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Solvability and asymptotic stability of a class of nonlinear functional-integral equation with feedback control

Payam Nasertayoob

Department of Mathematics and Computer Science, Amirkabir University of Technology (Polytechnic), Hafez Ave., P. O. Box 15914, Tehran, Iran.

Abstract

Using the technique of measure of noncompactness we prove the existence, asymptotic stability and global attractivity of a class of nonlinear functional-integral equation with feedback control. We will also include a class of examples in order to indicate the validity of the assumptions.

Keywords: Functional integral equation, measure of noncompactness, asymptotic stability, feedback control, fixed point.
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1. Introduction and Preliminaries

Recently, measures of noncompactness have been successfully applied to investigate the behavior of nonlinear functional integral equations [3, 5–8, 11]. Argyros [1] and Deimling [10] considered the equation

$$x(t) = f(t, x(t)) \int_0^1 \phi(t, s, x(s))ds. \tag{1.1}$$

In 2003, Banas and Rzepka [5, 6] investigated the Volterra counterpart of Eq. (1.1) on nubounded interval, i.e.,

$$x(t) = f(t, x(t)) + \int_0^t \phi(t, s, x(s))ds, \tag{1.2}$$

and

$$x(t) = f(t, x(t)) \int_0^t \phi(t, s, x(s))ds. \tag{1.3}$$

Email address: nasertayoob@aut.ac.ir (Payam Nasertayoob)

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In 2009, using Schauder fixed point theorem and based on the measure of noncompactness technique Banas and Rzepka [7] studied the existence and asymptotic stability of the nonlinear quadratic Volterra integral equation

$$x(t) = p(t) + f(t, x(t)) \int_0^t \phi(t, s, x(s))ds. \tag{1.4}$$

Very recently, Liu *at al.* [11] studied the more general nonlinear functional-integral equation

$$x(t) = f(t, x(t), \int_0^t \phi(t, s, x(a(s)), x(b(s)))ds). \tag{1.5}$$

On the other hands, in the more realistic situation, the biological systems or ecosystems are continuously perturbed via unpredictable forces. These perturbations are generally results of the change in the system's parameters. In the language of the control theory, these perturbation functions may be regarded as control variables and, consequently, one should ask the question that whether or not a system can withstand those unpredictable perturbations which persist for a finite periodic time.
In this paper we consider the following functional integral-equation with feedback control

$$\begin{aligned} x(t) &= f(t, x(t), \int_0^{\beta(t)} \phi(t, s, x(a(s)), x(b(s)), v(s))ds), \\ \frac{dv}{dt} &= -\alpha(t)v(t) + g(t, x(t)), \quad t \in \mathbb{R}_+, \end{aligned} \tag{1.6}$$

where $a = a(t)$, $b = b(t)$, $\phi = \phi(t, s, x_1, x_2, v)$, $f = f(t, x, w)$, $\beta = \beta(t)$, $\alpha = \alpha(t)$ and $g = g(t, x)$ are given, while $x = x(t)$ and $v = v(t)$ are unknown functions. Applying the Darbo fixed point theorem associated with measure of noncompactness defined by Banaś [2], and following the method of Chen [9] for study of the global attractivity of Lotka-Volterra competition system with feedback control we establish the existence, asymptotically stability and global attractivity of solutions for functional-integral equation with feedback control (1.6).

2. Preliminaries

In this section, we present some definitions and results which will be needed further on. Let $\mathbb{R}_+ = [0, \infty)$ and $(E, \|.\|)$ be an infinite dimensional Banach space with zero element θ. We write $B(x, r)$ to denote the closed ball centered at x with radius r and $\overline{X}$, $ConvX$ to denote the closer and closed convex hall of X. Let m_E denote the family of all nonempty bounded subsets of E and n_E indicate the family of all relatively compact sets. We use the following definition of a measure of noncompactness [4].

