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Solutions for some nonlinear functional-integral equations with applications

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Abstract

In the present manuscript, we prove some results concerning the existence of solutions for some nonlinear functional-integral equations which contains various integral and functional equations that considered in nonlinear analysis and its applications. By utilizing the techniques of noncompactness measures, we operate the fixed point theorems such as Darbo's theorem in Banach algebra concerning the estimate on the solutions. The results obtained in this paper extend and improve essentially some known results in the recent literature. We also provide an example of nonlinear functional-integral equation to show the ability of our main result.

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1. Introduction and Preliminaries

Since the year 1922, Banach contraction principle, due to its simplicity and applicability, has became a very popular tool in modern analysis, especially in nonlinear analysis including its applications to differential and integral equations, variational inequality theory, complementarity problems, equilibrium problems, minimization problems and many others. Also, many authors have improved, extended and generalized this contraction principle in several ways. Existence of fixed points in ordered metric spaces has been initiated in 2004 by Ran and Reurings [17] further studied by Nieto and Rodriguez-Lopez [15]. Samet and Vetro [18] introduced the notion of fixed point of N order in case of single-valued mappings. It should be noted that

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through the coupled fixed point (for $N = 2$) and tripled fixed point (for $N = 3$) technique, we cannot solve a system with the following form:

$$\begin{aligned} x^3 + 6yz - 9x + 12 &= 0, \\ y^3 + 6xz - 9y + 12 &= 0, \\ z^3 + 6yx - 9z + 12 &= 0. \end{aligned}$$

In particular for $N = 3$ (Tripled case) i.e., Let $(X, \preceq)$ be partially ordered set and (X, d) be a complete metric space. We consider the following partial order on the product space $X^3 = X \times X \times X$.
It is well-known that the differential and integral equations that arise in many physical problems are mostly nonlinear and fixed point theory provides a powerful tool for obtaining the solutions of such equations which otherwise are difficult to solve by other ordinary methods.

Beside this Maleknejad et al. [13, 14] examined the existence of solutions for the nonlinear functional-integral equations (for short NLFIE) of the form

$$x(t) = g(t, x(t)) + f\left(t, \int_0^t u(t, s, x(s))ds, x(\alpha(t))\right) \tag{1.1}$$

and

$$x(t) = f(t, x(\alpha(t))) \int_0^t u(t, s, x(s))ds, \tag{1.2}$$

respectively, by availing the Darbo fixed-point theorem with suitable combination of measure of noncompactness defined in [1]. Banaś and Sadarangani [7] as well as Maleknejad et al. [12] discussed the existence of solutions for NLFIE

$$f\left(t, \int_0^t v(t, s, x(s))ds, x(\alpha(t))\right) .g\left(t, \int_0^a u(t, s, x(s))ds, x(\beta(t))\right). \tag{1.3}$$

Banaś and Rzepka [5, 6] dealt the existence of solutions of NLFIE and nonlinear quadratic Volterra integral equation

$$x(t) = f(t, x(t)) \int_0^t u(t, s, x(s))ds, \tag{1.4}$$

$$x(t) = p(t) + f(t, x(t)) \int_0^t v(t, s, x(s))ds, \tag{1.5}$$

respectively. The popular nonlinear Volterra integral equation and Urysohn integral equation are given as follows

$$x(t) = a(t) + \int_0^t u(t, s, x(s))ds, \tag{1.6}$$

$$x(t) = b(t) + \int_0^a v(t, s, x(s))ds, \tag{1.7}$$

respectively. Dhage [9] discussed the following nonlinear integral equation

$$x(t) = a(t) \int_0^a v(t, s, x(s))ds + \left(\int_0^t u(t, s, x(s))ds\right) \left(\int_0^a v(t, s, x(s))ds\right), \tag{1.8}$$

Moreover, the familiar quadratic integral equation of Chandrasekhar type [8] has the form

$$x(t) = 1 + x(t) \int_0^a \frac{t}{t+s} \phi(s) x(s) ds, \tag{1.9}$$

which is applicable in the theories of radiative transfer, neutron transport and kinetic energy of gases (see [8, 11]). In this paper, we study the existence of solutions of NLFIE

$$\begin{aligned} x(t) = &\left(q(t) + f(t, x(t), x(\theta(t))) + F\left(t, x(t), \int_0^t u(t, s, x(a(s)))ds, x(b(t))\right)\right) \\ &\times \left(r(t) + g(t, x(t), x(\vartheta(t))) + G\left(t, x(t), \int_0^a v(t, s, x(c(s)))ds, x(d(t))\right)\right) \end{aligned} \tag{1.10}$$

for $t \in [0, a]$. It is worthwhile mentioning that up to now equations 1.1-1.9 are a particular case of equation 1.10. Moreover, NLFIE 1.10 also involve with the functional equation of the first order having the form $x(t) = f(t, x(t), x(\theta(t)))$ and $x'(t) = g(t, x(t), x(\vartheta(t)))$. This paper investigates existence of solutions of NLFIE 1.10 under some relevant results of fixed point theorem for the product of two operators which satisfies the Darbo condition with suitable combination of a measure of noncompactness in the Banach algebra of continuous functions in the interval $[0, a]$. The existence results are interesting in themselves although their solutions are continuous and stable.

