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Nash equilibrium strategy for two-person zero-sum matrix games on credibility space

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Abstract

In this paper, firstly, we obtain the credibility measure of fuzzy trapezoidal variables. Also, we attain the expected value of fuzzy trapezoidal variables. Then, based on these theorems, we present the expected value Nash equilibrium strategy of the fuzzy games. In other words we extend the expected model to fuzzy trapezoidal variables and improve the previous researches in this area. However in some cases, the game don't have the Nash equilibrium strategy. Therefore, we investigate the existence of Pareto Nash equilibrium and weak Pareto Nash equilibrium strategies in this cases.

Keywords: Matrix game, fuzzy payoffs, Nash equilibrium, fuzzy trapezoidal variables.

1. Introduction

Modern game theory was developed by the mathematician John Neumann in the Mid-1940's. Neuman and Morgenstern [9] published the the book, that was considered to be the seminal work of game theory. Today, the idea of Nash equilibrium is central concept in game theory. In 1951, noncooperative game theory was presented by Nash [10].
Nash proves that if we approve mixed strategies, then every game with a finite number of players has at least one Nash equilibrium.
In the game theory, the Nash equilibrium is a solution concept of a non-cooperative game in volving two or more players in which each player is assumed to know the strategies of other players and no player hass anything to gain by changing only his or her own strategy [11]. But many situations often are not crisp and deterministic, for example some information and knowlage are usually represented by comparative deal. Zadeh [15] exhibited the idea of fuzziness that it's type of subjective uncertainty. He is pioneer in this category. Then fuzzy number became useful tool to gauge with incomplete information in engineering, social

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and economics. Dubios and Prade [5] used this idea in their work. Many of researches such as Zimmermann [16], Sakawa [12] and Yazenin [13], [14] applied this theory to optimization problems. Lately, Liu [6] founded theory in the uncertain environments and called such a theory uncertain programming. In 2003, Meada [8] constructed kind of concepts of minimax equilibrium strategies and investigated their properties. Li Cunlin and Zhang Qiang [3] investigated two-person zero-sum games in fuzzy environment. They obtained Nash equilibrium of this kind of game. They also obtained pareto Nash equilibrium strategy for fuzzy matrix game. Maeda and Cun Lin presented several kinds of equilibrium strategy for two-person zero-sum matrix games in the symmetric tringular fuzzy environment. Then, Bapi Dutta[2] extended this models in trapezoidal fuzzy environment. So, Ding Jian et al.[4] establshed the expected model of two-person zero-sum matrix games with fuzzy payoffs.
In this paper, we extend the expected model to fuzzy trapezoidal variables and present the expected value Nash eqilibrium strategy of the fuzzy games. This method simplify the previous ways.

2. Preliminaries

In this section, we give some basic concepts and results of credibility space, credibility measure, fuzzy variables and the expected value of fuzzy variables which are used throughout the paper.
Let P be a nonempty set, and $\mathcal{P}$ the power set of P (i.e., the larggest - algebra over P). Each element in $\mathcal{P}$ is called an event. In order to present an axiomatic definition of credibility, it is necessary to assign to each event A a number $Cr\{A\}$ which indicates the credibility that A will occur. In order to ensure that the number $Cr\{A\}$ has certain mathematical properties which we intuitively expect a credibility to have, we accept the following four axioms:

- Axiom 1. (Normality) $Cr\{P\} = 1$.
- Axiom 2. (Monotonicity) $Cr\{A\} \leq Cr\{B\}$ whenever $A \subset B$.
- Axiom 3. (Self-Duality) $Cr\{A\} + Cr\{A^c\} = 1$ for any event A.
- Axiom 4. (Maximality) $Cr\{\bigcup_i A_i\} = \sup_i Cr\{A_i\}$ for any events $\{A_i\}$ with $\sup_i Cr\{A_i\} < 0.5$.

Definition 2.1 ([7])**.** The set function $Cr : P \mapsto [0, 1]$ is called a credibility measure, if it satisfies the normality, monotonicity, self-duality, and maximality axioms.

Definition 2.2. Let P be a nonempty set, $\mathcal{P}$ the power set of P , and Cr a credibility measure. Then the triplet $(P, \mathcal{P}(P), Cr)$ is called a credibility space.

