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Twisted $(\alpha, \beta) - \psi-$contractive type mappings and applications in Partial ordered metric spaces

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Abstract

The purpose of this paper is to discuss the existence and uniqueness of fixed points for new classes of mappings defined on a 0−complete partial ordered metric space. The obtained results generalize some recent theorems in the literature. Several applications and interesting consequences of our theorems are also given. ©2017 All rights reserved.

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1. Introduction

Matthews [9] introduced the notion of a partial metric space as a part of the study of denotational semantics of dataflow networks. He showed that the Banach contraction mapping theorem can be generalized to the partial metric context for applications in program verification. Subsequently, several authors [2, 3, 6, 7, 10, 11, 12, 14] derived fixed point theorems in partial metric spaces. See also the presentation by Bukatin et al. [1] where the motivation for introducing non-zero distance (i.e., the "distance" p where $p(x, x) \neq 0$) is explained, which is also leading to interesting research in foundations of topology.

In view of the above considerations, the principal motivation of this paper is to relate some results in the literature by discussing the existence and uniqueness of fixed points for new classes of mappings defined on a complete metric space. In particular, we use our results to obtain fixed points for some new classes of cyclic mappings and cyclic ordered mappings. We conclude the paper by giving an application of proved results in solving functional equations. The following definitions and details can be seen, e.g., in [1, 5, 9, 10, 13].

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Definition 1.1. A partial metric on a nonempty set X is a function $p : X \times X \to \mathbb{R}^+$ such that for all $x, y, z \in X$:

(p_1) $x = y \Longleftrightarrow p(x, x) = p(x, y) = p(y, y)$,
(p_2) $p(x, x) \leq p(x, y)$,
(p_3) $p(x, y) = p(y, x)$,
(p_4) $p(x, y) \leq p(x, z) + p(z, y) - p(z, z)$.

The pair (X, p) is called a partial metric on X.

It is clear that, if $p(x, y) = 0$, then from (p_1) and (p_2) $x = y$. But if $x = y, p(x, y)$ may not be 0. Each partial metric p on X generates a T_0 topology τ_p on X which has a base, the family of open p-balls $\{B_p(x, \epsilon) : x \in X, \epsilon > 0\}$, where $B_p(x, \epsilon) = \{y \in X : p(x, y) < p(x, x) + \epsilon\}$ for all $x \in X$ and $\epsilon > 0$. A sequence $\{x_n\}$ in (X, p) converges to a point $x \in X$ (in the sense of τ_p) if $\lim_{n\to\infty} p(x, x_n) = p(x, x)$. This will be denoted as $x_n \to x$ as $n \to \infty$ or $\lim_{n\to\infty} x_n = x$. If $f : X \to X$ is continuous at $x \in X$ (with respect to τ_p), then for each sequence $\{x_n\}$ in X, we have

$$x_n \to x_0 \quad as \quad fx_n \to fx_0.$$

If p is a partial metric on X, then the function $p^s : X \times X \to \mathbb{R}^+$ given by

$$p^s(x, y) = 2p(x, y) - p(x, x) - p(y, y) \tag{1.1}$$

is a metric on X. Furthermore, $\lim_{n\to\infty} p^s(x_n, x) = 0$ if and only if

$$p(x, x) = \lim_{n\to\infty} p(x_n, x) = \lim_{n,m\to\infty} p(x_n, x_m).$$

Example 1.2. A paradigmatic example of a partial metric space is the pair $(\mathbb{R}^+, p)$, where $p(x, y) = \max\{x, y\}$ for all $x, y \in \mathbb{R}^+$. The corresponding metric is

$$p^s(x, y) = 2\max\{x, y\} - x - y = |x - y|.$$

Example 1.3. If (X, d) is a metric space and $c \geq 0$ is arbitrary, then

$$p(x, y) = d(x, y) + c$$

defines a partial metric on X and the corresponding metric is $p^s(x, y) = 2d(x, y)$.

Other examples of partial metric spaces which are interesting from a computational point of view may be found in [4, 9].

Definition 1.4. Let (X, p) be a partial metric space. Then:

(1) a sequence $\{x_n\}$ in (X, p) is called a Cauchy sequence if $\lim_{n,m\to\infty} p(x_n, x_m)$ exists (and is finite).
(2) the space (X, p) is said to be complete if every Cauchy sequence $\{x_n\}$ in X converges, with respect to τ_p, to a point $x \in X$ such that $p(x, x) = \lim_{n,m\to\infty} p(x_n, x_m)$.
(3) a sequence $\{x_n\}$ in (X, p) is called $0-$Cauchy if $\lim_{n,m\to\infty} p(x_n, x_m) = 0$.
(4) the space (X, p) is said to be $0-$complete if every $0-$Cauchy sequence in X converges (in τ_p) to a point $x \in X$ such that $p(x, x) = 0$.

Lemma 1.5. *Let (X, p) be a partial metric space. Then:*

(a) $\{x_n\}$ is a Cauchy sequence in (X, p) if and only if it is a Cauchy sequence in the metric space (X, p^s).
(b) the space (X, p) is complete if and only if the metric space (X, p^s) is complete.

(c) every $0-$*Cauchy sequence in* (X, p) *is Cauchy in* (X, p^s).