2. Definitions and preliminaries

This section is devoted to revise some data which will be required in our further circumstances. Let E is a real Banach space with the norm $\|\cdot\|$ and zero element θ'. Symbolically $B(x, r)$ represents the closed ball centered at x and with radius r, as well as we indicates by B_r the ball $B(\theta', r)$. The notation M_E appears for the family of all nonempty and bounded subsets of E and notation N_E also appears for its subfamily consisting of all relatively compact subsets. Additionally, if $X(\neq \phi) \subset E$ then the symbols $\overline{X}$, $ConvX$ in consideration of the closure and convex closure of X, respectively. We exercise the definition on the concept of a measure of noncompactness [1] as follows.

Definition 2.1. Let $X \in M_E$ and

$$\mu(X) = \inf\left\{\delta > 0 : X = \cup_{i=1}^m X_i \;\; with \;\; diam(X_i) \leq \delta, i = 1, 2, \ldots m\right\},$$

where for a fixed number $t \in [0, a]$, we denote

$$diamX(t) = \sup\{|x(t) - y(t)| : x, y \in X\},$$

Clearly, $0 \leq \mu(X) < 1$. $\mu(X)$ is called the Kuratowski measure of noncompactness.

Theorem 2.2. *Let $X, Y \in M_E$ and $\lambda \in \mathbb{R}$. Then*

(i) $\mu(X) = 0$ *if and only if* $X \in N_E$,
(ii) $X \subseteq Y) \Longrightarrow \mu(X) \subseteq \mu(Y)$,
(iii) $\mu(\overline{X}) = \mu(ConvX) \leq \mu(X)$,
(iv) $\mu(X \cup Y) = \max\{\mu(X), \mu(Y)\}$,
(v) $\mu(\lambda X) = |\lambda|\mu(X)$, *where* $\lambda X = \{\lambda x : x \in X\}$,
(vi) $\mu(X + Y) \leq \mu(X) + \mu(Y)$, *where* $X + Y = \{x + y : x \in X, y \in Y\}$,
(vii) $|\mu(X) - \mu(Y)| \leq 2d_h(X, Y)$, *where* $d_h(X, Y)$ *denotes the Hausdorff metric of* X *and* Y *, i.e.*

$$d_h(X, Y) = \max\{\sup_{y \in Y} d(y, X), \sup_{x \in X} d(x, Y)\},$$

where $d(.,.)$ *is the distance from an element of* E *to a set of* E.

Furthermore, every function $\mu : M_E \to [0, 1)$, satisfying conditions (i)-(vi) of Theorem 2.2, will be called a regular measure of noncompactness in the Banach space E (cf. [5]).
Now let us theorize that is a nonempty subset of a Banach space E and $S : \Omega \to E$ is a continuous operator, which transforms bounded subsets of Ω to bounded ones. Additionally, let μ be a regular measure of noncompactness in E.

Definition 2.3. The continuous operator S satisfies the Darbo condition with a constant K' with respect to measure μ provided

$$\mu(SX) \leq K'\mu(X),$$

for each $X \in M_E$ such that $x \subset \Omega$.
If $K' < 1$, then S i s called a contraction with respect to μ.

In the continuation, consider the space $C[0, a]$ is consisting of all real functions defined and continuous on the interval $[0, a]$. The space $C[0, a]$ is equipped with standard norm

$$\|x\| = \sup\{|x(t)| : t \in [0, a]\},$$

Evidently, the space $C[0, a]$ has also the structure of Banach algebra. Taking into our considerations, we will utilize a regular measure of noncompactness defined in [2] (cf. also [1]). Let us fix a set $X \in M_{C[0,a]}$. For $x \in X$ and for a given $\epsilon > 0$ denote by $w(x, \epsilon)$ the modulus of continuity of x, i.e.,

$$w(x, \epsilon) = \sup\{|x(t) - x(s)| : t, s \in [0, a], |t - s| \leq \epsilon\}.$$

Further, put

$$w(X, \epsilon) = \sup\{w(x, \epsilon) : x \in X\}, \qquad w_0(X) = \lim_{\epsilon \to 0} w(X, \epsilon).$$

The function $w_0(X)$ is a regular measure of noncompactness in the space $C[0, a]$, which can be shown in [6]. For our purposes we will require the following lemma and theorem [10, 2].