Definition 2.3 ([4])**.** A fuzzy variable is a (measurable) function from a credibility space$(P, \mathcal{P}(P), Cr)$to the set of real numbers $\mathbb{R}$.

Definition 2.4 ([7])**.** Let $\tilde{x}$ be a fuzzy variable on credibility space $(P, \mathcal{P}(P), Cr)$, its membership function $\mu_{\tilde{x}}(t)$ is derived from the credibility measure

$$\mu_{\tilde{x}}(t) = (2Cr\{\theta \in P|\tilde{x}(\theta) = t\}) \wedge 1, t \in \mathbb{R}.$$

Theorem 2.5 ([7])**.** *Let $\tilde{x}$ be a fuzzy variable on credibility space $(P, \mathcal{P}(P), Cr)$, its membership function is $\mu_{\tilde{x}}(t)$. For event B , then*

$$Cr(\tilde{x} \in B) = \frac{1}{2}(\sup_{t\in B} \mu_{\tilde{x}}(t) + 1 - \sup_{t\notin B} \mu_{\tilde{x}}(t))$$

3. Fuzzy variable and the Expected value of Fuzzy variable

In this section, we give some basic concepts and results of fuzzy trapezoidal variables. Then we prove the expected value of fuzzy trapezoidal variables.

Definition 3.1 ([2]). A fuzzy trapezoidal variable $\tilde{a} = (a, b, h, k)$ is a fuzzy set defined on the space of real number sets $\mathbb{R}$, whose membership function $\mu_{\tilde{a}} : \mathbb{R} \longrightarrow [0, 1]$ as following:

$$\mu_{\tilde{a}}(x) = \begin{cases} \frac{x-(a-h)}{h} & a-h \leq x \leq a, \\ 1 & a \leq x \leq b, \\ \frac{(b+k)-x}{k} & b \leq x \leq b+k, \\ 0 & otherwise, \end{cases}$$

where $h > 0$ is the left extention and $k > 0$ is the right extention.

Theorem 3.2. *Let $\tilde{a} = (a, b, h, k)$ be a fuzzy trapezoidal variable and $x \in \mathbb{R}$, then*

$$Cr\{\tilde{a} \geq x\} = \begin{cases} 1, & x < a-h, \\ 1 - \frac{1}{2}\mu_{\tilde{x}}(t), & a-h \leq x \leq a, \\ \frac{1}{2}, & a \leq x \leq b, \\ \frac{1}{2}\mu_{\tilde{x}}(t), & b \leq x \leq b+k, \\ 0, & x > b+k. \end{cases}$$

Proof. By Theorem 2.5, we have

$$Cr(\tilde{a} \geq x) = \frac{1}{2}(\sup_{x \leq t} \mu_{\tilde{a}}(t) + 1 - \sup_{x > t} \mu_{\tilde{a}}(t)).$$

So, we consider following conditions.
If $x \leq a - h$ then for $t = a$ we get

$$\begin{aligned} Cr(\tilde{a} \geq x) &= \frac{1}{2}(\sup_{x \leq t} \mu_{\tilde{a}}(t) + 1 - \sup_{x > t} \mu_{\tilde{a}}(t)) \\ &= \frac{1}{2}(1 + 1 - 0) \\ &= 1. \end{aligned}$$

If $a - h \leq x \leq a$ since the membership function of $\tilde{a}$ is strictly decreasing, then

$$\begin{aligned} Cr(\tilde{a} \geq x) &= \frac{1}{2}(\sup_{x \leq t} \mu_{\tilde{a}}(t) + 1 - \sup_{x > t} \mu_{\tilde{a}}(t)) \\ &= \frac{1}{2}(1 + 1 - \mu_{\tilde{a}}(t)) \\ &= 1 - \mu_{\tilde{a}}(t). \end{aligned}$$

In the interval of $[a, b]$ the membership function of $\tilde{a}$ is monotonous, thus

$$\begin{aligned} Cr(\tilde{a} \geq x) &= \frac{1}{2}(\sup_{x \leq t} \mu_{\tilde{a}}(t) + 1 - \sup_{x > t} \mu_{\tilde{a}}(t)) \\ &= \frac{1}{2}(1 + 1 - 1) \\ &= \frac{1}{2}. \end{aligned}$$