Definition 2.1. A mapping $\mu : m_E \to \mathbb{R}_+$ is said to be a measure of noncompactness in E if it satisfies the following conditions
(1) The family $\ker \mu = \{X \in m_E : \mu(X) = 0\}$ is nonempty and $\ker \mu \subseteq n_E$.
(2) $X \subseteq Y \Rightarrow \mu(X) \le \mu(Y)$.
(3) $\mu(\bar{X}) = \mu(X)$.
(4) $\mu(convX) = \mu(X)$.
(6) $\mu(\lambda X + (1-\lambda)Y) \le \lambda\mu(X) + (1-\lambda)\mu(Y) for \lambda \in [0, 1]$. If (X_n) is a sequence of close sets from m_E such that $X_{n+1} \subset X_n$, $(n = 1, 2, ...)$ and if $\lim_{n\to\infty} \mu(X_n) = 0$, then the intersection set $X_\infty = \bigcap_{n=1}^{\infty}$ is nonempty.

We will need the following fixed point theorem of darbo type [4].

Theorem 2.2. *Let Λ be a nonempty bounded closed convex subset of the space E and let $F : \Lambda \to \Lambda$ be a continuous operator such that $\mu(FA) \le k\mu(A)$ for each each nonempty subset A of Λ, where $k \in [0, 1)$ is a constant. Then F has at least one fixed point in Λ.*

We will work in the Banach space $BC(\mathbb{R}_+)$ consisting of all bounded and continuous functions on $\mathbb{R}_+$. The space $BC(\mathbb{R}_+)$ is equipped with the standard norm $\|x\| = \sup\{|x(t)| : t \in \mathbb{R}_+\}$.
Let X be a nonempty bounded subset of $BC(\mathbb{R}_+)$ and T be a positive number. For $x \in X$ and $\varepsilon \geq 0$, define

$$\omega^T(x,\varepsilon) = \sup\{|x(t) - x(s)| : s,t \in [0,T],\ \ |t-s| \leq \varepsilon\},$$

$$\omega^T(X,\varepsilon) = \sup\{\omega^T(x,\varepsilon) :\ x \in X\}, \qquad \omega_0^T(X) = \lim_{\varepsilon\to 0} \omega^T(X,\varepsilon)$$

$$\omega_0(X) = \lim_{T\to\infty} \omega_0^T(X), \qquad X(t) = \{x(t) : x \in X\},$$

$$\mathrm{diam} X(t) = \sup\{|x(t) - y(t)| : x, y \in X\}$$

and

$$\mu(X) = \omega_0(X) + \limsup_{t\to\infty} \mathrm{diam} X(t). \tag{2.1}$$

Banaś has shown in [2] that μ is a measure of noncompactness in the space $BC(\mathbb{R}_+)$.

Definition 2.3. The solutions of the system (1.6) are said to be locally attractive if there exist a ball $B(x_0,r)$ in the space $BC(\mathbb{R}_+)$ such that for arbitrary solutions $(x(t),v(t))$ and $(y(t),u(t))$ of system (1.6) belong to $B(x_0,r) \times BC(\mathbb{R}_+)$ we have that

$$\lim_{t\mapsto\infty} |x(t) - y(t)| = 0, \tag{2.2}$$

$$\lim_{t\mapsto\infty} |v(t) - u(t)| = 0. \tag{2.3}$$

The solution $(x(t),v(t))$ of the system (1.6) is said to be globally attractive if (2.2) hold for each solution $(y(t),u(t))$ of (1.6).

Definition 2.4. The solutions of (1.6) is said to be asymptotically stable (or uniformly locally attractive) if there exist a ball $B(x_0,r) \subseteq BC(\mathbb{R}_+)$ such that for any solutions $(x(t),v(t))$ and $(y(t),u(t))$ of system (1.6) belong to $B(x_0,r) \times BC(\mathbb{R}_+)$, condition (2.2) is uniformly satisfied with respect to $B(x_0,r)$, i.e., for each $\varepsilon > 0$ there exist $T > 0$ such that

$$|x(t) - y(t)| \leq \varepsilon,$$

where $x, y \in B(x_0,r)$, and $t \geq T$. In addition

$$\lim_{t\mapsto\infty} |v(t) - u(t)| = 0.$$

We emphasize that the definition above of stability is a combination of the definitions of asymptotic stability of solutions of nonlinear functional integral equations [5] and global attractivity of differential systems with positive solutions[9]. In fact, positive solutions of the second equation in system (1.6) under proper conditions are globally attractive. In the following Section, in order to transform the system (1.6) into one functional integral equation we will consider these conditions.