(d) if (X, p) *is complete, then it is* $0-$*complete.*

The converse assertions of (c) and (d) do not hold as the following easy example shows:

Example 1.6. The space $X = [0, \infty) \cap \mathbb{Q}$ with the partial metric $p(x, y) = \max\{x, y\}$ is $0-$complete, but is not complete (since $p^s(x, y) = |x - y|$ and (X, p^s) is not complete). Moreover, the sequence $\{x_n\}$ with $x_n = 1$ for each $n \in \mathbb{N}$ is a Cauchy sequence in (X, p), but it is not a $0-$Cauchy sequence.

Denote with Ψ the family of nondecreasing functions $\Psi : [0, +\infty) \to [0, +\infty)$ such that $\Sigma_{n=1}^{\infty} \psi^n(t) < +\infty$ for all $t > 0$, where ψ^n is the n^{th} iterate of ψ. The next lemma is obvious.

Lemma 1.7. *If* $\psi \in \Psi$, *then* $\psi(0) = 0$ *and* $\psi(t) < t$ *for all* $t > 0$.

The following definitions play a key role in our paper.

Definition 1.8. Let $f : X \to X$ and $\alpha : X \times X \to [0, +\infty)$. We say that f is an $\alpha-$admissible mapping if for all $x, y \in X$,

$$\alpha(x, y) \geq 1 \Longrightarrow \alpha(fx, fy) \geq 1.$$

Definition 1.9. Let $f : X \to X$ and $\alpha, \beta : X \times X \to [0, +\infty)$. We say that f is a twisted $(\alpha, \beta)-$ admissible mapping if for all $x, y \in X$,

$$\alpha(x, y) \geq 1 \Longrightarrow \alpha(fy, fx) \geq 1,$$

$$\beta(x, y) \geq 1 \Longrightarrow \beta(fy, fx) \geq 1.$$

Definition 1.10. Let (X, p) be a partial ordered metric space and $f : X \to X$ be a twisted $(\alpha, \beta)-$admissible mapping. Then f is said to be a

(a) twisted $(\alpha, \beta) - \psi-$contractive mapping of type (I), if

$$\alpha(x, y)\beta(x, y)p(fx, fy) \leq \psi(p(x, y)) \tag{1.2}$$

holds for all $x, y \in X$, where $\psi \in \Psi$.

(b) twisted $(\alpha, \beta) - \psi-$contractive mapping of type (II), if there is $0 < l \leq 1$ such that

$$(\alpha(x, y)\beta(x, y) + l)^{p(fx,fy)} \leq (1 + l)^{\psi(p(x,y))} \tag{1.3}$$

holds for all $x, y \in X$, where $\psi \in \Psi$.

(c) twisted $(\alpha, \beta) - \psi-$contractive mapping of type (III), if there is $l \geq 1$ such that

$$(p(fx, fy) + l)^{\alpha(x,y)\beta(x,y)} \leq \psi(p(x, y)) + l \tag{1.4}$$

holds for all $x, y \in X$, where $\psi \in \Psi$.

2. Main Results

In this section, we give some theorems linking the above concepts.

Theorem 2.1. *Let* (X, p) *be a* $0-$*complete partial ordered metric space and let* $f : X \to X$ *be a continuous twisted* $(\alpha, \beta) - \psi-$*contractive mapping of type (I) or (II) or (III). If there exists* $x_0 \in X$ *such that* $\alpha(x_0, fx_0) \geq 1$ *and* $\beta(x_0, fx_0) \geq 1$, *then* f *has a fixed point in* X.

Proof. Let $x_0 \in X$ such that $\alpha(x_0, fx_0) \geq 1$ and $\beta(x_0, fx_0) \geq 1$. Define a sequence $\{x_n\}$ by $x_n = f^n x_0 = fx_{n-1}$ for all $n \in \mathbb{N}$. Since f is a twisted $(\alpha, \beta)-$admissible mapping and

$$\beta(x_0, x_1) = \beta(x_0, fx_0) \geq 1,$$

then

$$\beta(x_2, x_1) = \beta(fx_1, fx_0) \geq 1$$

which implies

$$\beta(x_2, x_3) = \beta(fx_1, fx_2) \geq 1.$$

By continuing this process, we get $\beta(x_{2n}, x_{2n+1}) \geq 1$ and $\beta(x_{2n}, x_{2n-1}) \geq 1$ for all $n \in N$. Similarly, we have $\alpha(x_{2n}, x_{2n+1}) \geq 1$ and $\alpha(x_{2n}, x_{2n-1}) \geq 1$ for all $n \in \mathbb{N}$.

Now, we distinguish the following cases:

Case-(a) Let f be a twisted $(\alpha, \beta) - \psi-$contractive mapping of type (I). Then by 1.2 with $x = x_{2n}$ and $y = x_{2n+1}$ we have

$$\begin{aligned} p(x_{2n+1}, x_{2n+2}) &\leq \alpha(x_{2n}, x_{2n+1})\beta(x_{2n}, x_{2n+1})p(x_{2n+1}, x_{2n+2}) \\ &\leq \psi(p(x_{2n}, x_{2n+1})). \end{aligned}$$

Then $p(x_{2n+1}, x_{2n+2}) \leq \psi(p(x_{2n}, x_{2n+1}))$. Similarly, by 1.2 with $x = x_{2n}$ and $y = x_{2n-1}$ we have $p(x_{2n+1}, x_{2n}) \leq \psi(p(x_{2n}, x_{2n-1}))$. In view of these inequalities, for all $n \in \mathbb{N}$ we can deduce that $p(x_n, x_{n+1}) \leq \psi^n(p(x_0, x_1))$.