Lemma 2.4. *Let D be a bounded, closed and convex subset of E. If operator $S : D \to D$ is a strict set contraction, then S has a fixed point in D.*

Theorem 2.5. *Let us suppose that is a nonempty, bounded, convex and closed subset of $C[0, a]$ and the operators P and T transform continuously the set into $C[0, a]$, just like that $P(\Omega)$ and $T(\Omega)$ are bounded. Furthermore, let the operator $S = P.T$ transform into itself. If the each operators P and T satisfies the Darbo condition on the set with the constants K_1 and K_2, respectively, then the operator S satisfies the Darbo condition on Ω with the constant*

$$\|P(\Omega)\|K_2 + \|T(\Omega)\|K_1.$$

Remark 2.6. In Theorem 2.5, if $\|P(\Omega)\|K_2 + \|T(\Omega)\|K_1 < 1$, then S is a contraction with respect to the measure w_0 and has at least one fixed point in the set Ω.

Now we will identify solutions of the integral equation 1.10.

3. Main Result

In this section, we will study the solvability of NLFIE 1.10 for $x \in C[0, a]$, under the following hypotheses.

(A1) The function $q : [0, a] \to \mathbb{R}$ is continuous and bounded with $k = \sup_{t\in[0,a]} |q(t)|$.

(A2) The functions $f : [0, a] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}, F, G : [0, a] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are continuous and there exists nonnegative constants l, m such that

$$|f(t, 0, 0)| \leq l,$$

$$|g(t, 0, 0)| \leq l,$$

$$|F(t, 0, 0, 0)| \leq m,$$

$$|G(t, 0, 0, 0)| \leq m.$$

(A3) There exists the continuous functions $a_j : [0,a] \to [0,a]$, for $j = 1, 2, \dots 10$ such that

$$|f(t, x_1, x_2) - f(t, y_1, y_2)| \le a_1(t)|x_1 - y_1| + a_2(t)|x_2 - y_2|,$$

$$|g(t, x_1, x_2) - g(t, y_1, y_2)| \le a_3(t)|x_1 - y_1| + a_4(t)|x_2 - y_2|,$$

$$|F(t, x_1, y_1, x_2) - F(t, x_3, y_2, x_4)| \le a_5(t)|x_1 - x_3| + a_6(t)|y_1 - y_2| + a_7(t)|x_2 - x_4|,$$

$$|G(t, x_1, y_1, x_2) - G(t, x_3, y_2, x_4)| \le a_8(t)|x_1 - x_3| + a_9(t)|y_1 - y_2| + a_{10}(t)|x_2 - x_4|,$$

for all $t \in [0,a]$ and $x_1, x_2, x_3, x_4, y_1, y_2 \in \mathbb{R}$.

(A4) The functions $u = u(t, s, x(\acute{a}(s)))$ and $v = v(t, s, x(c(s)))$ act continuously from the set $[0,a] \times [0,a] \times \mathbb{R}$ into $\mathbb{R}$. Moreover, the functions $\theta, \acute{a}, b, c$ and d transform continuously the interval $[0,a]$ into itself.

(A5) There exists a nonnegative constant K such that

$$K = \max_j \{a_j(t) : t \in [0,a]\},$$

for $j = 1, 2, \dots 10$.

(A6) (Sublinear condition) There exists the constants ξ and η such that

$$|u(t, s, x(\acute{a}(s)))| \le \xi + \eta|x|,$$

$$|v(t, s, x(c(s)))| \le \xi + \eta|x|,$$

for all $t, s \in [0,a]$ and $x \in \mathbb{R}$.

(A7) $4\sigma\tau < 1$, for $\sigma = 4K + Ka\eta$ and $\tau = k + l + Ka\xi + m$.

Now we can formulate the main result of this paper.

Theorem 3.1. *Under the assumptions (A1) - (A7), NLFIE 1.10 has at least one solution in the Banach algebra $C = C[0,a]$.*

Proof. To prove this result using Theorem 2.5, we consider the operators P and T on the Banach algebra $C[0,a]$ in the following way:

$$(Px)(t) = q(t) + f(t, x(t), x(\theta(t))) + F\left(t, x(t), \int_0^t u(t, s, x(\acute{a}(s)))ds, x(b(t))\right),$$

$$(Tx)(t) = r(t) + g(t, x(t), x(\vartheta(t))) + G\left(t, x(t), \int_0^t v(t, s, x(c(s)))ds, x(d(t))\right),$$

for $t \in [0,a]$. Now, taking into account the assumptions (A1), (A2) and (A4), it is clear that P and T transforms the Banach algebra $C[0,a]$ into itself. Now, the operator S defined on the algebra $C[0,a]$ as follows

$$Sx = (Px).(Tx).$$

Definitely, S transform $C[0,a]$ into itself. Next, let us fix $x \in C[0,a]$, then using our imposed assumptions for $t \in [0, a]$, we obtain