3. Main results

First we will consider system (1.6) under the following assumptions:
(H_1) $f : \mathbb{R}_+ \times \mathbb{R}^2 \to \mathbb{R}$ is continuous, $f(t,0,0) \in BC(\mathbb{R}_+)$ and there exist continuous functions m_1, m_2 and nondecreasing continuous function $\chi : \mathbb{R}_+ \to \mathbb{R}_+$ with $\chi(0) = 0$ such that

$$|f(t,x_1,y_1) - f(t,x_2,y_2)| \leq m_1(t)|x_1 - x_2| + \chi(m_2(t)|y_1 - y_2|), \tag{3.1}$$

and

$$M_1 = \sup\{m_1(t) : \ t \in \mathbb{R}_+\} < 1.$$

(H_2) function $g : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}_+$ satisfy the Lipschitz condition with respect to the second variable, i.e.,there is a constant k such that

$$|g(t, y) - g(t, x)| \leq k|y - x|, \qquad t \in \mathbb{R}_+. \tag{3.2}$$

(H_3) the functions $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ and $g : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}_+$ are continuous and

$$\alpha_L = \inf\{\alpha(t) : t \in \mathbb{R}_+\} > 0,$$

$$\alpha_M = \sup\{\alpha(t) : t \in \mathbb{R}_+\} < \infty,$$

$$g_L = \inf\{g(t, x(t)) : t \in \mathbb{R}_+, , x \in BC(\mathbb{R}_+)\} > 0,$$

$$g_M = \sup\{g(t, x(t)) : t \in \mathbb{R}_+, , x \in BC(\mathbb{R}_+)\} < \infty.$$

Remark 3.1.

For any function x belong to $BC(\mathbb{R}_+)$, the solution of the second equation in (1.6), denoted by $v_x(t)$, can be expressed as follow

$$v_x(t) = v_x(0) \exp\{-\int_0^t \alpha(s)ds\} + \int_0^t [g(s, x(s)) \exp\{-\int_s^t \alpha(\theta)d\theta]ds. \tag{3.3}$$

Chen [9] showed that under assumption (H_3) and with the positive initial condition $v_x(0) > 0$ the solution $v_x(t)$ is bounded above and blow by positive constants and globally attractive. Thus, we may define the following operator on the space $BC(\mathbb{R}_+)$,

$$(Fx)(t) = f(t, x(t), \int_0^{\beta(t)} \phi(t, s, x(a(s)), x(b(s)), v_x(s))ds), \quad t \in \mathbb{R}_+, \tag{3.4}$$

Therefor, the existence problem of solutions of system (1.6) can be converted into problem of existence of fixed point of operator (3.4).
From now on and for the sake of simplicity we will use the following notations

$$\Delta(t, x, y) = m_2(t) \int_0^{\beta(t)} |\phi(t, s, x(a(s)), x(b(s)), v_x(s)) - \phi(t, s, y(a(s)), y(b(s)), v_y(s))|ds,$$

$$\lambda(t, X) = \sup\{\Delta(t, x, y) : x, y \in X \subseteq BC(\mathbb{R}_+)\}. \tag{3.5}$$

(H_4) the function $\phi : \mathbb{R}_+^2 \times \mathbb{R}^3 \to \mathbb{R}$ is continuous and there exist positive constants M such that

$$\Delta(t, x, y) \leq M, \qquad t \in \mathbb{R}_+, \tag{3.6}$$

for any x, $y \in BC(\mathbb{R}_+)$ and $t \in \mathbb{R}_+$. Also

$$\lim_{t\to\infty} \lambda(t, BC(\mathbb{R}_+)) = 0. \tag{3.7}$$

Theorem 3.2. *Under assumptions (H_1)-(H_4), system (1.6) has at least one solution $(x(t), v(t)) \in BC(\mathbb{R}_+)\times BC(\mathbb{R}_+)$. Moreover, under positive initial condition $v(0) > 0$ solutions of (1.6) are asymptotically stable.*

