function, $q = (q_{mnk})$ be a triple analytic of positive real numbers and $\left[\chi^{3q}_{f\mu}, \|(d(X_1,0), d(X_2,0), \ldots, d(X_{n-1},0))\|^{\varphi}_p\right]^{I^3}_{\theta_{rst}}$ is a paranormed space with respect to the paranorm defined by*

$$g(x) = \inf\left\{\left[f_{mnk}\left(\|\mu_{mnk}(X), (d(X_1,0), d(X_2,0), \ldots, d(X_{n-1},0))\|_p\right)\right]^{q_{mnk}} \leq 1\right\}.$$

*Proof.* Clearly $g(X) \geq 0$ for $X = (X_{mnk}) \in \left[\chi^{3q}_{f\mu}, \|(d(X_1,0), d(X_2,0), \ldots, d(X_{n-1},0))\|^{\varphi}_p\right]^{I^3}_{\theta_{rst}}$. Since $f_{mnk}(0) = 0$, we get $g(0) = 0$.

Conversely, suppose that $g(X) = 0$, then

$$\inf\left\{\left[f_{mnk}\left(\|\mu_{mnk}(X), (d(X_1,0), d(X_2,0), \ldots, d(X_{n-1},0))\|_p\right)\right]^{q_{mnk}} \leq 1\right\}.$$

Suppose that $\mu_{mnk}(X) \neq 0$ for each $m,n,k \in \mathbb{N}$. Then

$$\|\mu_{mnk}(X), (d(X_1,0), d(X_2,0), \ldots, d(X_{n-1},0))\|^{\varphi}_p \to \infty.$$

It follows that

$$\left(\left[f_{mnk}\left(\|\mu_{mnk}(X), (d(X_1,0), d(X_2,0), \ldots, d(X_{n-1},0))\|_p\right)\right]^{q_{mnk}}\right)^{1/H} \to \infty,$$

which is a contradiction. Therefore $\mu_{mnk}(X) = 0$. Let

$$\left(\left[f_{mnk}\left(\|\mu_{mnk}(X), (d(X_1,0), d(X_2,0), \ldots, d(X_{n-1},0))\|_p\right)\right]^{q_{mnk}}\right)^{1/H} \leq 1$$

and

$$\left(\left[f_{mnk}\left(\|\mu_{mnk}(Y), (d(X_1,0), d(X_2,0), \ldots, d(X_{n-1},0))\|_p\right)\right]^{q_{mnk}}\right)^{1/H} \leq 1.$$

Then by using Minkowski's inequality, we have

$$\left(\left[f_{mnk}\left(\|\mu_{mn}(X+Y), (d(X_1,0), d(X_2,0), \ldots, d(X_{n-1},0))\|_p\right)\right]^{q_{mnk}}\right)^{1/H}$$

$$\leq \left(\left[f_{mnk}\left(\|\mu_{mnk}(X),(d(X_1,0),d(X_2,0),\ldots,d(X_{n-1},0))\|_p\right)\right]^{q_{mnk}}\right)^{1/H} + \left(\left[f_{mnk}\left(\|\mu_{mnk}(Y),(d(X_1,0),d(X_2,0),\ldots,d(X_{n-1},0))\|_p\right)\right]^{q_{mnk}}\right)^{1/H}.$$

So we have

$$g(X+Y) = \inf\left\{\left[f_{mnk}\left(\|\mu_{mnk}(X+Y),(d(X_1,0),d(X_2,0),\ldots,d(X_{n-1},0))\|_p\right)\right]^{q_{mnk}} \leq 1\right\}$$
$$\leq \inf\left\{\left[f_{mnk}\left(\|\mu_{mnk}(X),(d(X_1,0),d(X_2,0),\ldots,d(X_{n-1},0))\|_p\right)\right]^{q_{mnk}} \leq 1\right\} + \inf\left\{\left[f_{mnk}\left(\|\mu_{mnk}(Y),(d(X_1,0),d(X_2,0),\ldots,d(X_{n-1},0))\|_p\right)\right]^{q_{mnk}} \leq 1\right\}.$$

Therefore,

$$g(X+Y) \leq g(X) + g(Y).$$

Finally we prove that the scalar multiplication is continuous. Let $\lambda$ be any complex number. By definition,

$$g(\lambda X) = \inf\left\{\left[f_{mnk}\left(\|\mu_{mnk}(\lambda X),(d(X_1,0),d(X_2,0),\ldots,d(X_{n-1},0))\|_p\right)\right]^{q_{mnk}} \leq 1\right\}.$$

Then

$$g(\lambda X) = \inf\left\{((|\lambda| t)^{q_{mnk}/H} : \left[f_{mnk}\left(\|\mu_{mn}(\lambda X),(d(X_1,0),d(X_2,0),\ldots,d(X_{n-1},0))\|_p\right)\right]^{q_{mnk}} \leq 1\right\},$$

where $t = \frac{1}{|\lambda|}$. Since $|\lambda|^{q_{mnk}} \leq \max\left(1, |\lambda|^{\text{supp}_{mnk}}\right)$, we have

$$g(\lambda X) \leq \max\left(1, |\lambda|^{\text{supp}_{mnk}}\right) (???) \inf\left\{t^{q_{mnk}/H} : \left[f_{mnk}\left(\|\mu_{mnk}(\lambda X),(d(X_1,0),d(X_2,0),\ldots,d(X_{n-1},0))\|_p\right)\right]^{q_{mnk}} \leq 1\right\}.$$

This completes the proof. □

**Proposition 5.3.** *If* $0 < q_{mnk} < p_{mnk} < \infty$ *for each* $m$, $n$, *and* $k$, *then*

$$\left[\Lambda^{3q}_{f\mu}, \|\mu_{mnk}(X),(d(X_1,0),d(X_2,0),\ldots,d(X_{n-1},0))\|_p^{\varphi}\right]^{I^3}_{\theta_{rst}} \subseteq \left[\Lambda^{3p}_{f\mu}, \|\mu_{mnk}(X),(d(X_1,0),d(X_2,0),\ldots,d(X_{n-1},0))\|_p^{\varphi}\right]^{I^3}_{\theta_{rst}}.$$

