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Analytical Solutions of a Class of Generalised Lane-Emden Equations: Power Series Method Versus Adomian Decomposition Method

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Abstract

In this paper, we obtain highly accurate analytical solutions of a class of strongly nonlinear Lane-Emden equations using a power series method and the Adomian decomposition method. The nonlinear term of the proposed problem involves the integer powers of a continuous real-valued function $\Lambda(y(x))$. In each of the proposed methods, a unified result is presented for the function $\Lambda(y(x))$. The particular cases of the trigonometric functions $\Lambda(y(x)) = \tan y(x)$, $\sec y(x)$ and the hyperbolic functions $\Lambda(y(x)) = \tanh y(x)$, $\operatorname{sech} y(x)$ are considered explicitly using the proposed methods. Lane-Emden equations involving the first integer powers of these trigonometric and hyperbolic functions are given as examples to illustrate the reliability, efficiency and accuracy of the proposed methods. Numerical comparisons of the results obtained show excellent agreements between the two methods, an indication that both methods are accurate, effective, reliable and convenient in solving singular strongly nonlinear ordinary differential equations with appropriate initial conditions.

Keywords: Standard Lane-Emden equation, Strongly nonlinear Lane-Emden equation, Elementary functions, Integer powers of elementary functions, Power series method, Adomian decomposition method, Series solution.
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1. Introduction

The standard and classical Lane-Emden equation (LEE), a singular nonlinear second order ordinary differential equation, first introduced in 1870 by Lane and studied in 1907 by Emden, is the one specified

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by the initial value problem ([32], [48])

$$\begin{aligned} & y''(x) + \frac{2}{x}y'(x) + y^m(x) = 0, \quad 0 < x \leq 1; \\ & y(0) = 1, \quad y'(0) = 0, \end{aligned} \tag{1.1}$$

where real $m \geq 0$. This problem models the behaviour of spherical cloud ([32, 39]). LEE is of vital importance in the modelling of reaction-diffusion process and the solution of such LEE is useful in the optimisation of this process ([49]). In quantum mechanics and astrophysics, the values of m are physically significant and motivating and lie in the interval $[0, 5]$ ([39]). Clearly, equation (1.1) is linear when $m = 0, 1$ and it follows that the analytical solutions of the corresponding equations are achievable in closed-form. By extension, a closed-form solution is also obtainable for $m = 5$ (see, e.g., [32]).

Another classical and well-studied LEE is a pair of initial value problems (see, e. g., [14, 48])

$$\begin{aligned} & y''(x) + \frac{2}{x}y'(x) + e^y = 0, \quad 0 < x \leq 1, \\ & y(0) = 0, \quad y'(0) = 0; \end{aligned} \tag{1.2}$$

and

$$\begin{aligned} & y''(x) + \frac{2}{x}y'(x) + e^{-y} = 0, \quad 0 < x \leq 1, \\ & y(0) = 0, \quad y'(0) = 0. \end{aligned} \tag{1.3}$$

The problem (1.2) illustrates the isothermal gas spheres embedded in a pressurised medium at the maximum possible mass allowing for hydrostatic equilibrium; while the initial value problem (1.3) describes Richardson's model of thermionic current – the density and electric force of an electron gas in the neighbourhood of a hot body in thermal equilibrium ([18, 37]).

In recent years, LEE has further been studied by a number of researchers due to its important applications in mathematical physics, mathematical chemistry and astrophysics to include a broader nonlinear term ([27, 28, 38, 46, 48]):

$$\begin{aligned} & y''(x) + \frac{2}{x}y'(x) + f(y(x)) = 0 \\ & y(0) = A, \quad y'(0) = B, \end{aligned} \tag{1.4}$$

where A and B are constants, and $f(y)$ is a real-valued continuous function. Equation (1.4) with several variation of $f(y(x))$ has been used to model various occurences that include theory of stellar structure, thermal explosions, the thermal behaviour of a spherical cloud of gas, isothermal gas spheres and thermionic currents ([15, 18, 37]).