$$
\begin{aligned}
& |(Sx)(t)| = |(Px)(t)| \times |(Tx)(t)| \\
= \; & | \; q(t) + f(t, x(t), x(\theta(t))) + F\left(t, x(t), \int_0^t u(t, s, x(á(s)))ds, x(b(t))\right) | \\
& \times | \; r(t) + g(t, x(t), x(\vartheta(t))) + G\left(t, x(t), \int_0^t v(t, s, x(c(s)))ds, x(d(t))\right) | \\
\leq \; & \{k + |f(t, x(t), x(\theta(t))) - f(t, 0, 0)| + |f(t, 0, 0)| \\
& + |F\left(t, x(t), \int_0^t u(t, s, x(á(s)))ds, x(b(t))\right) - F(t, 0, 0, 0)| + |F(t, 0, 0, 0)|\Big\} \\
& \times \{k + |g(t, x(t), x(\vartheta(t))) - g(t, 0, 0)| + |g(t, 0, 0)| \\
& + |G\left(t, x(t), \int_0^a v(t, s, x(c(s)))ds, x(d(t))\right) - G(t, 0, 0, 0)| + |G(t, 0, 0, 0)|\Big\} \\
\leq \; & \left(k + a_1(t)|x(t)| + a_2(t)|x(\theta(t))| + l + a_5(t)|x(t)| + a_6(t)\int_0^t |u(t, s, x(á(s)))|ds + a_7(t)|x(b(t))| + m\right) \\
& \times \left(k + a_3(t)|x(t)| + a_4(t)|x(\vartheta(t))| + l + a_8(t)|x(t)| + a_9(t)\int_0^a |v(t, s, x(c(s)))|ds + a_{10}(t)|x(d(t))| + m\right) \\
& \times\{k + 4K\|x\| + l + Ka(\xi + \eta\|x\|) + m\}.\{k + 4K\|x\| + l + Ka(\xi + \eta\|x\|) + m\} \\
\leq \; & \{(4K + Ka\eta)\|x\| + k + l + Ka\xi + m\}^2.
\end{aligned}
$$

Let $\sigma = 4K + Ka\eta$ and $\tau = k + l + ka\xi + m$, then from the above estimate, it follows that

$$\|Px\| \leq \sigma\|x\| + \tau, \tag{3.1}$$

$$\|Tx\| \leq \sigma\|x\| + \tau, \tag{3.2}$$

$$\|Sx\| \leq (\sigma\|x\| + \tau)^2, \tag{3.3}$$

for $x \in C[0, a]$.

From estimate 3.3, we conclude that the operator S maps the ball $B_r \subset C[0, a]$ into itself for $r_1 \leq r \leq r_2$, where

$$r_1 = \frac{1 - 2\sigma\tau - \sqrt{1 - 4\sigma\tau}}{2\sigma^2},$$

$$r_2 = \frac{1 - 2\sigma\tau + \sqrt{1 - 4\sigma\tau}}{2\sigma^2}.$$

In the following, we will assume that $r = r_1$. Moreover, let us observe that from estimates 3.1 and 3.2, we obtain

$$\|PB_r\| \leq \sigma r + r, \tag{3.4}$$

$$\|PB_r\| \leq \sigma r + r. \tag{3.5}$$

Now, we have to prove that the operator P is continuous on the ball B_r. To do this, fix $\epsilon > 0$ and take arbitrary $x, y \in B_r$ such that $\|x-y\| \leq \epsilon$. Then for $t \in [0,a]$, we have

$$\begin{aligned}
&|(Px)(t)-(Py)(t)| \\
&\leq |f(t,x(t),x(\theta(t)))-f(t,y(t),y(\theta(t)))| \\
&+|F\left(t,x(t),\int_0^t u(t,s,x(\acute{a}(s)))ds,x(b(t))\right)-F\left(t,y(t),\int_0^t u(t,s,y(\acute{a}(s)))ds,y(b(t))\right)| \\
&\leq a_1(t)|x(t)-y(t)|+a_2(t)|x(\theta(t))-y(\theta(t))| \\
&+|F\left(t,x(t),\int_0^t u(t,s,x(\acute{a}(s)))ds,x(b(t))\right)-F\left(t,x(t),\int_0^t u(t,s,x(\acute{a}(s)))ds,y(b(t))\right)| \\
&+|F\left(t,x(t),\int_0^t u(t,s,x(\acute{a}(s)))ds,y(b(t))\right)-F\left(t,y(t),\int_0^t u(t,s,y(\acute{a}(s)))ds,y(b(t))\right)| \\
&\leq a_1(t)|x(t)-y(t)|+a_2(t)|x(\theta(t))-y(\theta(t))|+a_5(t)|x(t)-y(t)| \\
&+a_7(t)|x(\theta(t))-y(\theta(t))|+a_6(t)\int_0^t |u(t,s,x(\acute{a}(s)))-u(t,s,y(\acute{a}(s)))|ds \\
&\leq 4K\|x-y\|+Kaw(u,\epsilon) \\
&\leq 4K\epsilon+Kaw(u,\epsilon),
\end{aligned}$$

where

$$w(u,\epsilon)=\sup\{|u(t,s,x)-u(t,s,y)| : t,s \in [0,a], x,y \in [-r,r], \|x-y\| \leq \epsilon\}.$$