*Proof.* The proof is standard, so we omit it. □

**Proposition 5.4.**

(i) *If* $0 < \inf q_{mnk} \leq q_{mnk} < 1$, *then*

$$\left[\Lambda^{3q}_{f\mu}, \|\mu_{mnk}(X),(d(X_1,0),d(X_2,0),\ldots,d(X_{n-1},0))\|_p^{\varphi}\right]^{I^3}_{\theta_{rst}} \subset \left[\Lambda^{3}_{f\mu}, \|\mu_{mnk}(X),(d(X_1,0),d(X_2,0),\ldots,d(X_{n-1},0))\|_p^{\varphi}\right]^{I^3}_{\theta_{rst}}.$$

(ii) *If* $1 \leq q_{mnk} \leq \sup q_{mnk} < \infty$, *then*

$$\left[\Lambda^{3}_{f\mu}, \|\mu_{mnk}(X),(d(X_1,0),d(X_2,0),\ldots,d(X_{n-1},0))\|_p^{\varphi}\right]^{I^3}_{\theta_{rst}} \subset \left[\Lambda^{3q}_{f\mu}, \|\mu_{mnk}(X),(d(X_1,0),d(X_2,0),\ldots,d(X_{n-1},0))\|_p^{\varphi}\right]^{I^3}_{\theta_{rst}}.$$

*Proof.* The proof is standard, so we omit it. □

**Proposition 5.5.** *Let $f' = \left(f'_{mnk}\right)$ and $f'' = \left(f''_{mnk}\right)$ are Musielak-Orlicz functions, we have*

$$\left[\Lambda^{3q}_{f'\mu}, \|\mu_{mnk}(X), (d(X_1,0), d(X_2,0), \ldots, d(X_{n-1},0))\|^{\varphi}_{p}\right]^{I^3}_{\theta_{rst}}$$
$$\bigcap \left[\Lambda^{3q}_{f''\mu}, \|\mu_{mn}(X), (d(X_1,0), d(X_2,0), \ldots, d(X_{n-1},0))\|^{\varphi}_{p}\right]^{I^3}_{\theta_{rs}}$$
$$\subseteq \left[\Lambda^{3q}_{f'+f''\mu}, \|\mu_{mnk}(X), (d(X_1,0), d(X_2,0), \ldots, d(X_{n-1},0))\|^{\varphi}_{p}\right]^{I^3}_{\theta_{rst}}.$$

*Proof.* The proof is easy so we omit it. □

**Proposition 5.6.** *For any Musielak-Orlicz function $f = (f_{mnk})$, let $q = (q_{mnk})$ be a triple analytic of positive real numbers. Then*

$$\left[\chi^{3q}_{f\mu}, \|\mu_{mnk}(X), (d(X_1,0), d(X_2,0), \ldots, d(X_{n-1},0))\|^{\varphi}_{p}\right]^{I^3}_{\theta_{rst}}$$
$$\subset \left[\Lambda^{3q}_{f\mu}, \|\mu_{mnk}(X), (d(X_1,0), d(X_2,0), \ldots, d(X_{n-1},0))\|^{\varphi}_{p}\right]^{I^3}_{\theta_{rs}}.$$

*Proof.* The proof is easy so we omit it. □

**Proposition 5.7.** *Let $\left[\Lambda^{3q}_{f\mu}, \|\mu_{mnk}(X), (d(X_1,0), d(X_2,0), \ldots, d(X_{n-1},0))\|^{\varphi}_{p}\right]^{I^3}_{\theta_{rst}}$ is solid.*

*Proof.* Let $X = (X_{mnk}) \in \left[\Lambda^{3q}_{f\mu}, \|\mu_{mnk}(X), (d(X_1,0), d(X_2,0), \ldots, d(X_{n-1},0))\|^{\varphi}_{p}\right]^{I^3}_{\theta_{rst}}$, i.e.,

$$\sup_{mn} \left[\Lambda^{3q}_{f\mu}, \|\mu_{mnk}(X), (d(X_1,0), d(X_2,0), \ldots, d(X_{n-1},0))\|^{\varphi}_{p}\right]^{I^3}_{\theta_{rs}} < \infty.$$

Let $(\alpha_{mnk})$ be triple sequence of scalars such that $|\alpha_{mnk}| \leq 1$ for all $m, n, k \in \mathbb{N} \times \mathbb{N} \times \mathbb{N}$. Then we get

$$\sup_{mnk} \left[\Lambda^{3q}_{f\mu}, \|\mu_{mnk}(\alpha X), (d(X_1,0), d(X_2,0), \ldots, d(X_{n-1},0))\|^{\varphi}_{p}\right]^{I^3}_{\theta_{rst}}$$
$$\leq \sup_{mn} \left[\Lambda^{3q}_{f\mu}, \|\mu_{mnk}(X), (d(X_1,0), d(X_2,0), \ldots, d(X_{n-1},0))\|^{\varphi}_{p}\right]^{I^3}_{\theta_{rst}}.$$

This completes the proof. □

**Proposition 5.8.** *The sequence space $\left[\Lambda^{3q}_{f\mu}, \|\mu_{mnk}(X), (d(X_1,0), d(X_2,0), \ldots, d(X_{n-1},0))\|^{\varphi}_{p}\right]^{I^3}_{\theta_{rst}}$ is monotone.*

*Proof.* The proof follows from Proposition 5.9. □

**Proposition 5.9.** *If $f = (f_{mnk})$ be any Orlicz function, then*

$$\left[\Lambda^{3q}_{f\mu}, \|\mu_{mnk}(X), (d(X_1,0), d(X_2,0), \ldots\ldots, d(X_{n-1},0))\|^{\varphi^*}_{p}\right]^{I^3}_{\theta_{rst}}$$
$$\subset \left[\Lambda^{3q}_{f\mu}, \|\mu_{mnk}(X), (d(X_1,0), d(X_2,0), \ldots, d(X_{n-1},0))\|^{\varphi^{**}}_{p}\right]^{I^3}_{\theta_{rst}}$$

*if and only if* $\sup_{r,s,t\geq 1} \frac{\varphi^*_{rst}}{\varphi^{**}_{rst}} < \infty$.