If $b \leq x \leq a$ the membership function of $\tilde{a}$ is strictly increasing, hence

$$\begin{aligned} Cr(\tilde{a} \geq x) &= \frac{1}{2}(\sup_{x \leq t} \mu_{\tilde{a}}(t) + 1 - \sup_{x > t} \mu_{\tilde{a}}(t)) \\ &= \frac{1}{2}(\mu_{\tilde{a}}(t) + 1 - 1) \\ &= \frac{1}{2}\mu_{\tilde{a}}(t). \end{aligned}$$

□

One of the ways to solve the fuzzy problems is to use the average value of fuzzy variables. So, is defined an expected operator for fuzzy variabels. L. Baoding and L. Kui [1] presented credibility measures. Based on it, they defined the expectation of the fuzzy variable as following form.

Definition 3.3 ([1]). Let $\tilde{x}$ be a fuzzy variable. Then the expected value of $\tilde{x}$ is defined by

$$E_c(\tilde{x}) = \int_0^{+\infty} Cr\{\tilde{x} \geq t\}dt - \int_{-\infty}^{0} Cr\{\tilde{x} \leq t\}dt,$$

provided that at least one of the two integrals is finite.

In the following, basic properties about the expected value of fuzzy variables is given.

Theorem 3.4 ([4]). *Let $\tilde{x}$, $\tilde{y}$ be a fuzzy variable, $\alpha, \beta \in \mathbb{R}$ is a constant, then*

$$E_c(\alpha\tilde{x} + \beta\tilde{y}) = \alpha E_c(\tilde{x}) + \beta E_c(\tilde{y}).$$

Theorem 3.5. *Let $\tilde{a} = (a, b, h, k)$ is a fuzzy trapezoidal variable, then*

$$E_c(\tilde{a}) = \frac{2a + 2b - h + k}{4}.$$

Proof. By using the Definition 3.3 and to purpose that $Cr\{\tilde{a} \leq t\} = 0$ for $t \leq 0$ and $a - h \geq 0$, we get

$$\begin{aligned} E_c(\tilde{a}) &= \int_0^{+\infty} Cr\{\tilde{a} \geq t\}dt - \int_{-\infty}^{0} Cr\{\tilde{a} \leq t\}dt \\ &= \int_0^{\infty} Cr\{\tilde{a} \geq t\}dt \\ &= \int_0^{a-h} 1dt + \int_{a-h}^{a} (1 - \frac{1}{2}\mu_{\tilde{a}}(t))dt + \int_a^b \frac{1}{2}dt + \int_b^{b+k} \mu_{\tilde{a}}(t) + \int_{b+k}^{+\infty} 0dt \\ &= \frac{1}{2}\big(a + b - \int_{a-h}^{a} \mu_{\tilde{a}}(t)\big)dt + \int_b^{b+k} \mu_{\tilde{a}}(t)dt) \\ &= \frac{1}{2}(a + b - \frac{h}{2} + \frac{k}{2}) = \frac{2a + 2b - h + k}{4}. \end{aligned}$$

So, complete the proof. □

Nash introduced non-cooperative games based on the idea that each player has a well-defined quantitative utility function. Actually, decision making problems always is made in uncertain environments. In this paper is used fuzzy variables to present the payoffs of the players, so the payoffs of strategy is modeled by a trapezoidal fuzzy variable $\tilde{a} = (a, b, h, k)$. So, expected payoffs of the players are fuzzy variables.
In this section, we shall consider two-person zero-sum games with fuzzy payoffs. Firstly, we define some useful notations. Let I, J denote players and let $M = \{1, 2, ..., m\}$ and $N = \{1, 2, ..., n\}$ be the sets of all

pure strategies available for player I and J, respectively. The sets of all mixed strategies available for players I and J by

$$S_I \equiv \{(x_1, x_2, ..., x_m) \in \mathbb{R}^m | x_i \geq 0, i = 1, 2, ..., m, \sum_{i=1}^{m} x_i = 1\},$$

$$S_J \equiv \{(y_1, y_2, ..., y_n) \in \mathbb{R}^n | y_j \geq 0, i = 1, 2, ..., n, \sum_{j=1}^{n} y_j = 1\}.$$