Case-(b) Let f be a twisted $(\alpha, \beta) - \psi$-contractive mapping of type (II). Then by 1.3 with $x = x_{2n}$ and $y = x_{2n+1}$ we have

$$\begin{aligned} (1 + l)^{p(x_{2n+1}, x_{2n+2})} &\leq (\alpha(x_{2n}, x_{2n+1})\beta(x_{2n}, x_{2n+1}) + l)^{p(x_{2n+1}, x_{2n+2})} \\ &\leq (1 + l)^{\psi(p(x_{2n}, x_{2n+1}))}. \end{aligned}$$

Then

$$p(x_{2n+1}, x_{2n+2}) \leq \psi(p(x_{2n}, x_{2n+1})).$$

Similarly, by 1.3 with $x = x_{2n}$ and $y = x_{2n-1}$ we have

$$p(x_{2n+1}, x_{2n}) \leq \psi(p(x_{2n}, x_{2n-1})).$$

Again, for all $n \in \mathbb{N}$ we can deduce that

$$p(x_n, x_{n+1}) \leq \psi^n(p(x_0, x_1)).$$

Case-(c) Let f be a twisted $(\alpha, \beta) - \psi$-contractive mapping of type (III). Then by 1.4 with $x = x_{2n}$ and $y = x_{2n+1}$ we have

$$\begin{aligned} p(x_{2n+1}, x_{2n+2}) + l &\leq (p(x_{2n+1}, x_{2n+2}) + l)^{\alpha(x_{2n}, x_{2n+1})\beta(x_{2n}, x_{2n+1})} \\ &\leq \psi(p(x_{2n}, x_{2n+1})) + l. \end{aligned}$$

Then

$$p(x_{2n+1}, x_{2n+2}) \leq \psi(p(x_{2n}, x_{2n+1})).$$

Similarly, by 1.4 with $x = x_{2n}$ and $y = x_{2n-1}$ we have

$$p(x_{2n+1}, x_{2n}) \leq \psi(p(x_{2n}, x_{2n-1})).$$

Thus, in all cases, for all $n \in \mathbb{N}$ we can deduce easily that

$$p(x_n, x_{n+1}) \leq \psi^n(p(x_0, x_1)).$$

Fix $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$\Sigma_{n \geq N} \psi^n(p(x_0, x_1)) < \epsilon.$$

Let $m, n \in N$ with $m > n \geq N$. Then, by using the triangular inequality, we get

$$p(x_n, x_m) \leq \Sigma_{k=n}^{m-1} p(x_k, x_{k+1}) \leq \Sigma_{n \geq N} \psi^n(p(x_0, x_1)) < \epsilon$$

and consequently $\lim_{m,n \to +\infty} p(x_n, x_m) = 0$. Hence $\{x_n\}$ is a $0-$Cauchy sequence. Since X is $0-$complete, then there is $z \in X$ such that $x_n \to z$ as $n \to +\infty$. Finally, since f is continuous then we have

$$fz = \lim_{n \to +\infty} fx_n = \lim_{n \to +\infty} x_{n+1} = z$$

and so z is a fixed point of f . □

Similarly, one can obtain the same conclusion under an alternative assumption. Precisely, in the following theorem we omit the continuity condition on f but use an adjunctive condition on X.

Theorem 2.2. *Let (X, p) be a $0-$complete partial ordered metric space and let $f : X \to X$ be a twisted $(\alpha, \beta) - \psi$-contractive mapping of type (I) or (II) or (III). Also suppose that the following conditions hold:*

(i) there exists $x_0 \in X$ such that $\alpha(x_0, fx_0) \geq 1$ and $\beta(x_0, fx_0) \geq 1$,

(ii) if $\{x_n\}$ is a sequence in X such that $\alpha(x_{2n}, x_{2n+1}) \geq 1$ and $\beta(x_{2n}, x_{2n+1}) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$ and $x_n \to x$ as $n \to +\infty$, then $\alpha(x_{2n}, x) \geq 1$ and $\beta(x_{2n}, x) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$.

Then f has a fixed point.

Proof. Let $x_0 \in X$ such that $\alpha(x_0, fx_0) \geq 1$ and $\beta(x_0, fx_0) \geq 1$. Proceeding as in the proof of Theorem 2.1, we know that there is $z \in X$ such that $x_n \to z$ as $n \to +\infty$, $\alpha(x_{2n}, x_{2n+1}) \geq 1$ and $\beta(x_{2n}, x_{2n+1}) \geq 1$. We shall prove that $z = fz$. Assume to the contrary that $z \neq fz$. From (ii) we have $\alpha(x_{2n}, z) \geq 1$ and $\beta(x_{2n}, z) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$. Now, we distinguish the following cases:

Case-(a) Let f be a twisted $(\alpha, \beta) - \psi-$contractive mapping of type (I). Then by 1.3 with $x = x_{2n}$ and $y = z$ we have

$$\begin{aligned} p(fx_{2n}, fz) &\leq \alpha(x_{2n}, z)\beta(x_{2n}, z)p(fx_{2n}, fz) \\ &\leq \psi(p(x_{2n}, z)). \end{aligned}$$