*Proof.* Let $x \in \left[\Lambda^{3q}_{f\mu}, \|\mu_{mnk}(X), (d(X_1,0), d(X_2,0), \ldots, d(X_{n-1},0))\|^{\varphi^*}_p\right]^{I^3}_{\theta_{rst}}$ and $N = \sup_{r,s,t\geq 1} \frac{\varphi^*_{rst}}{\varphi^{**}_{rst}} < \infty$. Then we get

$$\left[\Lambda^{3q}_{f\mu}, \|\mu_{mnk}(X), (d(X_1,0), d(X_2,0), \ldots, d(X_{n-1},0))\|^{\varphi^{**}_{rst}}_p\right]^{I^3}_{\theta_{rst}}$$
$$= N\left[\Lambda^{3q}_{f\mu}, \|\mu_{mnk}(X), (d(X_1,0), d(X_2,0), \ldots, d(x_{n-1},0))\|^{\varphi^*_{rst}}_p\right]^{I^3}_{\theta_{rst}} = 0.$$

Thus $x \in \left[\Lambda^{3q}_{f\mu}, \|\mu_{mn}(X), (d(X_1,0), d(X_2,0), \ldots, d(X_{n-1},0))\|^{\varphi^{**}}_p\right]^{I^3}_{\theta_{rst}}$. Conversely, suppose that

$$\left[\Lambda^{3q}_{f\mu}, \|\mu_{mnk}(X), (d(X_1,0), d(X_2,0), \ldots, d(X_{n-1},0))\|^{\varphi^*}_p\right]^{I^3}_{N_\theta}$$
$$\subset \left[\Lambda^{3qu}_{f\mu}, \|\mu_{mnk}(X), (d(X_1,0), d(X_2,0), \ldots, d(X_{n-1},0))\|^{\varphi^{**}}_p\right]^{I^3}_{\theta_{rst}}$$

and $X \in \left[\Lambda^{3q}_{f\mu}, \|\mu_{mnk}(X), (d(X_1,0), d(X_2,0), \ldots, d(X_{n-1},0))\|^{\varphi^*}_p\right]^{I^3}_{\theta_{rst}}$. Then

$$\left[\Lambda^{3q}_{f\mu}, \|\mu_{mnk}(X), (d(X_1,0), d(X_2,0), \ldots, d(X_{n-1},0))\|^{\varphi^*}_p\right]^{I^3}_{\theta_{rst}} < \epsilon$$

for every $\epsilon > 0$. Suppose that $\sup_{r\geq 1, s\geq 1, t\geq 1} \frac{\varphi^*_{rst}}{\varphi^{**}_{rst}} = \infty$, then there exists a member $(r_{abc}s_{abc}t_{abc})$ such that $\lim_{r_{abc}s_{abc}t_{abc}\to\infty} \frac{\varphi^*_{r_{abc}s_{abc}t_{abc}}}{\varphi^{**}_{r_{abc}s_{abc}t_{abc}}} = \infty$. Hence, we have

$$\left[\Lambda^{3q}_{f\mu}, \|\mu_{mnk}(X), (d(X_1,0), d(X_2,0), \ldots, d(X_{n-1},0))\|^{\varphi^*_{rst}}_p\right]^{I^3}_{\theta_{rst}} = \infty.$$

Therefore $X \notin \left[\Lambda^{3q}_{f\mu}, \|\mu_{mnk}(X), (d(X_1,0), d(X_2,0), \ldots, d(X_{n-1},0))\|^{\varphi^{**}}_p\right]^{I^3}_{\theta_{rst}}$, which is a contradiction. This completes the proof. □

**Proposition 5.10.** *If $f = (f_{mnk})$ be any Musielak Orlicz function, then*

$$\left[\Lambda^{3q}_{f\mu}, \|\mu_{mnk}(X), (d(X_1,0), d(X_2,0), \ldots, d(X_{n-1},0))\|^{\varphi^*}_p\right]^{I^3}_{\theta_{rst}}$$
$$= \left[\Lambda^{3q}_{f\mu}, \|\mu_{mnk}(X), (d(X_1,0), d(X_2,0), \ldots, d(X_{n-1},0))\|^{\varphi^{**}}_p\right]^{I^3}_{\theta_{rst}}$$

*if and only if* $\sup_{r,s,t\geq 1} \frac{\varphi^*_{rst}}{\varphi^{**}_{rst}} < \infty, \sup_{r,s,t\geq 1} \frac{\varphi^{**}_{rst}}{\varphi^*_{rst}} > \infty$.

*Proof.* It is easy to prove, so we omit it. □

**Proposition 5.11.** $\left[\chi^{3q}_{f\mu}, \|\mu_{mnk}(X), (d(X_1,0), d(X_2,0), \ldots, d(X_{n-1},0))\|^{\varphi}_p\right]^{I^3}_{\theta_{rst}}$ *is not solid.*

*Proof.* The result follows from the following example. □

**Example 5.12.** Consider

$$X = (X_{mnk}) = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & & & \\ 1 & 1 & \cdots & 1 \end{pmatrix} \in \left[\chi^{3q}_{f\mu}, \|\mu_{mnk}(X), (d(X_1,0), d(X_2,0), \ldots, d(X_{n-1},0))\|^{\varphi}_p\right]^{I^3}_{\theta_{rst}}.$$

Let

$$\alpha_{mnk} = \begin{pmatrix} -1^{m+n+k} & -1^{m+n+k} & \cdots & -1^{m+n+k} \\ -1^{m+n+k} & -1^{m+n+k} & \cdots & -1^{m+n+k} \\ \vdots & \vdots & \ddots & \\ -1^{m+n+k} & -1^{m+n+k} & \cdots & -1^{m+n+k} \end{pmatrix}$$

for all $m, n, k \in \mathbb{N}$. Then $\alpha_{mnk} X_{mnk} \notin \left[\chi^{3q}_{f\mu}, \|\mu_{mnk}(X), (d(X_1,0), d(X_2,0), \ldots, d(X_{n-1},0))\|^{\varphi}_p\right]^{I^3}_{\theta_{rst}}$. Hence $\left[\chi^{3q}_{f\mu}, \|\mu_{mnk}(X), (d(X_1,0), d(X_2,0), \ldots, d(X_{n-1},0))\|^{\varphi}_p\right]^{I^3}_{\theta_{rst}}$ is not solid.