Let player I choose a mixed strategy $x \in S_I$ and player J choose mixed strategy $y \in S_J$. The variable $\tilde{a}_{ij} = (a_{ij}, b_{ij}, h_{ij}, k_{ij})$ indicates payoffs that player I receives and player J loses. The fuzzy payoff matrix of the game is given by

$$A = \begin{bmatrix} \tilde{a}_{11} & \cdots & \tilde{a}_{1n} \\ \vdots & \ddots & \vdots \\ \tilde{a}_{m1} & \cdots & \tilde{a}_{mn} \end{bmatrix}.$$

The fuzzy two-person zero-sum games is denoted by $\tilde{G} \equiv (S_I, S_J, \tilde{A})$. The values $x^T \tilde{A} y = \sum_{i=1}^{m} \sum_{j=1}^{n} x_i \tilde{a}_{ij} y_j$ are the expected that player I receives and player J loses. In the rest of this paper, we set $\tilde{A} = (\tilde{a}_{ij}), A = (a_{ij}), B = (b_{ij}), H = (h_{ij}), K = (h_{ij})$, where $\tilde{A}$ is $m \times n$ matrix whose (i, j)-th element is $\tilde{a}_{ij}$, A is $m \times n$ matrix whose (i, j)-th element is a_{ij} , H is $m \times n$ matrix whose (i, j)th element is h_{ij} and so B is $m \times n$ matrix whose (i, j)th element is b_{ij}.
According above statement, if the payoffs of the player are fuzzy variables, the expected incomes of the player, for example $x^T \tilde{a} y$, are fuzzy variables. The expected value model of the fuzzy two-person zero-sum games is presented as below.

Definition 3.6 ([4])**.** A pair $(\widehat{x}, \widehat{y}) \in S_I \times S_J$ is the expected Nash equilibrium strategy for the game $\tilde{G}$, if it holds that

$$i)\ \ E_c(x^T \tilde{A} \widehat{y}) \leq E_c(\widehat{x}^T \tilde{A} \widehat{y}), \quad \forall x \in S_I,$$
$$ii)\ E_c(\widehat{x}^T \tilde{A} \widehat{y}) \leq E_c(x^T \tilde{A} y), \quad \forall y \in S_J.$$

The point $E(\widehat{x}^T \tilde{A} \widehat{y})$ is the value of the game.

Theorem 3.7. *Let $\tilde{G}$ be a two-person zero-sum game with fuzzy payoffs, the pair $(\widehat{x}, \widehat{y})$ is the expected Nash equilibrium strategy of $\tilde{G}$ if and only if the followings hold:*

$$i)\ \ x^T(\frac{1}{2}A + \frac{1}{2}B - \frac{1}{4}H + \frac{1}{4}K)\widehat{y} \leq \widehat{x}(\frac{1}{2}A + \frac{1}{2}B - \frac{1}{4}H + \frac{1}{4}K)\widehat{y},$$
$$ii)\ \widehat{x}^T(\frac{1}{2}A + \frac{1}{2}B - \frac{1}{4}H + \frac{1}{4}K)\widehat{y} \leq \widehat{x}^T(\frac{1}{2}A + \frac{1}{2}B - \frac{1}{4}H + \frac{1}{4}K)y.$$

Proof. First, we suppose that $(\widehat{x}, \widehat{y})$ is the expected Nash equilibrium strategy of $\tilde{G}$ then we have