Case-(b) Let f be a twisted $(\alpha, \beta) - \psi-$contractive mapping of type (II). Then by 1.3 with $x = x_{2n}$ and $y = z$ we have

$$\begin{aligned} (1 + l)^{p(fx_{2n}, fz)} &\leq (\alpha(x_{2n}, z)\beta(x_{2n}, z) + l)^{p(fx_{2n}, fz)} \\ &\leq (1 + l)^{\psi(p(x_{2n}, z))}, \end{aligned}$$

that implies

$$p(fx_{2n}, fz) \leq \psi(p(x_{2n}, z)).$$

Case-(c) Let f be a twisted $(\alpha, \beta) - \psi-$contractive mapping of type (III). Then by 1.3 with $x = x_{2n}$ and $y = z$ we have

$$\begin{aligned} p(fx_{2n}, fz) + l &\leq (p(fx_{2n}, fz) + l)^{\alpha(x_{2n}, z)\beta(x_{2n}, z)} \\ &\leq \psi(p(x_{2n}, z)) + l, \end{aligned}$$

that implies

$$p(fx_{2n}, fz) \leq \psi(p(x_{2n}, z)).$$

Therefore, in all cases, by using the triangular inequality, we can write

$$\begin{aligned} p(z, fz) &\leq p(z, fx_{2n}) + p(fx_{2n}, fz) - p(fx_{2n}, fx_{2n}) \\ &= p(z, x_{2n+1}) + p(fx_{2n}, fz) - p(x_{2n+1}, x_{2n+1}) \\ &\leq p(z, x_{2n+1}) + \psi(p(x_{2n}, z)) - p(x_{2n+1}, x_{2n+1}). \end{aligned}$$

By taking the limit as $n \to +\infty$ in the above inequality, since ψ is continuous in $t = 0$, we have $p(z, fz) = 0$, that is, $z = fz$. □

Example 2.3. Let $X = \mathbb{R}$ be endowed with the partial ordered metric $p(x, y) = \max\{x, y\}$ for all $x, y \in X$ and $f : X \to X$ be defined by

$$fx = \begin{cases} \frac{-1}{4}x, & if \quad x \in [-1, 1], \\ \sqrt[3]{\frac{x+1}{x^2+1}}, & if \quad x \in \mathbb{R} \setminus [-1, 1]. \end{cases}$$

Define also $\alpha, \beta : X \times X \to [0, +\infty)$ by

$$\alpha(x, y) = \beta(x, y) = \begin{cases} 1, & if \quad x \in [0, 1] \quad and \quad y \in [-1, 0], \\ 0 & otherwise, \end{cases}$$

and $\psi : [0, +\infty) \to [0, +\infty)$ by $\psi(t) = \frac{1}{2}t$ for all $t \geq 0$. We prove that Theorem 2.2 can be applied to f.

Proof. Let $\alpha(x, y) \geq 1$ for $x, y \in X$. Then $x \in [0, 1]$ and $y \in [-1, 0]$, and so $fy \in [0, 1]$ and $fx \in [-1, 0]$, that is, $\alpha(fy, fx) \geq 1$. Also, assume $\beta(x, y) \geq 1$ for $x, y \in X$. Therefore $x \in [0, 1]$ and $y \in [-1, 0]$, and hence $fy \in [0, 1]$ and $fx \in [-1, 0]$, that is, $\beta(fy, fx) \geq 1$. Clearly, $\alpha(0, f0) \geq 1$ and $\beta(0, f0) \geq 1$. Now, let $\{x_n\}$ be a sequence in X such that $\alpha(x_{2n}, x_{2n+1}) \geq 1$ and $\beta(x_{2n}, x_{2n+1}) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$ and $x_n \to x$ as $n \to +\infty$. This implies that $\{x_{2n+1}\} \subset [-1, 0]$ and $\{x_{2n}\} \subset [0, 1]$. Thus, $x = 0$ and so $\alpha(x_{2n}, x) \geq 1$ and $\beta(x_{2n}, x) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$. Moreover, for $x \in [0, 1]$ and $y \in [-1, 0]$ we have

$$\begin{aligned} \alpha(x, y)\beta(x, y)p(fx, fy) &= \max\{fx, fy\} \\ &= \frac{1}{4}\max\{x, y\} \\ &\leq \frac{1}{2}\max\{x, y\} = \psi(p(x, y)). \end{aligned}$$

Otherwise, $\alpha(x, y)\beta(x, y) = 0$ and 1.2 trivially holds. Then f is a twisted $(\alpha, \beta) - \psi-$contractive mapping of type (I) and, by Theorem 2.2, f has a fixed point. □

Now, we give examples to prove validity of Theorem 2.2.

Example 2.4. Let X, d, α and β be as in Example 2.3 and $f : X \to X$ be defined by

$$fx = \begin{cases} \frac{-1}{4\pi}(x), & if \quad x \in [-1, 1], \\ \frac{x^2 - \cos(x^5)}{x + \sin x}, & if \quad x \in \mathbb{R} \setminus [-1, 1]. \end{cases}$$

Define $\psi : [0, +\infty) \to [0, +\infty)$ by $\psi(t) = \frac{1}{4}t$ for all $t \geq 0$. We prove that Theorem 2.2 can be applied to f.