**Proposition 5.13.** $\left[\chi^{3q}_{f\mu}, \|\mu_{mnk}(X), (d(X_1,0), d(X_2,0), \ldots, d(X_{n-1},0))\|^{\varphi}_p\right]^{I^3}_{\theta_{rst}}$ *is not monotone.*

*Proof.* The proof follows from Proposition 5.14. □

**Proposition 5.14.** *Let* $\left[\chi^{3q}_{f\mu}, \|\mu_{mnk}(X), (d(X_1,0), d(X_2,0), \ldots, d(X_{n-1},0))\|^{\varphi}_p\right]^{I^3}_{\theta_{rst}}$ *and* $\left[\chi^{3q}_{f\mu}, \|\mu_{mnk}(Y), \left(d(X_1,0), d(X_2,0), \ldots, d(X_{n-1},0)\right)\|^{\varphi}_p\right]^{I^3}_{\theta_{rst}}$ *be three elements* $\epsilon_3 S^{BF}_0$ *such that*

$$\left[\chi^{3q}_{f\mu}, \|\mu_{mnk}(X), (d(X_1,0), d(X_2,0), \ldots, d(X_{n-1},0))\|^{\varphi}_p\right]^{I^3}_{\theta_{rst}} <^{P-\Gamma^3} (???) \left[\Gamma^{3q}_{f\mu}, \|\mu_{mnk}(Y), (d(X_1,0), d(X_2,0), \ldots, d(X_{n-1},0))\|^{\varphi}_p\right]^{I^3}_{\theta_{rst}},$$

*then there exists an element* $\left[\chi^{3q}_{f\mu}, \|\mu_{mnk}(Z), (d(X_1,0), d(X_2,0), \ldots, d(X_{n-1},0))\|^{\varphi}_p\right]^{I^3}_{\theta_{rst}} \in_3 S^{BF}_0$ *such that*

$$\left[\chi^{3q}_{f\mu}, \|\mu_{mnk}(X), (d(X_1,0), d(X_2,0), \ldots, d(X_{n-1},0))\|^{\varphi}_p\right]^{I^3}_{\theta_{rst}} <^{P-\chi^3} \left[\chi^{3q}_{f\mu}, \|\mu_{mnk}(Z), (d(X_1,0), d(X_2,0), \ldots, d(X_{n-1},0))\|^{\varphi}_p\right]^{I^3}_{\theta_{rst}} <^{P-\chi^3} \left[\chi^{3q}_{f\mu}, \|\mu_{mn}(Y), (d(X_1,0), d(X_2,0), \ldots, d(X_{n-1},0))\|^{\varphi}_p\right]^{I^3}_{\theta_{rst}}.$$

*Proof.* Let $\left[\chi^{3q}_{f\mu}, \|\mu_{mnk}(X), (d(X_1,0), d(X_2,0), \ldots, d(X_{n-1},0))\|^{\varphi}_p\right]^{I^3}_{\theta_{rst}}$, and $\left[\chi^{3q}_{f\mu}, \|\mu_{mnk}(Y), \left(d(X_1,0), d(X_2,0), \ldots, d(X_{n-1},0)\right)\|^{\varphi}_p\right]^{I^3}_{\theta_{rst}} \in_3 S^{BF}_0$ be such that

$$\left[\chi^{3q}_{f\mu}, \|\mu_{mnk}(X), (d(X_1,0), d(X_2,0), \ldots, d(X_{n-1},0))\|^{\varphi}_p\right]^{I^3}_{\theta_{rst}} <^{P-\chi^3} \left[\chi^{3q}_{f\mu}, \|\mu_{mnk}(Y), (d(X_1,0), d(X_2,0), \ldots, d(X_{n-1},0))\|^{\varphi}_p\right]^{I^3}_{\theta_{rst}}.$$

Define the sequence

$$\left[\chi^{3q}_{f\mu}, \|\mu_{mnk}(Z), (d(X_1,0), d(X_2,0), \ldots, d(X_{n-1},0))\|^{\varphi}_p\right]^{I^3}_{\theta_{rst}} <^{P-\chi^3} \left[\chi^{3q}_{f\mu}, \|\mu_{mnk}(Z), (d(X_1,0), d(X_2,0), \ldots, d(X_{n-1},0))\|^{\varphi}_p\right]^{I^3}_{\theta_{rst}}$$

as follows

$$\left[\chi^{3q}_{f\mu}, \|\mu_{mnk}(Z), (d(X_1,0), d(X_2,0), \ldots, d(X_{n-1},0))\|^{\varphi}_p\right]^{I^3}_{\theta_{rst}}$$

$$= \left[\chi^{3q}_{f\mu}, \left\|\mu_{mnk}\left(X^{1/5}\right), \left(d\left(X_1,0\right), d\left(X_2,0\right), \ldots, d\left(X_{n-1},0\right)\right)\right\|^{\varphi}_{p}\right]^{I^3}_{\theta_{rst}}$$
$$\otimes \left[\chi^{3q}_{f\mu}, \left\|\mu_{mnk}\left(Y^{4/5}\right), \left(d\left(X_1,0\right), d\left(X_2,0\right), \ldots, d\left(X_{n-1},0\right)\right)\right\|^{\varphi}_{p}\right]^{I^3}_{\theta_{rst}}.$$

This implies that

$$\left[\chi^{3q}_{f\mu}, \left\|\mu_{mnk}\left(X\right), \left(d\left(X_1,0\right), d\left(X_2,0\right), \ldots, d\left(X_{n-1},0\right)\right)\right\|^{\varphi}_{p}\right]^{I^3}_{\theta_{rs}}$$
$$<^{P-\chi^3} \left[\chi^{3q}_{f\mu}, \left\|\mu_{mnk}\left(Z\right), \left(d\left(X_1,0\right), d\left(X_2,0\right), \ldots, d\left(X_{n-1},0\right)\right)\right\|^{\varphi}_{p}\right]^{I^3}_{\theta_{rs}}$$
$$<^{P-\chi^3} \left[\chi^{3q}_{f\mu}, \left\|\mu_{mnk}\left(Y\right), \left(d\left(X_1,0\right), d\left(X_2,0\right), \ldots, d\left(X_{n-1},0\right)\right)\right\|^{\varphi}_{p}\right]^{I^3}_{\theta_{rst}}.$$

□

**Theorem 5.15.** *Let*

$$\left[\chi^{3q}_{f\mu}, \left\|\mu_{mnk}\left(X\right), \left(d\left(X_1,0\right), d\left(X_2,0\right), \ldots, d\left(X_{n-1},0\right)\right)\right\|^{\varphi}_{p}\right]^{I^3}_{\theta_{rst}}$$
$$<^{P-\chi^3} \left[\chi^{3q}_{f\mu}, \left\|\mu_{mnk}\left(Y\right), \left(d\left(X_1,0\right), d\left(X_2,0\right), \ldots, d\left(X_{n-1},0\right)\right)\right\|^{\varphi}_{p}\right]^{I^3}_{\theta_{rst}}$$