A variety of analytical methods have been used to solve LEE (1.4) of which (1.1) is a standard and classical case. Solutions of LEE have been obtained via: the series method (see e.g., [32], [36]), and in particular, Taylor series ([24]); homotopy perturbation method ([16, 42, 43, 50]); an integral operator([51]). The Adomian decomposition method (ADM) was used in [48] to investigate (1.1) and (1.4), where the nonlinear functions $f(y)$ are trigonometric, hyperbolic and exponential functions (see also [1]). In [19], the authors used variational iteration method (VIM) to solve differential equations arising in astrophysics of which LEE is an important special case (see also [23]). In [11], the coefficient of y' in LEE was re-written in the form of a new function such that the equation could be solved analytically in terms of the new function. In [26], all real solutions of LEE (1.1) for $m = 5$ were achieved in the form of Jacobian and Weierstrass elliptic functions. The solution of LEE within a reproducing kernel method and group preserving scheme was examined in [22]. See [25] for the Lagrangian formulation of LEE (1.4) and the subsequent analytical solutions via Noether symmetry classification and reduction.

For other methods of obtaining approximate analytical solutions and numerical solutions of Lane-Emden equations, see [30] (collocation method, CM); [29] (series expansion method, SEM and conformable HomotopyAdomian decomposition method, CHADM); [34] (conformable residual power series method, CRPSM); [41] (orthonormal Bernoulli's polynomials method, OBPM); [31, Subsection 5.2.2] (power series method, PSM). Other prominent papers that also discussed approximate solutions to LEEs include [13, 17, 40, 6, 39]. A hybrid numerical method combining with Chebyshev wavelets and a finite difference technique was used to solve LEEs in [33]. A shifted Jacobi-Gauss spectral collocation method (JSCM) was used in [12] to solve a class of nonlinear second order ordinary differential equations of Lane-Emden type; notable special cases such as shifted Chebyshev polynomials of the first and second kinds were considered for numerical comparison purposes. For Ulam stability for a nonlinear differential equation of Lane-Emden type with anti periodic conditions, see [5, 21, 44].

In a very recent paper [8], the first author of the present paper considered, using a power series method, approximate analytical solutions of a class of Lane-Emden equations whose nonlinear terms are given by Jacobi polynomials, of which Gegenbauer polynomials, Legendre polynomials, and Chebyshev polynomials of the first, second, third & fourth kinds are important special cases. In [9], which is also of recent, the approximate analytical solutions of a class of Lane-Emden equation whose nonlinear terms are described by trigonometric, hyperbolic and exponential functions were obtained using the Mittag-Leffler function method (see also [7]). In [48] (see also [1, 9, 21, 33, 35, 39]) the authors considered Lane-Emden equations with nonlinear terms given by the trigonometric functions $f(y) = \sin y, f(y) = \cos y$; and the hyperbolic functions $f(y) = \sinh y, f(y) = \cosh y$. In [10], the authors presented analytical and numerical solutions of Lane-Emden equations involving the integer powers of $\sin y, \cos y$; and $\sinh y, \cosh y$. In [10], a unified result is presented for an integer power of any continuous real-valued function, compared with the case by case computations for the nonlinear functions $f(y)$ presented in, e.g., [48].

In this paper, we use a power series method (PSM) and the Adomian decomposition method (ADM) to obtain analytical solutions of a class of generalised Lane-Emden equations. The nonlinearity of the proposed problem is strong and is given by the integer power of a continuous real-valued function $\Lambda(y(x))$, namely, $f(y(x)) = \Lambda^m(y(x))$, for integer $m \geq 0$, real $x > 0$. The special cases $\Lambda(y(x)) = \tan y(x)$, $\sec y(x)$; $\Lambda(y(x)) = \tanh y(x)$, $\operatorname{sech} y(x)$ are considered explicitly using the proposed methods. The results obtained in this paper, therefore, generalise several results in the literature including the aforementioned ones. The first powers with $m = 1, 2, 3, 4$ are explicitly presented as examples to illustrate the proposed methods. Numerical and graphical comparisons of results show a high level of agreements between the two proposed methods, which depict that the present methods are reliably efficient in solving a class of singular strongly nonlinear ordinary differential equations with appropriate initial conditions.