Proof. Proceeding as in the proof of Example 2.3, we deduce that f is a twisted $(\alpha-\beta)-$admissible mapping and that the conditions (i) and (ii) of Theorem 2.2 hold. Moreover, if $x \in [0,1], y \in [-1,0]$ and $0 < l \leq 1$ we have

$$\begin{aligned}(\alpha(x,y)\beta(x,y)+l)^{p(fx,fy)} &= (1+l)^{p(fx,fy)} \\ &= (1+l)^{\frac{1}{4\pi}\max\{x,y\}} \\ &\leq (1+l)^{\frac{3}{4\pi}\max\{x,y\}} \\ &\leq (1+l)^{\frac{1}{4}\max\{x,y\}} \\ &= (1+l)^{\psi(p(x,y))}.\end{aligned}$$

Otherwise, $\alpha(x,y)\beta(x,y) = 0$ and 1.3 trivially holds. Hence, f is a twisted $(\alpha,\beta)-\psi-$contractive mapping of type (II) and by Theorem 2.2, f has a fixed point. □

Example 2.5. Let $X = \mathbb{R}$ be endowed with the usual metric $d(x,y) = |x-y|$ for all $x,y \in X$ and $f : X \to X$ be defined by

$$fx = \begin{cases} \frac{1}{8}x, & if \quad x \in [0,1], \\ \frac{1}{x} - \frac{1}{1+x}, & if \quad x \in (1,\infty). \end{cases}$$

Define also $\alpha, \beta : X \times X \to [0,+\infty)$ by

$$\alpha(x,y) = \beta(x,y) = \begin{cases} 1, & if \quad x,y \in [0,1], \\ 0 & otherwise, \end{cases}$$

and $\psi : [0,+\infty) \to [0,+\infty)$ by $\psi(t) = \frac{1}{2}t$ for all $t \geq 0$. We prove that Theorem 2.2 can be applied to f.

Proof. Proceeding as in the proof of Example 2.3, we deduce that f is a twisted $(\alpha,\beta)-$admissible mapping and that the conditions (i) and (ii) of Theorem 2.2 hold. Moreover, if $x,y \in [0,1]$ and $l \geq 1$, we have

$$\begin{aligned}(p(fx,fy)+l)\alpha(x,y)\beta(x,y) &= \frac{1}{8}\max\{x,y\}+l \\ &= \frac{1}{8}\max\{x,y\}+l \\ &\leq \frac{1}{2}\max\{x,y\}+l = \psi(p(x,y))+l.\end{aligned}$$

Otherwise, $\alpha(x,y)\beta(x,y) = 0$ and 1.4 trivially holds. Hence, f is a twisted $(\alpha,\beta)-\psi-$contractive mapping of type (III) and, by Theorem 2.2, f has a fixed point. □

In the next result, we consider a hypothesis useful to obtain the uniqueness of the fixed point.

Theorem 2.6. *Assume that all the hypothesis of Theorem 2.1 (respectively Theorem 2.2) hold. Adding the following condition:*

(A) for all $x,y \in X$ with x,y there exists $v \in X$ such that $\alpha(x,v) \geq 1$, $\alpha(y,v) \geq 1$ and $\beta(x,v) \geq 1$, $\beta(y,v) \geq 1$,

we obtain the uniqueness of the fixed point of f.

Proof. Suppose that z and z^* are two fixed points of f such that z, z^*. By condition (A), there exists v such that $\alpha(z,v) \geq 1$ and $\alpha(z^*,v) \geq 1$. Therefore, since f is a twisted $(\alpha,\beta)-$admissible mapping, we deduce that $\alpha(f^{2n}z, f^{2n}v) \geq 1$, $\alpha(f^{2n-1}v, f^{2n-1}z) \geq 1$ and $\alpha(f^{2n}z^*, f^{2n}v) \geq 1$, $\alpha(f^{2n-1}v, f^{2n-1}z^*) \geq 1$. Similarly, we get $\beta(f^{2n}z, f^{2n}v) \geq 1$, $\beta(f^{2n-1}v, f^{2n-1}z) \geq 1$ and $\beta(f^{2n}z^*, f^{2n}v) \geq 1$, $\beta(f^{2n-1}v, f^{2n-1}z^*) \geq 1$.

Now, if f is a twisted $(\alpha, \beta) - \psi-$contractive mapping of type (I), then by 1.2 with $x = f^{2n}z$ and $y = f^{2n}v$ we have

$$\begin{aligned} p(ff^{2n}z, ff^{2n}v) &\leq \alpha(f^{2n}z, f^{2n}v)\beta(f^{2n}z, f^{2n}v)p(ff^{2n}z, ff^{2n}v) \\ &\leq \psi(p(f^{2n}z, f^{2n}v)). \end{aligned}$$

Similarly by 1.2 with $x = f^{2n-1}v$ and $y = f^{2n-1}z$ we get

$$p(ff^{2n-1}z, ff^{2n-1}v) \leq \psi(p(f^{2n-1}z, f^{2n-1}v)).$$

Hence for all $n \in \mathbb{N}$ we have

$$p(ff^n z, ff^n v) \leq \psi(p(f^n z, f^n v)),$$

or equivalently,

$$p(z, f^{n+1}v) \leq \psi^n(p(z, v)).$$

Of course, we get the same conclusion if we suppose that f is a twisted $(\alpha, \beta) - \psi-$contractive mapping of type (II) or (III) and so we omit the details. By taking the limit as $n \to +\infty$ in the above inequality we obtain

$$\lim_{n\to+\infty} p(z, f^{n+1}v) = 0.$$

Using a similar argument we also get

$$\lim_{n\to+\infty} p(z^*, f^{n+1}v) = 0.$$

From the last two limits and the triangular inequality we have

$$p(z, z^*) \leq \lim_{n\to+\infty} [p(z, f^{n+1}v) + p(z^*, f^{n+1}v) - p(f^{n+1}v, f^{n+1}v)] = 0,$$

that is, $z = z^*$. □

The following result is a consequence of Theorem 2.1.