In view of uniformly continuous of the function $u = u(t,s,x)$ on the bounded subset $[0,a] \times [0,a] \times [-r,r]$, we have that $w(u,\epsilon) \to 0$ as $\epsilon \to 0$. Thus, from the above inequality the operator P is continuous on B_r. Similarly, the operator T is also continuous on B_r. Hence, we conclude that S is continuous operator on B_r. Next, we prove that the operators P and T satisfies the Darbo condition with respect to the measure w_0, defined in Section 2, in the ball B_r. To do this, we take a nonempty subset X of B_r and $x \in X$. Let $\epsilon > 0$ be fixed and $t_1, t_2 \in [0,a]$ with $t_2 - t_1 \leq \epsilon$ and we can assume that $t_1 \leq t_2$. Then, taking into account our assumptions, it follows

$$\begin{aligned}
&|(Px)(t_2)-(Px)(t_1)| \\
&\leq |q(t_2)-q(t_1)|+|f(t_2,x(t_2),x(\theta(t_2)))-f(t_1,x(t_1),x(\theta(t_1)))| \\
&+|F\left(t_2,x(t_2),\int_0^{t_2} u(t_2,s,x(\acute{a}(s)))ds,x(b(t_2))\right)-F\left(t_1,x(t_1),\int_0^{t_1} u(t_1,s,x(\acute{a}(s)))ds,x(b(t_1))\right)| \\
&\leq w(q,\epsilon)+|f(t_2,x(t_2),x(\theta(t_2)))-f(t_2,x(t_1),x(\theta(t_1)))|+|f(t_2,x(t_1),x(\theta(t_1)))-f(t_1,x(t_1),x(\theta(t_1)))| \\
&+|F\left(t_2,x(t_2),\int_0^{t_2} u(t_2,s,x(\acute{a}(s)))ds,x(b(t_2))\right)|-F\left(t_2,x(t_1),\int_0^{t_1} u(t_1,s,x(\acute{a}(s)))ds,x(b(t_1))\right)| \\
&+|F\left(t_2,x(t_1),\int_0^{t_1} u(t_1,s,x(\acute{a}(s)))ds,x(b(t_1))\right)|-F\left(t_1,x(t_1),\int_0^{t_1} u(t_1,s,x(\acute{a}(s)))ds,x(b(t_1))\right)| \\
&\leq w(q,\epsilon)+a_1(t)|x(t_2)-x(t_1)|+a_2(t)|x(\theta(t_2))-x(\theta(t_1))|+w_f(\epsilon,.,.)+a_3(t)|x(t_2)-x(t_1)| \\
&+a_6(t)\int_0^{t_2} u(t_2,s,x(\acute{a}(s)))ds-\int_0^{t_1} u(t_1,s,x(\acute{a}(s)))ds|+a_7(t)|x(b(t_2))-x(b(t_1))|+w_F(\epsilon,.,.,.) \\
&\leq w(q,\epsilon)+2Kw(x,\epsilon)+Kw(x,w(\theta,\epsilon)) \\
&+w_f(\epsilon,.,.)+K\left\{\int_0^{t_1} |u(t_2,s,x(\acute{a}(s)))-u(t_1,s,x(\acute{a}(s)))|ds+\int_{t_1}^{t_2} |u(t_2,s,x(\acute{a}(s)))|ds\right\} \\
&+Kw(x,w(b,\epsilon))+w_F(\epsilon,.,.,.)w(Px,\epsilon) \leq w(q,\epsilon)+2Kw(x,\epsilon)+Kw(x,w(\theta,\epsilon)) \\
&+w_f(\epsilon,.,.)+K\{w_u(\epsilon,.,.)a+K'\epsilon\}+Kw(x,w(b,\epsilon))+w_F(\epsilon,.,.,.) \qquad (3.6)
\end{aligned}$$

where

$$
\begin{aligned}
w_f(\epsilon,.,.) &= \sup\{|f(t,x_1,x_2)-f(t_0,x_1,x_2)| : t,t_0\in[0,a], |t-t_0|\le\epsilon, x_1,x_2\in[-r,r]\},\\
w_u(\epsilon,.,.) &= \sup\{|u(t,s,x)-u(t_0,s,x)| : t,t_0\in[0,a], |t-t_0|\le\epsilon, x\in[-r,r]\},\\
w_F(\epsilon,.,.,.) &= \sup\{|F(t,x_1,y_1,x_2)-F(t_0,x_1,y_1,x_2)| : t,t_0\in[0,a],\\
&\qquad |t-t_0|\le\epsilon, x_1,x_2\in[-r,r], y_1\in[-K'a,K'a]\},\\
K' &= \sup\{|u(t,s,x)| : t,s\in[0,a], x\in[-r,r]\}.
\end{aligned}
$$