2. A Class of Strongly Nonlinear Lane-Emden Equations

This section presents the proposed strongly nonlinear Lane-Emden equation. To be precise, we are considering the initial value problem

$$\begin{aligned} &y''(x) + \frac{\omega}{x} y'(x) + \Lambda^m(y(x)) = 0, \quad \text{real } \omega \geq 0 \\ &y(0) = y_0, \quad y'(0) = 0 \quad (y_0 = 0 \text{ or } 1), \end{aligned} \tag{2.1}$$

with $0 < x \leq 1, m \in \mathbb{N}_0$. Obviously, the particular case $\Lambda(y(x)) = y(x), \omega = 2$; with the initial condition $y_0 = 1$ gives the standard Lane-Emden equation (1.1). Our approaches in solving the initial value problem (2.1) involve a power series method ([8]) and the Adomian decomposition method ([10]). For the power series method of solution, it is assumed that the function $\Lambda(y(x))$ admits a power series expansion so that one can use the generalised Cauchy product to obtain the integer power of the power series solution; whereas, in the case of the Adomian decomposition method, it is supposed that the power function $\Lambda(y(x))$ possesses derivatives of all orders evaluated at the given initial condition y_0.

Apart from presenting general and unified results for arbitrary nonlinear functions $\Lambda(y(x))$, we specialise the results to the cases of the trigonometric functions $\Lambda(y(x)) = \tan y(x)$, $\Lambda(y(x)) = \sec y(x)$; and the hyperbolic ones $\Lambda(y(x)) = \tanh y(x)$, $\Lambda(y(x)) = \operatorname{sech} y(x)$. For extensive discussions on models involving trigonometric and hyperbolic functions as well as the physical interpretations of their solutions, see [15, 18, 37].

3. A Power Series Method of Solution

In this section, we apply a power series method to obtain the approximate analytical solutions to the strongly nonlinear Lane-Emden initial value problem (2.1). As mentioned earlier, the function $\Lambda(y(x))$ is chosen in such a way that it can be expanded in a power series, in particular, the Maclaurin series.

Towards this end, let the function $\Lambda(y(x))$ be given by the power series

$$\Lambda(y(x)) = \sum_{p=0}^{\infty} a_p y^p(x), \tag{3.1}$$

where a_p, $p = 0, 1, 2, 3, \ldots$, are the expansion (Maclaurin) coefficients which are real.

We proceed by giving the following result that will be needed in the sequel.

Proposition 3.1 ([8, 9]). *Let the function $y(x)$ be given by a convergent series expansion about $x = 0$:*

$$y(x) = \sum_{k=0}^{\infty} \mathsf{b}_k x^k, \quad 0 < x \le 1. \tag{3.2}$$

Then the function $\Lambda(y(x))$ given by (3.1) admits the power series expansion

$$\Lambda(y(x)) = \sum_{\ell=0}^{\infty} \mathsf{c}_\ell x^\ell, \tag{3.3}$$

where the constants c_ℓ are given by

$$\mathsf{c}_\ell = \sum_{p=0}^{\infty} a_p \mathsf{B}_{\ell,p}. \tag{3.4}$$

Here the numbers $\mathsf{B}_{\ell,j}$, $j = 3, 4, 5, \ldots$ are given by the finite series

$$\mathsf{B}_{\ell,j} = \sum_{q_{j-1}=0}^{\ell} \sum_{q_{j-2}=0}^{q_{j-1}} \cdots \sum_{q_1=0}^{q_2} \mathsf{b}_{\ell-q_{j-1}} \mathsf{b}_{q_{j-1}-q_{j-2}} \cdots \mathsf{b}_{q_2-q_1} \mathsf{b}_{q_1}. \tag{3.5}$$

In particular, the special cases $\mathsf{B}_{\ell,j}$, $j = 0, 1, 2$, are given respectively by

$$\mathsf{B}_{\ell,0} = \begin{cases} 1, & \ell = 0, \\ 0, & \ell = 1, 2, 3, \cdots, \end{cases} \qquad \mathsf{B}_{\ell,1} = \mathsf{b}_\ell, \quad \mathsf{B}_{\ell,2} = \sum_{q_1=0}^{\ell} \mathsf{b}_{q_1} \mathsf{b}_{\ell-q_1}. \tag{3.6}$$

The value of $\mathsf{B}_{0,j}$, $= 1, 2, 3, \ldots$ depends on the number $y(0)$.