Proof. By considering the conditions of theorem we infer that $Fx \in BC(\mathbb{R}_+)$ for any $x \in BC(\mathbb{R}_+)$. By using the conditions (H_1) and (H_3), for arbitrary $t \in \mathbb{R}_+$, we have

$$\begin{aligned}|(Fx)(t)| &= |f(t, x(t), \int_0^{\beta(t)} \phi(t, s, x(a(s)), x(b(s)), v_x(s))ds) - f(t, 0, 0)| + |f(t, 0, 0)| \\ &\le m_1(t)|x(t)| + \chi(m_2(t) \int_0^{\beta(t)} |\phi(t, s, x(a(s)), x(b(s)), v_x(s))|ds) + |f(t, 0, 0)| \\ &\le M_1|x(t)| + D.\end{aligned}$$

Where

$$D = \sup\{f(t, 0, 0) : \ t \in \mathbb{R}_+\} + \chi(2M). \tag{3.8}$$

Hence $Fx \in BC(\mathbb{R}_+)$. Inequality (3.8) yields that F transforms the ball $B_r = B(0, r)$ into itself where $r = \frac{D}{1-M_1}$. Now we show that F is continuous on the ball B_r. Let $\varepsilon > 0$ and $x, y \in B_r$ such that $\|x-y\| \le \varepsilon$, then

$$\begin{aligned}|(Fx)(t) - (Fy)(t)| &= |f(t, x(t), \int_0^{\beta(t)} \phi(t, s, x(a(s)), x(b(s)), v_x(s))ds) \\ &\quad - f(t, y(t), \int_0^{\beta(t)} \phi(t, s, y(a(s)), y(b(s)), v_y(s))ds)| \\ &\le M_1|x(t) - y(t)| + \chi(\Delta(t, x, y)),\end{aligned} \tag{3.9}$$

where $\Delta(t, x, y)$ is defined in (3.5). Considering conditions (H_1) and (H_3) there exist $T \ge 0$ such that for $t \ge T$ we have

$$\chi(\Delta(t, x, y)) \le \varepsilon \tag{3.10}$$

and then from (3.10) and (3.9) we have

$$|(Fx)(t) - (Fy)(t)| \le (M_1 + 1)\varepsilon, \qquad t \ge T.$$

On the other hand, in view of Remark 3.1 and keeping in mined assumption (H_4) as well as under positive initial condition there exist positive numbers l and L such that $0 < l \le v_x(t) \le L$ for any $t \in \mathbb{R}_+$ and $x \in B_r$. Using the uniform continuity of ϕ on $[0, T] \times [0, \beta_T] \times [-r, r]^2 \times [l, L]$ where $\beta_T = \sup\{\beta(t) : \ t \in [0, T]\}$ and considering assumption(H_1) we obtain

$$\lim_{\varepsilon \to 0} \chi(\Delta(t, x, y)) = 0.$$

Thus F maps the ball B_r continuously into itself. Let X be a nonempty subset of B_r, we show that $\mu(FX) \le M_1\mu(X)$. To do this, fix arbitrary $\varepsilon > 0$ and $T > 0$. Let t_1 and $t_2 \in [0, T]$ with $|t_1 - t_2| \le \varepsilon$ and x belong to X. we have

$$\begin{aligned}|(Fx)(t_2) - (Fx)(t_1)| &= |f(t_2, x(t_2), \int_0^{\beta(t_2)} \phi(t_2, s, x(a(s)), x(b(s)), v_x(s))ds) \\ &\quad - f(t_2, x(t_1), \int_0^{\beta(t_2)} \phi(t_2, s, y(a(s)), y(b(s)), v_y(s))ds)| \\ &\quad + |f(t_2, x(t_1), \int_0^{\beta(t_2)} \phi(t_2, s, y(a(s)), y(b(s)), v_y(s))ds) \\ &\quad - f(t_1, x(t_1), \int_0^{\beta(t_2)} \phi(t_2, s, y(a(s)), y(b(s)), v_y(s))ds)|\end{aligned}$$