*and*

$$\left[\chi^{3q}_{f\mu}, \left\|\mu_{mnk}\left(Y\right), \left(d\left(X_1,0\right), d\left(X_2,0\right), \ldots, d\left(X_{n-1},0\right)\right)\right\|^{\varphi}_{p}\right]^{I^3}_{\theta_{rst}}$$
$$<^{P-\chi^3} \left[\chi^{3q}_{f\mu}, \left\|\mu_{mnk}\left(Z\right), \left(d\left(X_1,0\right), d\left(X_2,0\right), \ldots, d\left(X_{n-1},0\right)\right)\right\|^{\varphi}_{p}\right]^{I^3}_{\theta_{rst}},$$

*then*

$$\left[\chi^{3q}_{f\mu}, \left\|\mu_{mnk}\left(X\right), \left(d\left(X_1,0\right), d\left(X_2,0\right), \ldots, d\left(X_{n-1},0\right)\right)\right\|^{\varphi}_{p}\right]^{I^3}_{\theta_{rst}}$$
$$<^{P-\chi^3} \left[\chi^{3q}_{f\mu}, \left\|\mu_{mnk}\left(Z\right), \left(d\left(X_1,0\right), d\left(X_2,0\right), \ldots, d\left(X_{n-1},0\right)\right)\right\|^{\varphi}_{p}\right]^{I^3}_{\theta_{rst}}.$$

*Proof.* The proof is omitted as it is straightforward. □

**Theorem 5.16.** *Let $A$ be a non-negative $\chi^{3BF}$–$\chi^{3BF}$ summability matrix and let* $\left[\chi^{3q}_{f\mu}, \left\|\mu_{mnk}\left(X\right), \left(d\left(X_1,0\right), d\left(X_2,0\right), \ldots, d\left(X_{n-1},0\right)\right)\right\|^{\varphi}_{p}\right]^{I^3}_{\theta_{rst}}$ *and* $\left[\chi^{3q}_{f\mu}, \left\|\mu_{mnk}\left(Y\right), \left(d\left(X_1,0\right), d\left(X_2,0\right), \ldots, d\left(X_{n-1},0\right)\right)\right\|^{\varphi}_{p}\right]^{I^3}_{\theta_{rst}}$ *be two elements in ${}_3\ell^F$ such that*

$$\left[\chi^{3q}_{f\mu}, \left\|\mu_{mnk}\left(X\right), \left(d\left(X_1,0\right), d\left(X_2,0\right), \ldots, d\left(X_{n-1},0\right)\right)\right\|^{\varphi}_{p}\right]^{I^3}_{\theta_{rst}}$$
$$<^{P-\chi^3} \left[\chi^{3q}_{f\mu}, \left\|\mu_{mnk}\left(Y\right), \left(d\left(X_1,0\right), d\left(X_2,0\right), \ldots, d\left(X_{n-1},0\right)\right)\right\|^{\varphi}_{p}\right]^{I^3}_{\theta_{rst}}$$

*with*

$$\left[\chi^{3q}_{f\mu}, \left\|\mu_{mnk}\left(X\right), \left(d\left(X_1,0\right), d\left(X_2,0\right), \ldots, d\left(X_{n-1},0\right)\right)\right\|^{\varphi}_{p}\right]^{I^3}_{\theta_{rst}} \in_3 S^{BF}_0$$

*and*

$$\left[\chi^{3q}_{f\mu}, \|\mu_{mnk}(Y), (d(X_1,0), d(X_2,0), \ldots, d(X_{n-1},0))\|^{\varphi}_{p}\right]^{I^3}_{\theta_{rst}} \in_3 S^F_0$$

*for some* $\delta > 0$, *then*

$$\left[\chi^{3q}_{f\mu}, \|\mu_{mnk}(AX), (d(X_1,0), d(X_2,0), \ldots, d(X_{n-1},0))\|^{\varphi}_{p}\right]^{I^3}_{\theta_{rst}} <^{P-\chi^3} \left[\chi^{3q}_{f\mu}, \|\mu_{mnk}(AY), (d(X_1,0), d(X_2,0), \ldots, d(X_{n-1},0))\|^{\varphi}_{p}\right]^{I^3}_{\theta_{rst}}.$$

*Proof.* Since

$$\left[\chi^{3q}_{f\mu}, \|\mu_{mnk}(X), (d(X_1,0), d(X_2,0), \ldots, d(X_{n-1},0))\|^{\varphi}_{p}\right]^{I^3}_{\theta_{rst}} <^{P-\chi^3} \left[\chi^{3q}_{f\mu}, \|\mu_{mnk}(Y), (d(X_1,0), d(X_2,0), \ldots, d(X_{n-1},0))\|^{\varphi}_{p}\right]^{I^3}_{\theta_{rst}},$$

then there exists an analytic triple sequence $\left[\chi^{3q}_{f\mu}, \|\mu_{mnk}(Z), (d(X_1,0), d(X_2,0), \ldots, d(X_{n-1},0))\|^{\varphi}_{p}\right]^{I^3}_{\theta_{rst}}$ with Pringsheim's limit zero such that

$$\left[\chi^{3q}_{f\mu}, \|\mu_{mnk}(X), (d(X_1,0), d(X_2,0), \ldots, d(X_{n-1},0))\|^{\varphi}_{p}\right]^{I^3}_{\theta_{rst}} = \left[\chi^{3q}_{f\mu}, \|\mu_{mnk}(Y), (d(X_1,0), d(X_2,0), \ldots, d(X_{n-1},0))\|^{\varphi}_{p}\right]^{I^3}_{\theta_{rst}} \otimes \left[\chi^{3q}_{f\mu}, \|\mu_{mnk}(Z), (d(X_1,0), d(X_2,0), \ldots, d(X_{n-1},0))\|^{\varphi}_{p}\right]^{I^3}_{\theta_{rst}}.$$