Since, the functions $q=q(t), f=f(t,x_1,x_2)$ and $F=F(t,x_1,y_1,x_2)$ are uniformly continuous on the set $[0,a], [0,a]\times\mathbb{R}\times\mathbb{R}$ and $[0,a]\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}$, respectively, and the function $u=u(t,s,x)$ is also uniformly continuous on the set $[0,a]\times[0,a]\times\mathbb{R}$. Hence, we deduce that$w(q,\epsilon)\to 0, w_f(\epsilon,.,.)\to 0, w_u(\times,.,.)\to 0$ and $w_F(\epsilon,.,.,.)\to 0$ as $\epsilon\to 0$. Thus, from the above estimate (3.6) we conclude

$$w_0(PX)\le 4Kw_0(X). \tag{3.7}$$

Similarly, we can show that

$$w_0(TX)\le 2Kw_0(X). \tag{3.8}$$

Finally, from the estimates 3.4, 3.5, 3.6, 3.7 and keeping in mind Theorem 2.5, we conclude that the operator S satisfies the Darbo condition on B_r with respect to the measure w_0 with constant $4K(\sigma r+\tau)+2K(\sigma r+\tau)$. Thus, we have

$$
\begin{aligned}
6K(\sigma r+\tau) = 6K(\sigma r_1+\tau) &= 6K\left\{\sigma\left(\frac{(1-2\sigma r)-\sqrt{1-4\sigma\tau}}{2\sigma^2}\right)+\tau\right\}\\
&= \frac{3K}{\sigma}(1-\sqrt{1-4\sigma\tau}).
\end{aligned}
$$

Taking into account the assumption (A7), since $1-\sqrt{1-4\sigma\tau}<1$ and $\frac{3K}{\sigma}=\frac{3K}{4K+Ka\eta}<1$, therefore, the operator S is a contraction on B_r with respect to measure w_0. Thus, S has at least one fixed point in the ball B_r, by applying Theorem 2.5 and Remark 2.6. Consequently, the NLFIE 1.10 has at least one solution in the ball B_r. □

4. Example

Now, we begin with an example of a NLFIE and to illustrate the existence of its solutions by using Theorem 3.1.

Example 4.1. Consider the following NLFIE.

$$
\begin{aligned}
x(t) = \Big[&te^{-(t+3)}+\frac{t}{7(1+t)}\arctan|x(t)|+\frac{t}{16}\ln(1+|x(1-t)|)\\
&+\frac{1}{12}\int_0^t\Big(\frac{\cos(x(1-s))}{3}+2t\arctan\Big(\frac{|x(1-s)|}{1+|x(1-s)|}\Big)ds\Big]\\
&\times\Big[\cos\Big(\frac{te^{-t}}{1+t}\Big)+\frac{1-t}{3(1+t)}\arctan|x(t)|+\frac{t}{13}\ln(1+|x(1+t)|).\\
&+\frac{1}{17}\int_0^1\Big(\frac{t\sin x(\sqrt{s})}{3}+(1+t)\ln(1+|x(\sqrt{s})|)\Big)ds\Big]
\end{aligned} \tag{4.1}
$$

where $t\in[0,1]$. Observe that equation 4.1 is a particular case of equation 1.10. Let us take $q:[0,1]\to\mathbb{R}$, $f:[0,1]\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$, $F,G:[0,1]\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ and $u,v:[0,1]\times[0,1]\times\mathbb{R}\to\mathbb{R}$ and comparing 4.1 with equation 1.10, we get

$$q(t)=te^{-(t+3)}, f(t,x_1,x_2)=\frac{t}{7(1+t)}\arctan|x_1|+\frac{t}{16}\ln(1+|x_2|),$$

$$r(t) = \cos\left(\frac{te^{-t}}{1+t}\right), g(t, x_1, x_2) = \frac{1-t}{3(1+t)} \arctan|x(t)| + \frac{t}{13}\ln(1+|x(1+t)|),$$

$$F(t, x_1, y_1, x_2) = \frac{1}{12}y_1, \quad G(t, x_1, y_1, x_2) = \frac{1}{17}y_1,$$

$$u(t, s, x) = \frac{\cos x}{3} + 2t\arctan\left(\frac{|x|}{1+|x|}\right), \quad v(t, s, x) = \frac{t\sin x}{3} + (1+t)\ln(1+|x|),$$

then we can easily test that the assumptions of Theorem 3.1 are satisfied. In fact, we have that the function $q(t)$ is continuous and bounded on $[0, 1]$ with $k = e^{-4} = 0.0183156\ldots$. Thus, the assumption $(A1)$ is satisfied. Moreover, these functions are continuous and satisfies the assumption $(A3)$ with $a_1 = \frac{1}{14}, a_2 = \frac{1}{16}, a_3 = a_5 = a_6 = a_8 = 0, a_4 = \frac{1}{12}, a_7 = \frac{1}{17}$. In this case, we have

$$K = \max\{\frac{1}{14}, \frac{1}{16}, \frac{1}{9}, \frac{1}{13}, \frac{1}{12}, 0, \frac{1}{17}\} = \frac{1}{9}.$$

Further,

$$|f(t, 0, 0)| = 0, |g(t, 0, 0)| = 0, |F(t, 0, 0, 0)| = 0, |G(t, 0, 0, 0)| = 0,$$

$$|u(t, s, x)| \leq \frac{1}{3} + 2|x|, \quad |v(t, s, x)| \leq \frac{1}{3} + 2|x|.$$

It is observed that $l = m = 0, \xi = \frac{1}{3}, \eta = 2$ and $a = 1$. Finally, we see that

$$4\sigma\tau = 4(4K + Ka\eta)(k + l + Ka\xi + m) < 1.$$

Hence, all the assumptions from $(A1)$ to $(A7)$ are satisfied. Now, on the basis of result obtained in Theorem 3.1, we deduce that NLFIE 4.1 has at least one solution in Banach algebra $C[0, 1]$.