$$
\begin{aligned}
&+|f(t_1,x(t_1),\int_0^{\beta(t_2)}\phi(t_2,s,y(a(s)),y(b(s)),v_y(s))ds)\\
&-f(t_1,x(t_1),\int_0^{\beta(t_2)}\phi(t_1,s,y(a(s)),y(b(s)),v_y(s))ds)|\\
&+|f(t_1,x(t_1),\int_0^{\beta(t_2)}\phi(t_1,s,y(a(s)),y(b(s)),v_y(s))ds)\\
&-f(t_1,x(t_1),\int_0^{\beta(t_1)}\phi(t_1,s,y(a(s)),y(b(s)),v_y(s))ds)|\\
&\le M_1|x(t_2)-x(t_1)|+\omega_r^T(f,\varepsilon)\\
&+\chi(m_2(t_1)|\int_0^{\beta(t_2)}\phi(t_2,s,y(a(s)),y(b(s)),v_y(s))ds)\\
&-\phi(t_1,s,y(a(s)),y(b(s)),v_y(s))ds)|)\\
&+\chi(m_2(t_1)|\int_{\beta(t_1)}^{\beta(t_2)}\phi(t_1,s,y(a(s)),y(b(s)),v_y(s))ds)|)\\
&\le M_1\omega^T(x,\varepsilon)+\omega_r^T(f,\varepsilon)+\chi(M_2\beta_T\omega_r^T(\phi,\varepsilon))+\chi(M_2\overline{\phi}\omega^T(\beta,\varepsilon)),
\end{aligned}
$$

where

$$\beta_T=\sup\{\beta(t):\ t\in[0,T]\},\qquad M_2=\sup\{m_2(t):\ t\in[0,T]\},$$

$$\overline{\phi}=\sup\{\phi(t,s,x,y,z):t\in[0,T],s\in[0,\beta_T],x,y\in[-r,r],z\in[l,L]\},$$

$$\omega_r^T(f,\varepsilon)=\sup\{|f(t,x,y)-f(s,x,y)|:s,t\in[0,T],|s-t|\le\varepsilon,x\in[-r,r],y\in[-\beta_T\overline{\phi},\beta_T\overline{\phi}]\},$$

$$\omega_r^T(\phi,\varepsilon))=\sup\{|\phi(t_1,s,x,y,z)-\phi(t_2,s,x,y,z)|:t\in[0,T],|t_2-t_1|\le\varepsilon,s\in[0,\beta_T],x,y\in[-r,r],z\in[l,L]\}.$$

By using the above estimate we have

$$\omega^T(FX,\varepsilon)\le M_1\omega^T(X,\varepsilon)+\omega_r^T(f,\varepsilon)+\chi(M_2\beta_T\omega_r^T(\phi,\varepsilon))+\chi(M_2\overline{\phi}\omega^T(\beta,\varepsilon)).$$

From the continuity of f and ϕ on the sets $[0,T]\times[-r,r]\times[-\beta_T\overline{\phi},\beta_T\overline{\phi}]$ and $[0,T]\times[0,\beta_T]\times[-r,r]^2\times[l,L]$, respectively, we have $\omega_r^T(f,\varepsilon)\to 0$ and $\omega_r^T(\phi,\varepsilon)$ as $\varepsilon\to 0$. Moreover, based on condition H_1 and continuity of β we have $\omega^T(\beta,\varepsilon)\to 0$ and

$$\chi(M_2\beta_T\omega_r^T(\phi,\varepsilon))+\chi(M_2\overline{\phi}\omega^T(\beta,\varepsilon))\to 0,$$

as $\varepsilon\to 0$. Therefore, we obtain

$$\omega_0^T(FX)\le M_1\omega_0^T(X).$$

Consequently, by taking $T\to\infty$ we have

$$\omega_0(FX)\le M_1\omega_0(X).$$

On the other hand, for $x,y\in X$ and $t\in\mathbb{R}_+$ we get

$$|(Fx)(t)-(Fy)(t)|\le M_1|x(t)-y(t)|+\chi(\lambda(t,X)),$$

where $\lambda(t,X)$ is defined by (3.5). By considering conditions (H_1) and (H_4) we deduce that

$$\limsup_{t\to\infty}\operatorname{diam}(FX)(t)\le M_1\limsup_{t\to\infty}\operatorname{diam}(X)(t).\tag{3.11}$$