For each $a, b$ and $c$, we have

$$\frac{\left[\chi^{3q}_{f\mu}, \|\mu_{mnk}(AX), (d(X_1,0), d(X_2,0), \ldots, d(X_{n-1},0))\|^{\varphi}_{p}\right]^{I^3}_{\theta_{rst}}}{\left[\chi^{3q}_{f\mu}, \|\mu_{mn}(AY), (d(X_1,0), d(X_2,0), \ldots, d(X_{n-1},0))\|^{\varphi}_{p}\right]^{I^3}_{\theta_{rst}}}$$
$$= \frac{\sup_{r,s,t\geq a,b,c}\left[\chi^{3q}_{f\mu}, \|\mu_{mnk}(AX), (d(X_1,0), d(X_2,0), \ldots, d(X_{n-1},0))\|^{\varphi}_{p}\right]^{I^3}_{\theta_{rst}}}{\sup_{r,s,t\geq a,b,c}\left[\chi^{3q}_{f\mu}, \|\mu_{mnk}(AY), (d(X_1,0), d(X_2,0), \ldots, d(X_{n-1},0))\|^{\varphi}_{p}\right]^{I^3}_{\theta_{rst}}}$$
$$= \frac{\sup_{r,s,t\geq a,b,c}\sum^{\infty,\infty}_{m,n,k=1,1,1}\left[\chi^{3q}_{f\mu}, \|\mu_{mnk}(AX), (d(X_1,0), d(X_2,0), \ldots, d(X_{n-1},0))\|^{\varphi}_{p}\right]^{I^3}_{\theta_{rst}}}{\sup_{r,s,t\geq a,b,c}\sum^{\infty,\infty}_{m,n,k=1,1,1}\left[\chi^{3q}_{f\mu}, \|\mu_{mnk}(AY), (d(X_1,0), d(X_2,0), \ldots, d(X_{n-1},0))\|^{\varphi}_{p}\right]^{I^3}_{\theta_{rst}}}$$
$$= \frac{\sup_{r,s,t\geq k,\ell}\sum^{\infty,\infty}_{m,n,k=1,1,1}\left[\chi^{3q}_{f\mu}, \|\mu_{mnk}(A(Y\otimes Z)), (d(X_1,0), d(X_2,0), \ldots, d(X_{n-1},0))\|^{\varphi}_{p}\right]^{I^3}_{\theta_{rst}}}{\sup_{r,s,t\geq a,b,c}\sum^{\infty,\infty}_{m,n,k=1,1,1}\left[\chi^{3q}_{f\mu}, \|\mu_{mnk}(AY), (d(X_1,0), d(X_2,0), \ldots, d(X_{n-1},0))\|^{\varphi}_{p}\right]^{I^3}_{\theta_{rst}}}$$
$$= \sup_{r,s,t\geq a,b,c}\sum^{\infty,\infty}_{m,n,k=1,1,1} a_{r,s,t,m,n,k}\left[\chi^{3q}_{f\mu}, \|\mu_{mnk}(Y), (d(X_1,0), d(X_2,0), \ldots, d(X_{n-1},0))\|^{\varphi}_{p}\right]^{I^3}_{\theta_{rst}}$$
$$\times \left[\chi^{3q}_{f\mu}, \|\mu_{mnk}(Z), (d(X_1,0), d(X_2,0), \ldots, d(X_{n-1},0))\|^{\varphi}_{p}\right]^{I^3}_{\theta_{rst}} / \delta \sup_{r,s,t\geq a,b,c}\sum^{\infty,\infty}_{m,n,k=1,1,1} a_{r,s,g,m,n,k}.$$

Since $Y$ and $Z$ are triple analytic sequences with $Z$ is in $\chi^{3F}$ and $A$ is a non-negative $\chi^{3F}-\chi^{3F}$ matrix, then

$$P-\sup_{r,s,t\geq a,b,c}\sum_{m,n,k=1,1,1}^{\infty,\infty} a_{r,s,t,m,n,k}\left[\chi^{3q}_{f\mu},\|\mu_{mnk}(Y),(d(X_1,0),d(X_2,0),\ldots,d(X_{n-1},0))\|^{\varphi}_{p}\right]^{I^3}_{\theta_{rst}}$$
$$\left[\chi^{3q}_{f\mu},\|\mu_{mnk}(Z),(d(X_1,0),d(X_2,0),\ldots,d(X_{n-1},0))\|^{\varphi}_{p}\right]^{I^3}_{\theta_{rst}}=0.$$

Hence

$$P-\lim\frac{\left[\chi^{3q}_{f\mu},\|\mu_{mnk}(AX),(d(X_1,0),d(X_2,0),\ldots,d(X_{n-1},0))\|^{\varphi}_{p}\right]^{I^3}_{\theta_{rst}}}{\left[\chi^{3q}_{f\mu},\|\mu_{mnk}(AY),(d(X_1,0),d(X_2,0),\ldots,d(X_{n-1},0))\|^{\varphi}_{p}\right]^{I^3}_{\theta_{rst}}}\leq 0. \tag{5.1}$$

In a similar manner we can establish

$$P-\lim\frac{\left[\chi^{3q}_{f\mu},\|\mu_{mnk}(AX),(d(X_1,0),d(X_2,0),\ldots,d(X_{n-1},0))\|^{\varphi}_{p}\right]^{I^3}_{\theta_{rst}}}{\left[\chi^{3q}_{f\mu},\|\mu_{mnk}(AY),(d(X_1,0),d(X_2,0),\ldots,d(X_{n-1},0))\|^{\varphi}_{p}\right]^{I^3}_{\theta_{rst}}}\geq 0. \tag{5.2}$$

From (5.1) and (5.2), we have

$$P-\lim\frac{\left[\chi^{3q}_{f\mu},\|\mu_{mnk}(AX),(d(X_1,0),d(X_2,0),\ldots,d(X_{n-1},0))\|^{\varphi}_{p}\right]^{I^3}_{\theta_{rst}}}{\left[\chi^{3q}_{f\mu},\|\mu_{mn}(AY),(d(X_1,0),d(X_2,0),\ldots,d(X_{n-1},0))\|^{\varphi}_{p}\right]^{I^3}_{\theta_{rst}}}=0,$$

which implies

$$\left[\chi^{3q}_{f\mu},\|\mu_{mnk}(AX),(d(X_1,0),d(X_2,0),\ldots,d(X_{n-1},0))\|^{\varphi}_{p}\right]^{I^3}_{\theta_{rst}}$$
$$<^{P-\chi^3}\left[\chi^{3q}_{f\mu},\|\mu_{mnk}(AY),(d(X_1,0),d(X_2,0),\ldots,d(X_{n-1},0))\|^{\varphi}_{p}\right]^{I^3}_{\theta_{rst}}.$$

□

**Theorem 5.17.** *Let* $\left[\chi^{3q}_{f\mu},\|\mu_{mnk}(X),(d(X_1,0),d(X_2,0),\ldots,d(X_{n-1},0))\|^{\varphi}_{p}\right]^{I^3}_{\theta_{rs}}\in S^{3BF}_{0}$ *and* $A$ *be a transformation such that*

$$\left[\chi^{3q}_{f\mu},\|\mu_{mnk}(AX),(d(X_1,0),d(X_2,0),\ldots,d(X_{n-1},0))\|^{\varphi}_{p}\right]^{I^3}_{\theta_{rst}}$$
$$<^{P-\chi^3}\left[\chi^{3q}_{f\mu},\|\mu_{mnk}(X),(d(X_1,0),d(X_2,0),\ldots,d(X_{n-1},0))\|^{\varphi}_{p}\right]^{I^3}_{\theta_{rst}}.$$