Remark 4.2.

- If we take

$$q(t) = r(t) = g(t, x(t), x(\vartheta(t))) = 0,$$

$$G\left(t, x(t), \int_0^a v(t, s, x(c(s)))ds, x(d(t))\right) = 1, \quad f(t, x(t), x(\theta(t))) = f(t, x(t))$$

and

$$F\left(t, x(t), \int_0^t u(t, s, x(a(s)))ds, x(b(t))\right) = F\left(t, \int_0^t u(t, s, x(a(s)))ds, x(b(t))\right),$$

then equation 1.10 is in the following form

$$x(t) = f(t, x(t)) + F\left(t, \int_0^t u(t, s, x(a(s)))ds, x(b(t))\right),$$

which was studied by K. Maleknejad et.al. [13].

- If we take

$$q(t) = r(t) = f(t, x(t), x(\theta(t))) = g(t, x(t), x(\vartheta(t))) = 0,$$

$$G\left(t, x(t), \int_0^a v(t, s, x(c(s)))ds, x(d(t))\right) = G\left(t, \int_0^a v(t, s, x(c(s)))ds, x(d(t))\right)$$

and

$$F\left(t, x(t), \int_0^t u(t, s, x(a(s)))ds, x(b(t))\right) = F\left(t, \int_0^t u(t, s, x(a(s)))ds, x(b(t))\right),$$

then equation 1.10 is in the following form

$$x(t) = F\left(t, x(t), \int_0^t u(t, s, x(a(s)))ds, x(b(t))\right) \times G\left(t, \int_0^a v(t, s, x(c(s)))ds, x(d(t))\right),$$

which was studied by J. Banas, B. Rzepka [5] and J. Bana's, K. Sadarangani [7].

- If we take
$$q(t) = r(t) = g(t, x(t), x(\vartheta(t))) = 0,$$
$$G\left(t, x(t), \int_0^a v(t, s, x(c(s)))ds, x(d(t))\right) = 1, \quad f(t, x(t), x(\theta(t))) = f(t, x(t))$$
and
$$F\left(t, x(t), \int_0^t u(t, s, x(a(s)))ds, x(b(t))\right) = \int_0^t u(t, s, x(a(s)))ds,$$
then equation 1.10 is in the following form
$$x(t) = f(t, x(t)) + \int_0^t u(t, s, x(a(s)))ds,$$
which was studied by J. Banas, B. Rzepka [5].
- If we take
$$q(t) = r(t) = g(t, x(t), x(\vartheta(t))) = 0,$$
$$G\left(t, x(t), \int_0^a v(t, s, x(c(s)))ds, x(d(t))\right) = 1, \quad f(t, x(t), x(\theta(t))) = 0$$
and
$$F\left(t, x(t), \int_0^t u(t, s, x(a(s)))ds, x(b(t))\right) = F\left(t, x(t)\right) \int_0^t u(t, s, x(a(s)))ds,$$
then equation 1.10 is in the following form
$$x(t) = F\left(t, x(t)\right) \int_0^t u(t, s, x(a(s)))ds,$$
which was studied by K. Maleknejad et.al. [14].
- If we take
$$q(t) = r(t) = g(t, x(t), x(\vartheta(t))) = 0,$$
$$G\left(t, x(t), \int_0^a v(t, s, x(c(s)))ds, x(d(t))\right) = 1, \quad f(t, x(t), x(\theta(t))) = 0$$
and
$$F\left(t, x(t), \int_0^t u(t, s, x(a(s)))ds, x(b(t))\right) = a(t) + \int_0^t u(t, s, x(a(s)))ds,$$
in equation 1.10, then we get the following well known nonlinear Volterra integral equation
$$x(t) = a(t) + \int_0^t u(t, s, x(a(s)))ds.$$
- If we take
$$q(t) = r(t) = g(t, x(t), x(\vartheta(t))) = 0,$$
$$F\left(t, x(t), \int_0^t u(t, s, x(a(s)))ds, x(b(t))\right) = 1, \quad f(t, x(t), x(\theta(t))) = 0$$
and
$$G\left(t, x(t), \int_0^t v(t, s, x(c(s)))ds, x(d(t))\right) = b(t) + \int_0^a u(t, s, x(s))ds,$$
in equation 1.10, then we obtain Urysohn integral equation
$$x(t) = b(t) + \int_0^a v(t, s, x(s))ds.$$