The following result given as a corollary follows immediately.

Corollary 3.2. *For $m \in \mathbb{N}_0$, we have*

$$\Lambda^m(y(x)) = \left(\sum_{\ell=0}^{\infty} \mathsf{c}_\ell x^\ell \right)^m = \sum_{\ell=0}^{\infty} \mathsf{C}_{\ell,m} x^\ell, \tag{3.7}$$

where the numbers $\mathsf{C}_{\ell,j}$, $j = 3, 4, 5, \dots$ *are given by the finite series*

$$\mathsf{C}_{\ell,j} = \sum_{q_{j-1}=0}^{\ell} \sum_{q_{j-2}=0}^{q_{j-1}} \cdots \sum_{q_1=0}^{q_2} \mathsf{c}_{\ell-q_{j-1}} \mathsf{c}_{q_{j-1}-q_{j-2}} \cdots \mathsf{c}_{q_2-q_1} \mathsf{c}_{q_1}. \tag{3.8}$$

Here the coefficients c_ℓ *are as given in* (3.4). *In particular, the special cases* $\mathsf{C}_{\ell,j}$, $j = 0, 1, 2$, *are given respectively by*

$$\mathsf{C}_{\ell,0} = \begin{cases} 1, & \ell = 0, \\ 0, & \ell = 1, 2, \cdots, \end{cases} \qquad \mathsf{C}_{\ell,1} = \mathsf{c}_\ell, \quad \mathsf{C}_{\ell,2} = \sum_{q_1=0}^{\ell} \mathsf{c}_{q_1} \mathsf{c}_{\ell-q_1}. \tag{3.9}$$

We now present the main result of this paper.

Theorem 3.3. *For* $0 < x \le 1$; $m \in \mathbb{N}_0$; *real* $\omega \ge 0$, *the Lane-Emden type problem*

$$\begin{aligned} & y''(x) + \frac{\omega}{x} y'(x) + \Lambda^m(y(x)) = 0 \\ & y(0) = y_0, \quad y'(0) = 0, \quad y_0 = 0 \text{ or } 1, \end{aligned} \tag{3.10}$$

admits an analytical solution given by the power series

$$y(x) = y_0 + \sum_{\ell=0}^{\infty} \mathcal{B}^m_{2\ell+2,\omega} x^{2\ell+2}, \tag{3.11}$$

where

$$\mathsf{b}_0 = y_0, \quad \mathcal{B}^m_{2\ell+2,\omega} := \mathsf{b}_{2\ell+2} = -\frac{\mathsf{C}_{2\ell,m}}{(2\ell+2)(\omega+2\ell+1)}, \quad \ell \ge 0. \tag{3.12}$$

The coefficients $\mathsf{C}_{k,m}$, $k = 0, 1, 2, 3, \dots; m = 1, 2, 3, \dots,$ *are as given in Corollary 3.2, with the following special cases:*

$$\mathsf{C}_{0,m} = (\mathsf{c}_0)^m = \begin{cases} (a_0 + a_1 + a_2 + \dots)^m, & y_0 = 1, \\ (a_0)^m, & y_0 = 0, \end{cases} \tag{3.13}$$

$$\mathsf{C}_{0,1} = \mathsf{c}_0 = \sum_{p=0}^{\infty} a_p \mathsf{B}_{0,p}, \quad \mathsf{B}_{0,p} = (\mathsf{b}_0)^p = (y_0)^p, \quad \mathsf{B}_{0,m} = \begin{cases} 1, & y_0 = 1, \\ 0, & y_0 = 0. \end{cases} \tag{3.14}$$

Proof. The starting point is to substitute the power series (3.2) into the equation (3.10) and then use Proposition 3.1; we see that

$$\begin{aligned} & \mathsf{b}_0 = y_0, \qquad \mathsf{b}_1 = 0 \\ & \sum_{k=2}^{\infty} k(k-1)\mathsf{b}_k x^{k-2} + n \sum_{k=1}^{\infty} k \mathsf{b}_k x^{k-2} + \sum_{\ell=0}^{\infty} \mathsf{C}_{\ell,m} x^\ell = 0. \end{aligned} \tag{3.15}$$