Corollary 2.7. *Let (X, p) be a $0-$complete partial ordered metric space and let $f : X \to X$ be a continuous mapping. If there exists $\psi \in \Psi$ such that*

$$p(fx, fy) \leq \psi(p(x, y)), \tag{2.1}$$

holds for all $x, y \in X$, then f has a unique fixed point in X.

Proof. By taking $\alpha(x, y) = \beta(x, y) = 1$ for all $x, y \in X$ in Theorem 2.1, we deduce that f has a fixed point in X. The uniqueness of the fixed point follows easily from 2.1 and so we omit the details. □

3. Cyclic Results

In this section, we prove some analogous fixed point results involving a cyclic mapping. First, for our further use, we adapt Definition 1.10 as follows:

Definition 3.1. Let (X, p) be a partial metric space and A, B be two nonempty and closed subsets of X. Let $\alpha : X \times X \to [0, +\infty)$ and $f : A \cup B \to A \cup B$, with $fA \subseteq B$ and $fB \subseteq A$, such that $\alpha(fy, fx) \geq 1$ if $\alpha(x, y) \geq 1$, where $x \in A$ and $y \in B$. Thus f is said to be a

(a) cyclic $\alpha - \psi-$contractive mapping of type (I), if

$$\alpha(x, y)p(fx, fy) \leq \psi(p(x, y)), \tag{3.1}$$

holds for all $x \in A$ and $y \in B$, where $\psi \in \Psi$.

(b) cyclic $\alpha-\psi-$contractive mapping of type (II), if there is $0 < l \leq 1$ such that

$$(\alpha(x,y)+l)^{p(fx,fy)} \leq (1+l)^{\psi(p(x,y))} \tag{3.2}$$

holds for all $x \in A$ and $y \in B$, where $\psi \in \Psi$.

(c) cyclic $\alpha-\psi-$contractive mapping of type (III), if there is $l \geq 1$ such that

$$(p(fx,fy)+l)^{\alpha(x,y)} \leq \psi(p(x,y))+l \tag{3.3}$$

holds for all $x \in A$ and $y \in B$, where $\psi \in \Psi$.

Now, we prove the following result for a continuous cyclic mapping:

Theorem 3.2. *Let (X,p) be a $0-$complete partial metric space and A,B be two nonempty and closed subsets of X such that $f : A \cup B \to A \cup B$ is a continuous cyclic $\alpha-\psi-$contractive mapping of type (I) or (II) or (III). If there exists $x_0 \in A$ such that $\alpha(x_0, fx_0) \geq 1$, then f has a fixed point in $A \cap B$.*

Proof. Let $Y = A \cup B$ and $\beta : Y \times Y \to [0,+\infty)$ be the function defined by

$$\beta(x,y) = \begin{cases} 1 & if \quad x \in A \quad and \quad y \in B, \\ 0 & otherwise. \end{cases}$$

Then (Y,p) is a $0-$complete partial metric space and f is a twisted $(\alpha,\beta)-$admissible mapping. Now, if $x_0 \in A$ is such that $\alpha(x_0,fx_0) \geq 1$, then also $\beta(x_0,fx_0) \geq 1$ and hence all the hypothesis of Theorem 2.1 hold with $X = Y$. Consequently, f has a fixed point in $A \cup B$, say z. Since $z \in A$ implies $z = fz \in B$ and $z \in B$ implies $z = fz \in A$, then $z \in A \cap B$. □

Next we prove result by elimination continuity condition on cyclic $\alpha-\psi-$contractive mappings.

Theorem 3.3. *Let (X,p) be a $0-$complete partial metric space and A,B be two nonempty and closed subsets of X such that $f : A \cup B \to A \cup B$ is a cyclic $\alpha-\psi-$contractive mapping of type (I) or (II) or (III). Also suppose that the following conditions hold:*

(i) there exists $x_0 \in A$ such that $\alpha(x_0,fx_0) \geq 1$;

(ii) if $\{x_n\}$ is a sequence in X such that $\alpha(x_{2n},x_{2n+1}) \geq 1$ for all $n \in \mathbb{N}\cup\{0\}$ and $x_n \to x$ as $n \to +\infty$, then $\alpha(x_{2n},x) \geq 1$ for all $n \in \mathbb{N}\cup\{0\}$.

Then f has a fixed point in $A \cap B$.

Proof. Let $Y = A \cup B$ and define the function $\beta : Y \times Y \to [0,+\infty)$ as in the proof of Theorem 3.2. Let $\{x_n\}$ be a sequence in Y such that $\alpha(x_{2n},x_{2n+1}) \geq 1$ and $\beta(x_{2n},x_{2n+1}) \geq 1$ for all $n \in \mathbb{N}\cup\{0\}$ and $x_n \to x$ as $n \to +\infty$. Then $x_{2n} \in A$ and $x_{2n+1} \in B$. Now, since B is closed, then $x \in B$ and hence $\alpha(x_{2n},x) \geq 1$ and $\beta(x_{2n},x) \geq 1$. We deduce that all the hypothesis of Theorem 2.2 are satisfied with $X = Y$ and hence f has a fixed point. □

Corollary 3.4. *Let (X,p) be a $0-$complete partial metric space and A,B be two nonempty and closed subsets of X such that $f : A \cup B \to A \cup B$ is continuous, $fA \subseteq B$ and $fB \subseteq A$. If there exists $\psi \in \Psi$ such that*

$$p(fx,fy) \leq \psi(p(x,y)), \tag{3.4}$$

holds for all $x \in A$ and $y \in B$, then f has a unique fixed point in $A \cap B$.