*Then there exists* $\left[\chi^{3q}_{f\mu},\|\mu_{mnk}(X),(d(X_1,0),d(X_2,0),\ldots,d(X_{n-1},0))\|^{\varphi}_{p}\right]^{I^3}_{\theta_{rst}}\in\chi^{3BF}$ *such that*

$$\left[\chi^{3q}_{f\mu},\|\mu_{mnk}(Y),(d(X_1,0),d(X_2,0),\ldots,d(X_{n-1},0))\|^{\varphi}_{p}\right]^{I^3}_{\theta_{rst}}$$
$$=\left[\chi^{3q}_{f\mu},\|\mu_{mnk}(Y),(d(X_1,0),d(X_2,0),\ldots,d(X_{n-1},0))\|^{\varphi}_{p}\right]^{I^3}_{\theta_{rst}}\quad(m,n,k)$$

*and*

$$\left[\chi^{3q}_{f\mu},\|\mu_{mnk}(AY),(d(X_1,0),d(X_2,0),\ldots,d(X_{n-1},0))\|^{\varphi}_{p}\right]^{I^3}_{\theta_{rst}}$$
$$<^{P-\chi^3}\left[\chi^{3q}_{f\mu},\|\mu_{mnk}(Y),(d(X_1,0),d(X_2,0),\ldots,d(X_{n-1},0))\|^{\varphi}_{p}\right]^{I^3}_{\theta_{rst}}.$$

*Proof.* Let $\left[\chi^{3q}_{f\mu}, \|\mu_{mnk}(X), (d(X_1,0), d(X_2,0), \ldots, d(X_{n-1},0))\|^{\varphi}_{p}\right]^{I^3}_{\theta_{rst}} \in S^{3BF}_{0}$. Then there exists a subset $B_1 \subset \mathbb{N}\times\mathbb{N}\times\mathbb{N}$ with $\delta_3(B_1) = 1$ such that

$$P - \left[\chi^{3q}_{f\mu}, \|\mu_{mnk}(X), (d(X_1,0), d(X_2,0), \ldots, d(X_{n-1},0))\|^{\varphi}_{p}\right]^{I^3}_{\theta_{rst}} = 0,$$

over $B_1$. Let $\left[\chi^{3q}_{f\mu}, \|\mu_{m_a n_b k_c}(X), (d(X_1,0), d(X_2,0), \ldots, d(X_{n-1},0))\|^{\varphi}_{p}\right]^{I^3}_{\theta_{rst}} \in S^{3BF}_{0}$. Then there exists a subset $B_2 \subset \mathbb{N}\times\mathbb{N}\times\mathbb{N}$ with $\delta_3(B_2) = 1$ such that

$$P - \left[\chi^{3q}_{f\mu}, \|\mu_{m_a n_b k_c}(X), (d(X_1,0), d(X_2,0), \ldots, d(X_{n-1},0))\|^{\varphi}_{p}\right]^{I^3}_{\theta_{rst}} = 0,$$

over $B_2$. Since

$$\left[\chi^{3q}_{f\mu}, \|\mu_{mnk}(AX), (d(X_1,0), d(X_2,0), \ldots, d(X_{n-1},0))\|^{\varphi}_{p}\right]^{I^3}_{\theta_{rst}}$$
$$<^{P-\chi^3} \left[\chi^{3q}_{f\mu}, \|\mu_{mnk}(X), (d(X_1,0), d(X_2,0), \ldots, d(X_{n-1},0))\|^{\varphi}_{p}\right]^{I^3}_{\theta_{rst}},$$

we have

$$P - \lim \frac{\left[\chi^{3q}_{f\mu}, \|\mu_{m_a n_b k_c}(X), (d(X_1,0), d(X_2,0), \ldots, d(X_{n-1},0))\|^{\varphi}_{p}\right]^{I^3}_{\theta_{rst}}}{\left[\chi^{3q}_{f\mu}, \|\mu_{mnk}(X), (d(X_1,0), d(X_2,0), \ldots, d(X_{n-1},0))\|^{\varphi}_{p}\right]^{I^3}_{\theta_{rst}}} = 0.$$

Then there exists a subset $B_3 \subset \mathbb{N}\times\mathbb{N}\times\mathbb{N}$ with $\delta_3(B_3) = 1$ such that

$$P - \lim \frac{\left[\chi^{3q}_{f\mu}, \|\mu_{m_a n_b k_c}(X), (d(X_1,0), d(X_2,0), \ldots, d(X_{n-1},0))\|^{\varphi}_{p}\right]^{I^3}_{\theta_{rst}}}{\left[\chi^{3q}_{f\mu}, \|\mu_{mnk}(X), (d(X_1,0), d(X_2,0), \ldots, d(X_{n-1},0))\|^{\varphi}_{p}\right]^{I^3}_{\theta_{rst}}} = 0,$$

over $B_3$. Let $D = B_1 \bigcap B_2 \bigcap B_3$. Then clearly $\delta_3(D) = 1$.

For $r \neq m_a, s \neq n_b, t \neq k_c \in \mathbb{N}\times\mathbb{N}\times\mathbb{N}$, let us define

$$\left[\chi^{3q}_{f\mu}, \|\mu_{rst}(Y), (d(X_1,0), d(X_2,0), \ldots, d(X_{n-1},0))\|^{\varphi}_{p}\right]^{I^3}_{\theta_{rst}} = \begin{cases} \frac{\left[\chi^{3q}_{f\mu}, \|\mu_{rst}(X), (d(X_1,0), d(X_2,0), \ldots, d(X_{n-1},0))\|^{\varphi}_{p}\right]^{I^3}_{\theta_{rst}}}{(rst)^{-3}}, & \text{if } (r,s,t) \in D, \\ 0, & \text{otherwise.} \end{cases}$$

and

$$\left[\chi^{3q}_{f\mu}, \|\mu_{m_a n_b k_c}(Y), (d(X_1,0), d(X_2,0), \ldots, d(X_{n-1},0))\|^{\varphi}_{p}\right]^{I^3}_{\theta_{rst}} = \begin{cases} \frac{\left[\chi^{3q}_{f\mu}, \|\mu_{m_a n_b k_c}(X), (d(X_1,0), d(X_2,0), \ldots, d(X_{n-1},0))\|^{\varphi}_{p}\right]^{I^3}_{\theta_{rst}}}{\left[\chi^{3q}_{f\mu}, \|\mu_{mnk}(Y), (d(X_1,0), d(X_2,0), \ldots, d(X_{n-1},0))\|^{\varphi}_{p}\right]^{I^3}_{\theta_{rst}} (mnk)^{-3}}, & \text{if } (a,b,c) \in D, \\ 0, & \text{otherwise.} \end{cases}$$