Introducing an appropriate change in index in equation (3.15) so that all terms in the equation have the same form, gives

$$\sum_{\ell=0}^{\infty} (\ell+2)(\ell+1)\mathsf{b}_{\ell+2} x^\ell + n \sum_{\ell=0}^{\infty} (\ell+2)\mathsf{b}_{\ell+2} x^\ell + \sum_{\ell=0}^{\infty} \mathsf{C}_{\ell,m} x^\ell = 0, \tag{3.16}$$

which implies that

$$\sum_{\ell=0}^{\infty}\left((\ell+2)(\ell+1)\mathsf{b}_{\ell+2}+n(\ell+2)\mathsf{b}_{\ell+2}+\mathsf{C}_{\ell,m}\right)x^{\ell}=0. \tag{3.17}$$

Equating the coefficients of each power $x^{\ell}, \ell=0,1,2,\cdots$ to zero we obtain the recurrence relation

$$\mathsf{b}_{\ell+2}=-\frac{\mathsf{C}_{\ell,m}}{(\ell+2)(\omega+\ell+1)}, \quad \ell=0,1,2,3,\dots, \tag{3.18}$$

where the coefficients $\mathsf{C}_{\ell,m}$ are as given in Corollary 3.2, and this completes the proof of the theorem. □

We proceed now with the explicit consideration and simplification of Theorem 3.3 when the functions $\Lambda(y)$ are the trigonometric functions $\tan y(x)$, $\sec y(x)$; and the hyperbolic functions $\tanh y(x)$, $\operatorname{sech} y(x)$. The Maclaurin series representations for these elementary functions will be employed. Interestingly, these Maclaurin coefficients are explicitly described by Bernoulli and Euler numbers.

3.1. Lane-Emden Equation Involving Tangent Function $\Lambda(y(x))=\tan y(x)$

This subsection treats Lane-Emden equations involving the integer power of the tangent function $\Lambda(y(x))=\tan y(x)$. Special Lane-Emden equations involving the first powers $\tan^m y(x), m=1,2,3,4$ are considered as examples.

Corollary 3.4. *For $m\in\mathbb{N}_0, \omega\in\mathbb{R}$, the initial value problem*

$$y''(x)+\frac{\omega}{x}y'(x)+\tan^m(y(x))=0, \quad y(0)=1, \;\; y'(0)=0, \quad 0<x\le 1 \tag{3.19}$$

admits the series solution

$$y(x)=1+\sum_{\ell=0}^{\infty}\mathcal{B}^m_{2\ell+2,\omega}x^{2\ell+2}, \tag{3.20}$$

where the coefficients $\mathcal{B}^m_{2\ell+2,\omega}$ are given by (3.12) with

$$\mathsf{C}_{0,m}=(a_1+a_3+a_5+a_7+\cdots)^m=\tan^m(1)=(1.55741)^m \tag{3.21}$$

$$\mathsf{C}_{\ell,m}=\sum_{s_{m-1}=0}^{\ell}\sum_{s_{m-2}=0}^{s_{m-1}}\cdots\sum_{s_1=0}^{s_2}\mathsf{c}_{\ell-s_{m-1}}\mathsf{c}_{s_{m-1}-s_{m-2}}\cdots\mathsf{c}_{s_2-s_1}\mathsf{c}_{s_1}, \quad \ell=2,4,6,\dots \tag{3.22}$$

$$\mathsf{C}_{\ell,1}=\mathsf{c}_{\ell}=a_1\mathsf{B}_{\ell,1}+a_3\mathsf{B}_{\ell,3}+a_5\mathsf{B}_{\ell,5}+a_7\mathsf{B}_{\ell,7}+\cdots \quad \ell=2,4,6,\dots. \tag{3.23}$$

Here the coefficients $a_{2k+1}, k=0,1,2,3,\dots$ are defined by

$$\tan y(x)=\sum_{k=0}^{\infty}a_{2k+1}y^{2k+1}(x)=\sum_{k=0}^{\infty}(-1)^k\frac{2^{2k+2}\left(2^{2k+2}-1\right)B_{2k+2}}{(2k+2)!}y^{2k+1}(x), \tag{3.24}$$

where B_j are the jth Bernoulli numbers.