Proof. By taking $\alpha(x,y) = 1$ for all $x \in A$ and $y \in B$ in Theorem 3.2, we deduce that f has a fixed point in $A \cap B$. The uniqueness of the fixed point follows easily from 3.4 and so we omit the details. Clearly, if $\psi(t) = ct$ for all $t > 0$ where $c \in (0,1)$, then Corollary 3.4 reduces to Theorem 1.1 of [8]. □

4. Cyclic ordered results

By using the similar arguments to those presented in the previous section, we are able to obtain results in the setting of partial ordered complete metric spaces.

Definition 4.1. Let $(X, p, \preceq)$ be a partial ordered metric space and A, B be two nonempty and closed subsets of X. Let $\alpha : X \times X \to [0, +\infty)$ and $f : A \cup B \to A \cup B$, with $fA \subseteq B$ and $fB \subseteq A$, such that $\alpha(fy, fx) \geq 1$ if $\alpha(x, y) \geq 1$, where $x \in A$ and $y \in B$. Then f is said to be a

(a) cyclic ordered $\alpha - \psi-$contractive mapping of type (I), if

$$\alpha(x, y)p(fx, fy) \leq \psi(p(x, y)) \tag{4.1}$$

holds for all $x \in A$ and $y \in B$ with $x \preceq y$, where $\psi \in \Psi$.

(b) cyclic ordered $\alpha - \psi-$contractive mapping of type (II), if there is $0 < l \leq 1$ such that

$$(\alpha(x, y) + l)^{p(fx,fy)} \leq (1 + l)^{\psi(p(x,y))} \tag{4.2}$$

holds for all $x \in A$ and $y \in B$ with $x \preceq y$, where $\psi \in \Psi$.

(c) cyclic ordered $\alpha - \psi-$contractive mapping of type (III), if there is $l \geq 1$ such that

$$(p(fx, fy) + l)^{\alpha(x,y)} \leq \psi(p(x, y)) + l \tag{4.3}$$

holds for all $x \in A$ and $y \in B$ with $x \preceq y$, where $\psi \in \Psi$.

Theorem 4.2. *Let $(X, p, \preceq)$ be a partial ordered complete metric space and A, B be two nonempty and closed subsets of X such that $f : A \cup B \to A \cup B$ is a decreasing continuous cyclic ordered $\alpha - \psi$-contractive mapping of type (I) or (II) or (III). If there exists $x_0 \in A$ such that $\alpha(x_0, fx_0) \geq 1$ and $x_0 \preceq fx_0$, then f has a fixed point in $A \cap B$.*

Proof. Consider the 0−complete partial metric space (Y, p) where $Y = A \cup B$ and define the function $\beta : Y \times Y \to [0, +\infty)$ by

$$\beta(x, y) = \begin{cases} 1 & if \quad x \in A \quad and \quad y \in B \quad with \quad x \preceq y, \\ 0 & otherwise. \end{cases} \tag{4.4}$$

Clearly, 1.2 (respectively, 1.3 or 1.4) holds for all $x, y \in Y$. Let $\beta(x, y) \geq 1$ for $x, y \in X$, then $x \in A$ and $y \in B$ with $x \preceq y$. It follows that $fx \in B$ and $fy \in A$ with $fy \preceq fx$, since f is decreasing. Therefore $\beta(fy, fx) \geq 1$, that is, f is a twisted $(\alpha, \beta)-$admissible mapping. Now, let $\alpha(x_0, fx_0) \geq 1$ with $x_0 \in A$ and $x_0 \preceq fx_0$. From $x_0 \in A$ we have $fx_0 \in B$ with $x_0 \preceq fx_0$, that is, $\beta(x_0, fx_0) \geq 1$. Then all the hypothesis of Theorem 2.1 hold with $X = Y$ and f has a fixed point in $A \cup B$, say z. Since $z \in A$ implies $z = fz \in B$ and $z \in B$ implies $z = fz \in A$, then $z \in A \cap B$. □

Theorem 4.3. *Let $(X, p, \preceq)$ be a 0−complete partial metric space and A, B be two nonempty and closed subsets of X such that $f : A \cup B \to A \cup B$ is a cyclic ordered $\alpha - \psi-$contractive mapping of type (I) or (II) or (III). Also suppose that the following conditions hold:*

(i) there exists $x_0 \in A$ such that $\alpha(x_0, fx_0) \geq 1$ and $x_0 \preceq fx_0$;

(ii) if $\{x_n\}$ is a sequence in X such that $\alpha(x_{2n}, x_{2n+1}) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$ and $x_n \to x$ as $n \to +\infty$, then $\alpha(x_{2n}, x) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$;

(iii) if $\{x_n\}$ is a sequence in X such that $x_{2n} \preceq x_{2n+1}$ for all $n \in \mathbb{N} \cup \{0\}$ and $x_n \to x$ as $n \to +\infty$, then $x_{2n} \preceq x$ for all $n \in \mathbb{N} \cup \{0\}$.

Then f has a fixed point in $A \cap B$.