Thus, based on definition of μ we obtain

$$\mu(FX)\le M_1\mu(X),\tag{3.12}$$

where $M_1 \in [0,1)$. Thus, based on Theorem 2.2 operator F has at least one fixed point $x \in B_r$. In addition, under the assumptions of Theorem 2.2 it can be deduced that the set $fixF$ of fixed points of the operator F is a member of the kernel $\ker \mu$ [2]. In fact, it follows from (3.7) that for given $\varepsilon \geq 0$ there exist $T \geq 0$ such that

$$\chi(\lambda(t, BC(\mathbb{R}_+))) \leq (1 - M_1)\varepsilon, \qquad t \geq T, \tag{3.13}$$

thus, if x and y are any fixed points of operator F belong to B_r, in view of conditions (H_1) and (H_4), we obtain

$$\begin{aligned}
|y(t) - x(t)| &= |(Fy)(t) - (Fx)(t)| \\
&\leq |f(t, y(t), \int_0^{\beta(t)} \phi(t, s, y(a(s)), y(b(s)), v_y(s))ds) \\
&- f(t, x(t), \int_0^{\beta(t)} \phi(t, s, x(a(s)), x(b(s)), v_x(s))ds)| \\
&\leq m_1(t)|y(t) - x(t)| + \chi(\lambda(t, B_r)) \\
&\leq M_1|y(t) - x(t)| + \chi(\lambda(t, BC(\mathbb{R}_+))) \\
&\leq M_1\varepsilon + (1 - M_1)\varepsilon = \varepsilon, \qquad t \geq T,
\end{aligned} \tag{3.14}$$

On the other hand, for $x,\ y \in B_r$ and $v = v_x,\ u = u_y \in BC(\mathbb{R}_+)$ with positive initial condition and assumption $\xi(t) = u(t) - v(t)$, taking into account condition (H_3) we have

$$\begin{aligned}
\dot{\xi} &= -\alpha(t)\xi(t) + g(t, x) - g(t, y) \\
&\leq -\alpha(t)\xi(t) + |g(t, x) - g(t, y)| \\
&\leq -\alpha(t)\xi(t) + k|x(t) - y(t)|.
\end{aligned}$$

In view of (3.14), for $t \geq T$ we obtain

$$\dot{\xi} \leq -\alpha(t)\xi(t) + k\varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we have

$$\dot{\xi} \leq -\alpha(t)\xi(t), \qquad t \geq T.$$

Similar to Lemma 1.4 of [9], noticing the fact $\alpha_L > 0$, it follows

$$\begin{aligned}
|\xi(t)| &\leq |\xi(0)| \exp\{-\int_T^t \alpha(s)ds\} \\
&\leq |\xi(0)| \exp\{-\alpha_L(t - T)\} \to 0,\ as\ t \to \infty.
\end{aligned} \tag{3.15}$$

Combining (3.14) and (3.15), solutions of system (1.6) are asymptotically stable. This completes the proof.

4. Examples and remarks

Example 4.1.

Consider the following class of nonlinear functional-integral equations with feedback control

$$\begin{aligned}
x(t) &= h(t) + \frac{H(t^n x(t))}{1 + t^{n+1}} + G(\int_0^{t^d} \frac{v + s^3 \cos(ts^3 x^3(s^2 \sin s) - x^2(s^5) + v))}{1 + t^m + \sqrt{t}\cos^2(x(s^5))}), \\
\frac{dv}{dt} &= -\frac{1 + t^2}{4 + \cos t + t^2} v + \frac{2 + \cos t + x^2(t)}{5 + \cos t + 2x^2(t)}, \qquad t \geq 0,
\end{aligned} \tag{4.1}$$

where h is a continuous bounded function, H and G satisfy the Lipschitz condition with Lipschitz constants $k_1 \le 1$ and k_2, respectively, $n, m, d > 0$ and $4d < m$. Also

$$f(t,x,w) = h(t) + \frac{H(t^n x)}{1+t^{n+1}} + G(w), \;\; a(t) = t^2 \sin t, \;\; b(t) = t^5, \;\; \beta(t) = t^d,$$

$$\phi(t,s,y,z,v) = \frac{v + s^3 \cos(ts^3 y^3 - z^2 + v))}{1 + t^m + \sqrt{t}\cos^2 z},$$

$$\alpha(t) = \frac{1+t^2}{4+\cos t + t^2}, \qquad g(t,x) = \frac{2+\cos t + x^2}{5 + \cos t + 2x^2}.$$