Corollary 3.4 does not reveal explicitly the values of the coefficients $\mathsf{C}_{\ell,m}$ and by extension those of the expansion coefficients $\mathcal{B}^m_{2\ell+2,\omega}$ for certain values of the indices ℓ, m. In what follows we compute these coefficients for the first integers $m=1,2,3,4$. We present them as examples.

Example 3.5. Consider the Lane-Emden problem

$$y''(x)+\frac{\omega}{x}y'(x)+\tan(y(x))=0, \quad y(0)=1, \;\; y'(0)=0, \quad 0<x\le 1. \tag{3.25}$$

It suffices to compute the expansion coefficients $\mathcal{B}^1_{2\ell+2,\omega}$ of the series solution of (3.25). We do the computations for $\ell=0,1,2,3,4$ as follows.

- ($\ell=0$) In this case it is seen using the formula (3.21) (with $m=1$) that

$$\mathcal{B}^1_{2,\omega}=-\frac{\tan(1)}{2(\omega+1)}=-\frac{0.778704}{\omega+1}. \tag{3.26}$$

- ($\ell=1$) One sees clearly here that $\mathcal{B}^1_{4,\omega}=-\mathsf{C}_{2,1}/(4(\omega+3))$ with

$$\mathsf{C}_{2,1}=\mathsf{c}_2=a_1\mathsf{B}_{2,1}+a_3\mathsf{B}_{2,3}+a_5\mathsf{B}_{2,5}+a_7\mathsf{B}_{2,7}+\cdots. \tag{3.27}$$

Here the coefficients $\mathsf{B}_{2,m}$, $m=1,3,5,\ldots$ are given by

$$\mathsf{B}_{2,1}=\mathcal{B}^1_{2,\omega},\quad \mathsf{B}_{2,3}=3\mathcal{B}^1_{2,\omega},\quad \mathsf{B}_{2,5}=5\mathcal{B}^1_{2,\omega},\quad \mathsf{B}_{2,7}=7\mathcal{B}^1_{2,\omega},\quad \cdots; \tag{3.28}$$

and consequently, (3.27) gives

$$\begin{aligned}\mathsf{C}_{2,1}&=-\frac{\tan(1)}{2(\omega+1)}\left(a_1+3a_3+5a_5+7a_7+\cdots\right)\\ &=-\frac{\tan(1)}{2(\omega+1)}\sum_{m=0}^{\infty}\frac{(2m+1)(-1)^m2^{2m+2}\left(2^{2m+2}-1\right)B_{2m+2}}{(2m+2)!}\\ &=-\frac{\tan(1)}{2(\omega+1)}\left(4\csc^2(2)-\csc^2(1)\right)=-\frac{2.66746}{\omega+1}.\end{aligned}$$

As a result we obtain the value

$$\mathcal{B}^1_{4,\omega}=\frac{\tan(1)\left(4\csc^2(2)-\csc^2(1)\right)}{8(\omega+1)(\omega+3)}=\frac{0.666866}{(\omega+1)(\omega+3)}. \tag{3.29}$$

- ($\ell=2$) It is understood here that $\mathcal{B}^1_{6,\omega}=-\mathsf{C}_{4,1}/(6(\omega+5))$, where

$$\mathsf{C}_{4,1}=\mathsf{c}_4=a_1\mathsf{B}_{4,1}+a_3\mathsf{B}_{4,3}+a_5\mathsf{B}_{4,5}+a_7\mathsf{B}_{4,7}+\cdots \tag{3.30}$$

with

$$\begin{aligned}\mathsf{B}_{4,1}&=\mathcal{B}^1_{4,\omega},\quad \mathsf{B}_{4,3}=3\left(\left(\mathcal{B}^1_{2,\omega}\right)^2+\mathcal{B}^1_{4,\omega}\right),\quad \mathsf{B}_{4,5}=5\left(2\left(\mathcal{B}^1_{2,\omega}\right)^2+\mathcal{B}^1_{4,\omega}\right)\\ \mathsf{B}_{4,7}&=7\left(3\left(\mathcal{B}^1_{2,\omega}\right)^2+\mathcal{B}^1_{4,\omega}\right),\quad \cdots.\end{aligned} \tag{3.31}$$