Proof. Consider the 0−complete partial metric space (Y, p) where $Y = A \cup B$ and define the function $\beta : Y \times Y \to [0, +\infty)$ as in the proof of Theorem 4.2. Let $\{x_n\}$ be a sequence in X such that $\alpha(x_{2n}, x_{2n+1}) \geq 1$ and $\beta(x_{2n}, x_{2n+1}) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$ and $x_n \to x$ as $n \to +\infty$, then $x_{2n} \in A$ and $x_{2n+1} \in B$ with $x_{2n} \preceq x_{2n+1}$. Since B is closed and by (iii), we deduce that $x \in B$ and $x_{2n} \preceq x$, that is, $\beta(x_{2n}, x) \geq 1$. Since $\alpha(x_{2n}, x) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$, then all the hypothesis of Theorem 2.2 are satisfied and hence f has a fixed point. □

5. Application to functional equations

In this section, we denote by $B(W)$ the space of all bounded real-valued functions defined on the set W. Clearly, $B(W)$ endowed with the sup metric $p(h,k) = \sup_{x\in W}\{h(x), k(x)\}$ for all $h,k \in B(W)$, is a $0-$complete partial metric space.
In this setting, we discuss the problem of dynamic programming related to multistage process [6]. Indeed, this problem reduces to the problem of solving the functional equation

$$Q(x) = \sup_{y\in D}\{f(x,y) + K(x,y,Q(\tau(x,y)))\}, x \in W, \tag{5.1}$$

where $\tau : W \times D \to W$, $f : W \times D \to \mathbb{R}$, $K : W \times D \times \mathbb{R} \to \mathbb{R}$. Specifically, we will prove the following theorem:

Theorem 5.1. *Let $K : W \times D \times \mathbb{R} \to \mathbb{R}$ and $f : W \times D \to \mathbb{R}$ be two bounded functions and let $A : B(W) \to B(W)$ be defined by*

$$A(h)(x) = \sup_{y\in D}\{f(x,y) + K(x,y,h(\tau(x,y)))\}, \tag{5.2}$$

for all $h \in B(W)$ and $x \in W$. Assume that there exists $\theta : B(W) \times B(W) \to \mathbb{R}$ such that

(i) $\theta(h,k) \geq 0 \Longrightarrow \theta(A(k), A(h)) \geq 0$, where $h,k \in B(W)$,
(ii) $\max\{K(x,y,h(x)), K(x,y,k(x))\} \leq \psi(\max\{h(x), k(x)\})$, where $\psi \in \Psi$, $h,k \in B(W)$, $\theta(h,k) \geq 0, x \in W$ and $y \in D$,
(iii) if $\{h_n\}$ is a sequence in $B(W)$ such that $\theta(h_2n, h_{2n+1}) \geq 0$ for all $n \in \mathbb{N}\cup\{0\}$ and $h_n \to h^*$ as $n \to +\infty$, then $\theta(h_{2n}, h^*) \geq 0$ for all $n \in \mathbb{N}\cup\{0\}$,
(iv) there exists $h_0 \in B(W)$ such that $\theta(h_0, A(h_0)) \geq 0$.

Then the functional equation 5.1 has a bounded solution.

Proof. Note that $(B(W), p)$ is a 0-complete metric space. Let ϵ be an arbitrary positive number and $h_1, h_2 \in B(W)$ such that $\theta(h_1,h_2) \geq 0$, then there exist $y_1, y_2 \in D$ such that

$$A(h_1)(x) < f(x,y_1) + K(x,y_1,h_1(\tau(x,y_1))) + \epsilon, \tag{5.3}$$

$$A(h_2)(x) < f(x,y_2) + K(x,y_2,h_2(\tau(x,y_2))) + \epsilon, \tag{5.4}$$

$$A(h_1)(x) \geq f(x,y_2) + K(x,y_2,h_1(\tau(x,y_2))), \tag{5.5}$$

$$A(h_2)(x) \geq f(x,y_1) + K(x,y_1,h_2(\tau(x,y_1))). \tag{5.6}$$

Now, from 5.3 and 5.6, it follows easily that

$$\begin{aligned}\max\{A(h_1)(x), A(h_2)(x)\} &< \max\{K(x,y_1,h_1(\tau(x,y_1))), K(x,y_1,h_2(\tau(x,y_1)))\} + \epsilon\\ &\leq \max\{K(x,y_1,h_1(\tau(x,y_1))), K(x,y_1,h_2(\tau(x,y_1)))\} + \epsilon\\ &\leq \psi(\max\{h_1(x), h_2(x)\}) + \epsilon.\end{aligned}$$

Hence we get

$$\max\{A(h_1)(x), A(h_2)(x)\} < \psi(\max\{h_1(x), h_2(x)\}) + \epsilon. \tag{5.7}$$

This gives,

$$p(A(h_1), A(h_2)) \leq \psi(p(h_1, h_2)) + \epsilon. \tag{5.8}$$

Since $\epsilon > 0$ is arbitrary, then

$$p(A(h_1), A(h_2)) \leq \psi(p(h_1, h_2)).$$

Define

$$\alpha(h,k) = \beta(h,k) = \begin{cases} 1 \quad if \quad \theta(h,k) \geq 0, \quad where \quad h,k \in B(W), \\ 0 \quad otherwise. \end{cases}$$

Consequently, we have

$$\alpha(h_1, h_2)\beta(h_1, h_2)p(A(h_1), A(h_2)) \leq \psi(p(h_1, h_2)),$$

that is, A is a twisted $(\alpha, \beta) - \psi-$contractive mapping of type (I) with $\alpha(h,k) = 1$ for all $h, k \in B(W)$. Thus, by Theorem 2.2, A has a fixed point, that is, the functional equation 5.1 has a bounded solution. □

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