$$\begin{aligned} E_c(\widehat{x}\tilde{A}\widehat{y}) &= E_c(\sum_{i=1}^{m}\sum_{j=1}^{n} \widehat{x}(\tilde{a}_{ij})\widehat{y}) \\ &= \sum_{i=1}^{m}\sum_{j=1}^{n} \widehat{x} E_c(\tilde{a}_{ij})\widehat{y} \\ &= \sum_{i=1}^{m}\sum_{j=1}^{n} \widehat{x}(\frac{2a_{ij} + 2b_{ij} - h_{ij} + k_{ij}}{4})\widehat{y} \\ &= \frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{n} \widehat{x} a_{ij} \widehat{y} + \frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{n} \widehat{x} b_{ij} \widehat{y} - \frac{1}{4}\sum_{i=1}^{m}\sum_{j=1}^{n} \widehat{x} h_{ij} \widehat{y} \end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{4}\sum_{i=1}^{m}\sum_{j=1}^{n}\widehat{x}k_{ij}\widehat{y}\\
=&\frac{1}{2}\widehat{x}^T A\widehat{y}+\frac{1}{2}\widehat{x}^T B\widehat{y}-\frac{1}{4}\widehat{x}^T H\widehat{y}+\frac{1}{4}\widehat{x}^T K\widehat{y}\\
=&\widehat{x}^T(\frac{1}{2}A+\frac{1}{2}B-\frac{1}{4}H+\frac{1}{4}K)\widehat{y}.
\end{aligned}
$$

Analogously, we get

$$E_c(x^T\tilde{A}\widehat{y})=x^T(\frac{1}{2}A+\frac{1}{2}B-\frac{1}{4}H+\frac{1}{4}K)\widehat{y}.$$

From the definition of the expected Nash equilibrium strategy, we know that $E_c(x^T\tilde{A}\widehat{y})\leq E_c(\widehat{x}^T\tilde{A}\widehat{y})$ thus,

$$x^T(\frac{1}{2}A+\frac{1}{2}B-\frac{1}{4}H+\frac{1}{4}K)\widehat{y}\leq\widehat{x}^T(\frac{1}{2}A+\frac{1}{2}B-\frac{1}{4}H+\frac{1}{4}K)\widehat{y}.$$

Analogously, the condition (ii) holds. Overhand is obvious. □

If we adopt the following notations

$$A_0^L=A-H,\qquad B_0^U=B+K,$$

then we obtain following corollary that is direct result of previous theorem.

Corollary 3.8. *Let $\tilde{G}$ be a two-person zero-sum game with fuzzy payoffs. The pair $(\widehat{x},\widehat{y})$ is the expected Nash equilibrium strategy of $\tilde{G}$ if and only if the followings hold:*

$$
\begin{aligned}
&i)\ \ x^T\left(\frac{1}{4}(A_0^L+A+B+B_0^U)\right)\widehat{y}\leq\widehat{x}\left(\frac{1}{4}(A_0^L+A+B+B_0^U)\right)\widehat{y},\\
&ii)\ \widehat{x}^T\left(\frac{1}{4}(A_0^L+A+B+B_0^U)\right)\widehat{y}\leq\widehat{x}^T\left(\frac{1}{4}(A_0^L+A+B+B_0^U)\right)y.
\end{aligned}
$$

Example 3.9. Let $\tilde{G}$ be a fuzzy two-person zero-sum game with trapezoidal fuzzy payoff matrix $\tilde{A}$ given by

$$\tilde{A}=\begin{bmatrix}(50,60,10,20) & (80,96,16,32)\\(100,120,20,40) & (20,24,4,8)\end{bmatrix}.$$

Find the Nash equilibrium strategies forthe game $\tilde{G}$. By Theorem 3.7, if the pair (x,y) is the Nash equilibrium strategy of game $\tilde{G}$ it must satisfy the following inequalities

$$
\begin{aligned}
(0,1)\overline{F}y\leq x^T\overline{F}y,&\qquad (0,1)\overline{F}y\leq x^T\overline{F}y,\\
x^T\overline{F}y\leq x^T\overline{F}(1,0)^T,&\qquad x^T\overline{F}y\leq x^T\overline{F}(0,1)^T,
\end{aligned}
$$

where $\overline{F}=\begin{bmatrix}57.5 & 92\\115 & 23\end{bmatrix}$,$x=\begin{bmatrix}t\\1-t\end{bmatrix}$ and $y=\begin{bmatrix}s\\1-s\end{bmatrix}$. So the Nash equilibrium strategy of $\tilde{G}$ is given as below

$$\begin{cases}92(s-1)(1-t)-34.5(s-1)t\leq 0,\\57.5(t-1)s-69(t-1)(1-s)\leq 0,\end{cases}$$

then $(x,y)=\big((\frac{8}{11},\frac{3}{11}),(\frac{6}{11},\frac{5}{11})\big)$.

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