Thus one sees that

$$\begin{aligned}\mathsf{C}_{4,1}&=\mathcal{B}^1_{4,\omega}\left(a_1+3a_3+5a_5+7a_7+\cdots\right)+\left(\mathcal{B}^1_{2,\omega}\right)^2\left(3a_3+10a_5+21a_7+\cdots\right)\\ &=\frac{\tan(1)\left(4\csc^2(2)-\csc^2(1)\right)^2}{8(\omega+1)(\omega+3)}+\frac{\tan^2(1)}{4(\omega+1)^2}\sum_{m=1}^{\infty}(-1)^m\frac{m(2m+1)\left(2^{2m+2}-1\right)B_{2m+2}}{2^{-2m-2}(2m+2)!}.\end{aligned} \tag{3.32}$$

For computation purposes we shall truncate the infinite series at the Nth term, $N\in\mathbb{N}$, i.e., $1\le m\le N$. Now, here and in the subsequent computations we take $N=20$:

$$\sum_{m=1}^{N}(-1)^m\frac{m(2m+1)2^{2m+2}\left(2^{2m+2}-1\right)B_{2m+2}}{(2m+2)!}=5.33492. \tag{3.33}$$

Using this value in (3.32) we get

$$\mathsf{C}_{4,1}=\frac{5.51935\omega+11.9893}{(\omega+1)^2(\omega+3)}.$$

Hence we obtain

$$\mathcal{B}^1_{6,\omega}=-\frac{0.919892\omega+1.99822}{(\omega+1)^2(\omega+3)(\omega+5)}. \tag{3.34}$$

- ($\ell = 3$) Clearly here one has $\mathcal{B}^1_{8,\omega} = -\mathsf{C}_{6,1}/(8(\omega+7))$ with

$$\mathsf{C}_{6,1} = \mathsf{c}_6 = a_1\mathsf{B}_{6,1} + a_3\mathsf{B}_{6,3} + a_5\mathsf{B}_{6,5} + a_7\mathsf{B}_{6,7} + \cdots . \tag{3.35}$$

Here

$$\begin{aligned}\mathsf{B}_{6,1} =& \mathcal{B}^1_{6,\omega},\ \mathsf{B}_{6,3} = \left(\mathcal{B}^1_{2,\omega}\right)^3 + 6\mathcal{B}^1_{4,\omega}\mathcal{B}^1_{2,\omega} + 3\mathcal{B}^1_{6,\omega},\ \mathsf{B}_{6,5} = 10\left(\mathcal{B}^1_{2,\omega}\right)^3 + 20\mathcal{B}^1_{4,\omega}\mathcal{B}^1_{2,\omega} + 5\mathcal{B}^1_{6,\omega}\\ \mathsf{B}_{6,7} =& 35\left(\mathcal{B}^1_{2,\omega}\right)^3 + 42\mathcal{B}^1_{4,\omega}\mathcal{B}^1_{2,\omega} + 7\mathcal{B}^1_{6,\omega},\ \cdots .\end{aligned} \tag{3.36}$$

Thus one has

$$\begin{aligned}\mathsf{C}_{6,1} =& \mathcal{B}^1_{6,\omega}\left(a_1 + 3a_3 + 5a_5 + 7a_7 + \cdots\right) + \left(\mathcal{B}^1_{2,\omega}\right)^3\left(a_3 + 10a_5 + 35a_7 + \cdots\right)\\ &+ \mathcal{B}^1_{2,\omega}\mathcal{B}^1_{4,\omega}\left(6a_3 + 20a_5 + 42a_7 + \cdots\right)\\ =& \mathcal{B}^1_{6,\omega}\left(4\csc^2(2) - \csc^2(1)\right) + \left(\mathcal{B}^1_{2,\omega}\right)^3\sum_{m=1}^{\infty}(-1)^m\frac{m\left(4m^2-1\right)\left(2^{2m+2}-1\right)B_{2m+2}}{3\cdot 2^{-2m-2}(2m+2)!}\\ &+ \mathcal{B}^1_{2,\omega}\mathcal{B}^1_{4,\omega}\sum_{m=1}^{\infty}(-1)^m\frac{2m(2m+1)2^{2m+2}\left(2^{2m+2}-1\right)B_{2m+2}}{(2m+2)!}.\end{aligned}$$