Then we have $\left[\chi^{3q}_{f\mu}, \|\mu_{mnk}(Y), (d(X_1,0), d(X_2,0), \ldots, d(X_{n-1},0))\|^{\varphi}_{p}\right]^{I^3}_{\theta_{rst}} \in S^{3BF}_{0}$. Then we have

$$\left[\chi^{3q}_{f\mu}, \|\mu_{mnk}(X), (d(X_1,0), d(X_2,0), \ldots, d(X_{n-1},0))\|^{\varphi}_{p}\right]^{I^3}_{\theta_{rst}}$$

$$= \left[\chi^{3q}_{f\mu}, \|\mu_{mnk}(Y), (d(X_1,0), d(X_2,0), \ldots, d(X_{n-1},0))\|^{\varphi}_p\right]^{I^3}_{\theta_{rst}} \quad (m,n,k)$$

and this implies

$$A\left[\chi^{3q}_{f\mu}, \|\mu_{mnk}(Y), (d(X_1,0), d(X_2,0), \ldots, d(X_{n-1},0))\|^{\varphi}_p\right]^{I^3}_{\theta_{rst}}$$
$$<^{P-\chi^3} \left[\chi^{3q}_{f\mu}, \|\mu_{mnk}(Y), (d(X_1,0), d(X_2,0), \ldots, d(X_{n-1},0))\|^{\varphi}_p\right]^{I^3}_{\theta_{rst}}.$$

□

**Theorem 5.18.** *Let* $\left[\chi^{3q}_{f\mu}, \|\mu_{mnk}(X), (d(X_1,0), d(X_2,0), \ldots, d(X_{n-1},0))\|^{\varphi}_p\right]^{I^3}_{\theta_{rst}} \in S^{3BF}_0$ *and* $A$ *be a transformation such that*

$$\left[\chi^{3q}_{f\mu}, \|\mu_{mnk}(AX), (d(X_1,0), d(X_2,0), \ldots, d(X_{n-1},0))\|^{\varphi}_p\right]^{I^3}_{\theta_{rst}}$$
$$<^{P-\chi^3} \left[\chi^{3q}_{f\mu}, \|\mu_{mnk}(X), (d(X_1,0), d(X_2,0), \ldots, d(X_{n-1},0))\|^{\varphi}_p\right]^{I^3}_{\theta_{rst}}.$$

*Then there exists a* $\left[\chi^{3q}_{f\mu}, \|\mu_{mnk}(Y), (d(X_1,0), d(X_2,0), \ldots, d(X_{n-1},0))\|^{\varphi}_p\right]^{I^3}_{\theta_{rst}} \in S^{3BF}_0$ *such that*

$$\left[\chi^{3q}_{f\mu}, \|\mu_{mn}(X), (d(X_1,0), d(X_2,0), \ldots, d(X_{n-1},0))\|^{\varphi}_p\right]^{I^3}_{\theta_{rst}}$$
$$<^{P-\chi^3} \left[\chi^{3q}_{f\mu}, \|\mu_{mnk}(Y), (d(X_1,0), d(X_2,0), \ldots, d(X_{n-1},0))\|^{\varphi}_p\right]^{I^3}_{\theta_{rst}}$$

*and*

$$\left[\chi^{3q}_{f\mu}, \|\mu_{mnk}(AY), (d(X_1,0), d(X_2,0), \ldots, d(X_{n-1},0))\|^{\varphi}_p\right]^{I^3}_{\theta_{rst}}$$
$$<^{P-\chi^3} \left[\chi^{3q}_{f\mu}, \|\mu_{mnk}(Y), (d(X_1,0), d(X_2,0), \ldots, d(X_{n-1},0))\|^{\varphi}_p\right]^{I^3}_{\theta_{rst}}.$$

*Proof.* Consider

$$\left[\chi^{3q}_{f\mu}, \|\mu_{mnk}(Y), (d(X_1,0), d(X_2,0), \ldots, d(X_{n-1},0))\|^{\varphi}_p\right]^{I^3}_{\theta_{rst}}$$
$$= \left[\chi^{3q}_{f\mu}, \|\mu_{mnk}(X), (d(X_1,0), d(X_2,0), \ldots, d(X_{n-1},0))\|^{\varphi}_p\right]^{I^3}_{\theta_{rst}}$$

for all $m,n,k \in \mathbb{N}$. Then clearly $\left[\chi^{3q}_{f\mu}, \|\mu_{mnk}(Y), (d(X_1,0), d(X_2,0), \ldots, d(X_{n-1},0))\|^{\varphi}_p\right]^{I^3}_{\theta_{rs}} \in S^{3BF}_0$ such that

$$\left[\chi^{3q}_{f\mu}, \|\mu_{mnk}(X), (d(X_1,0), d(X_2,0), \ldots, d(X_{n-1},0))\|^{\varphi}_p\right]^{I^3}_{\theta_{rst}}$$
$$<^{P-\chi^3} \left[\chi^{3q}_{f\mu}, \|\mu_{mnk}(Y), (d(X_1,0), d(X_2,0), \ldots, d(X_{n-1},0))\|^{\varphi}_p\right]^{I^3}_{\theta_{rst}}$$

and

$$\left[\chi^{3q}_{f\mu}, \|\mu_{mnk}(AY), (d(X_1,0), d(X_2,0), \ldots, d(X_{n-1},0))\|^{\varphi}_p\right]^{I^3}_{\theta_{rst}}$$
$$<^{P-\chi^3} \left[\Gamma^{3q}_{f\mu}, \|\mu_{mnk}(Y), (d(X_1,0), d(X_2,0), \ldots, d(X_{n-1},0))\|^{\varphi}_p\right]^{I^3}_{\theta_{rst}}.$$

□

## 6. Conclusion

In this paper all the definitions are newly constructed and then construct with difference of triple sequence space of $\chi^3$, the new theorems are construct with some aspects. But our paper deals with metric condition of triple sequence space adopted with randomness and acceleration is a new contribution.

## Competing interests

The authors declare that there is no conflict of interests regarding the publication of this research paper.

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