- If we take

$$q(t) = r(t) = g(t, x(t), x(\vartheta(t))) = 0,$$

$$G\left(t, x(t), \int_0^a v(t, s, x(c(s)))ds, x(d(t))\right) = \int_0^a v(t, s, x(c(s)))ds, \quad f(t, x(t), x(\theta(t))) = 0$$

and

$$F\left(t, x(t), \int_0^t u(t, s, x(a(s)))ds, x(b(t))\right) = a(t) + \int_0^t u(t, s, x(a(s)))ds,$$

in equation 1.10, then we get the following form examined by B.C. Dhage [9]

$$x(t) = \left(a(t) + \int_0^t u(t, s, x(a(s)))ds\right)\left(\int_0^a v(t, s, x(c(s)))ds\right).$$

- Moreover, if

$$q(t) = r(t) = g(t, x(t), x(\vartheta(t))) = 0,$$

$$F\left(t, x(t), \int_0^t u(t, s, x(a(s)))ds, x(b(t))\right) = 1, \quad f(t, x(t), x(\theta(t))) = 0$$

and

$$G\left(t, x(t), \int_0^t v(t, s, x(c(s)))ds, x(d(t))\right) = 1 + \int_0^t v(t, s, x(c(s)))ds.x(d(t)),$$

$$x(d(t)) = t, \quad v(t, s, x(c(s))) = \frac{t}{t+s}\varphi(s),$$

in equation 1.10, then we obtain Urysohn integral equation

$$x(t) = 1 + x(t)\int_0^a \frac{t}{t+s}\varphi(s)x(s)ds.$$

The above equation is the famous quadratic integral equation of Chandrasekhar type [8] which is applied in the theories of radiative transfer, neutron transport and kinetic energy of gases.

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References

[1] J. Banaś, K. Goebel, *Measures of Noncompactness in Banach Spaces, Lecture Notes in Pure and Applied Mathematics*, **60**, Marcel Dekker, New York, (1980). 1, 2, 2

[2] J. Banaś, M. Lecko, *Fixed points of the product of operators in Banach algebra*, Panamer. Math. J., **12** (2002), 101–109. 2

[3] J. Banaś, M. Mursaleen, *Sequence spaces and measures of noncompactness with applications to differential and integral equations*, Springer, New York, (2014).

[4] J. Banaś, B. Rzepka, *An application of a measure of noncompactness in the study of asymptotic stability*, Appl. Math. Lett., **16** (2003), 1–6.

[5] J. Banaś, B. Rzepka, *On existence and asymptotic stability of solutions of a nonlinear integral equation*, J. Math. Anal. Appl., **284** (2003), 165–173. 1, 2, 4.2

[6] J. Banaś, B. Rzepka, *On local attractivity and asymptotic stability of solutions of a quadratic Volterra integral equation*, Appl. Math. Comput., **213** (2009), 102–111. 1

[7] J. Banaś, K. Sadarangani, *Solutions of some functional–integral equations in Banach algebra*, Math. Comput. Modelling, **38** (2003), 245–250. 1, 4.2

[8] S. Chandrasekhar, *Radiative Transfer*, Oxford Univ Press, London, (1950). 1, 1, 4.2

[9] B. C. Dhage, *On α–condensing mappings in Banach algebras*, Math. Student, **63** (1994), 146–152. 1, 4.2

[10] D. Guo, V. Lakshmikantham, X. Z. Liu, *Nonlinear Integral Equations in Abstract Spaces*, Kluwer, Dordrecht, (1996). 2

[11] S. Hu, M. Khavani, W. Zhuang, *Integral equations arising in the kinetic theory of gases*, Appl. Anal., **34** (1989), 261–266. 1

[12] K. Maleknejad, R. Mollapourasl, K. Nouri, *Study on existence of solutions for some nonlinear functional integral equations*, Nonlinear Anal., **69** (8) (2008), 2582–2588. 1

[13] K. Maleknejad, K. Nouri, R. Mollapourasl, *Existence of solutions for some nonlinear integral equations*, Commun. Nonlinear Sci. Numer. Simul., **14** (2009), 2559–2564. 1, 4.2

[14] K. Maleknejad, K. Nouri, R. Mollapourasl, *Investigation on the existence of solutions for some nonlinear functional- integral equations*, Nonlinear Anal., **71** (2009), e1575–e1578. 1, 4.2

[15] J. J. Nieto, R. Rodriguez-Lopez,*Contractive mapping theorems in partially ordered sets and applications to ordinary differential equations*, **22** (2005), 223–239. 1

[16] J. J. Nieto, R. Rodriguez-Lopez, *Existence and uniqueness of fixed point in partially ordered sets and applications to ordinary differential equations*, Acta Math. Sin.(Engl. Ser.), **23** (2007), 2205–2212. 1

[17] A. C. M. Ran, M. C. B. Reurings, *A fixed point theorem in partially ordered sets and some applications to matrix equations*, Proc. Amer. Math. Soc., **132** (2004), 1435–1443. 1

[18] B. Samet, C. Vetro,*Coupled fixed point theorems for multi-valued nonlinear contraction mappings in partially ordered metric spaces*, Nonlinear Anal., **74** (2011), 4260–4268. 1