Simple calculation shows that g satisfies the Lipschitz condition, $\alpha_L \ge \frac{1}{5}$, $\alpha_M \le 1$, $g_L \ge \frac{1}{4}$ $g_M \le 1$ and

$$\begin{aligned}
&\sup\{m_2(t)\int_0^{\beta(t)} |\phi(t,s,x(a(s)),x(b(s)),(\psi x)(t)) \\
&- \phi(t,s,y(a(s)),y(b(s)),(\Psi y)(t))|ds : x,y \in BC(R_+)\} \\
&\le \frac{2(L-l)(1+t^m) + 2L\sqrt{t} + (1+t^m+\sqrt{t})t^{4d}}{2(1+t^m)^2} \to 0, \quad as \quad t \to \infty,
\end{aligned}$$

that shows the assumptions of Theorem 3.2 are valid. Thus each integral equation of class (4.1) has at least one solution and solutions of each equation are asymptotically stable.

Remark 4.2.

In addition to conditions (H_1)-(H_4), let function f satisfies the following condition

$$\sup\{f(t,x,0) : t \in \mathbb{R}_+, x \in \mathbb{R}\} = W < \infty. \tag{4.2}$$

Then the solutions of system (1.6) are globally attractive. This follows from the fact that for any $x \in BC(R_+)$ we have

$$\begin{aligned}
|(Fx)(t)| &= |f(t,x(t),\int_0^{\beta(t)} \phi(t,s,x(a(s)),x(b(s)),v_x(s))ds) - f(t,x(t),0)| + |f(t,x(t),0)| \\
&\le \chi(m_2(t)\int_0^{\beta(t)} |\phi(t,s,x(a(s)),x(b(s)),v_x(s))|ds) + W \\
&\le \chi(2M) + W = R, \qquad t \in \mathbb{R}_+.
\end{aligned}$$

This inequality shows that operator F maps $BC(\mathbb{R}_+)$ into ball $B_R = B(0,R)$ and consequently $fixF \subseteq B_R$. In addition, similar to the proof of Theorem 3.2 we can show that $\mu(FX) \le M_1\mu(X)$, for any nonempty set $X \subseteq B_R$. Since $fixF$ is a member of the kernel $ker\mu$ we obtain

$$\begin{aligned}
&\lim_{t\mapsto\infty} |x(t) - y(t)| = 0, \qquad x, y \in fixF \\
&\lim_{t\mapsto\infty} |v(t) - u(t)| = 0,
\end{aligned}$$

thus solutions of system (1.6) are globally attractive.

Example 4.3.

Let in Example 4.1 function H be bounded and satisfies the Lipschitz condition with Lipschitz constants $k_1 \le 1$. Then function f satisfies the condition (4.2). It follows from Remark 4.2 that, in such a case, system (4.1) has at least a solution and solutions of system (4.1) are globally attractive. For instance, we can consider the following of nonlinear functional-integral equations with feedback control

$$x(t) = h(t) + \frac{\sin(t^n x(t))}{1+t^{n+1}} + \arctan\left(\int_0^{t^d} \frac{v + s^3\cos(ts^3 x^3(s^2\sin s) - x^2(s^5) + v))}{1+t^m+\sqrt{t}\cos^2(x(s^5))}\right),$$
$$\frac{dv}{dt} = -\frac{1+t^2}{4+\cos t+t^2}v + \frac{2+\cos t + x^2(t)}{5+\cos t+2x^2(t)}, \qquad t \geq 0, \tag{4.3}$$

where $H(t) = \sin(t)$ and $f(t, x, 0) = h(t) + \frac{\sin(t^n x(t))}{1+t^{n+1}}$. This system, under conditions $n, m, d > 0$ and $4d < m$ has at least one solution. Moreover, the solutions are globally attractive.

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