Similarly in this case we use the values

$$\begin{aligned}&\sum_{m=1}^{N}(-1)^m\frac{m\left(4m^2-1\right)2^{2m+2}\left(2^{2m+2}-1\right)B_{2m+2}}{3(2m+2)!} = 9.45039\\ &\sum_{m=1}^{N}(-1)^m\frac{2m(2m+1)2^{2m+2}\left(2^{2m+2}-1\right)B_{2m+2}}{(2m+2)!} = 10.66984\end{aligned} \tag{3.37}$$

to see that

$$\mathsf{C}_{6,1} = -\frac{13.1542\omega^2 + 78.9397\omega + 101.484}{(\omega+1)^3(\omega+3)(\omega+5)}, \tag{3.38}$$

which consequently gives

$$\mathcal{B}^1_{8,\omega} = \frac{1.64428\omega^2 + 9.86746\omega + 12.6856}{(\omega+1)^3(\omega+3)(\omega+5)(\omega+7)}. \tag{3.39}$$

- ($\ell = 4$) One understands here that $\mathcal{B}^1_{10,\omega} = -\mathsf{C}_{8,1}/(10(\omega+9))$, where

$$\mathsf{C}_{8,1} = \mathsf{c}_8 = a_1\mathsf{B}_{8,1} + a_3\mathsf{B}_{8,3} + a_5\mathsf{B}_{8,5} + a_7\mathsf{B}_{8,7} + \cdots \tag{3.40}$$

with

$$\begin{aligned}\mathsf{B}_{8,1} =& \mathcal{B}^1_{8,\omega},\quad \mathsf{B}_{8,3} = 3\mathcal{B}^1_{4,\omega}\left(\mathcal{B}^1_{2,\omega}\right)^2 + 6\mathcal{B}^1_{6,\omega}\mathcal{B}^1_{2,\omega} + 3\left(\mathcal{B}^1_{4,\omega}\right)^2 + 3\mathcal{B}^1_{8,\omega}\\ \mathsf{B}_{8,5} =& 5\left(\mathcal{B}^1_{2,\omega}\right)^4 + 30\mathcal{B}^1_{4,\omega}\left(\mathcal{B}^1_{2,\omega}\right)^2 + 20\mathcal{B}^1_{6,\omega}\mathcal{B}^1_{2,\omega} + 10\left(\mathcal{B}^1_{4,\omega}\right)^2 + 5\mathcal{B}^1_{8,\omega}\\ \mathsf{B}_{8,7} =& 35\left(\mathcal{B}^1_{2,\omega}\right)^4 + 105\mathcal{B}^1_{4,\omega}\left(\mathcal{B}^1_{2,\omega}\right)^2 + 42\mathcal{B}^1_{6,\omega}\mathcal{B}^1_{2,\omega} + 21\left(\mathcal{B}^1_{4,\omega}\right)^2 + 7\mathcal{B}^1_{8,\omega}, \cdots .\end{aligned} \tag{3.41}$$

We now have

$$\begin{aligned}\mathsf{C}_{8,1} =& \mathcal{B}^1_{8,\omega}\left(a_1 + 3a_3 + 5a_5 + 7a_7 + \cdots\right) + \left(\mathcal{B}^1_{4,\omega}\right)^2\left(3a_3 + 10a_5 + 21a_7 + \cdots\right)\\ &+ \mathcal{B}^1_{4,\omega}\left(\mathcal{B}^1_{2,\omega}\right)^2\left(3a_3 + 30a_5 + 105a_7 + \cdots\right) + \mathcal{B}^1_{6,\omega}\mathcal{B}^1_{2,\omega}\left(6a_3 + 20a_5 + 42a_7 + \cdots\right)\\ &+ \left(\mathcal{B}^1_{2,\omega}\right)^4\left(5a_5 + 35a_7 + \cdots\right)\end{aligned}$$

which gives

$$\begin{aligned}\mathsf{C}_{8,1} =&\mathcal{B}^1_{8,\omega}\left(4\csc^2(2)-\csc^2(1)\right)+\left(\mathcal{B}^1_{4,\omega}\right)^2\sum_{m=1}^{\infty}(-1)^m\frac{m(2m+1)2^{2m+2}\left(2^{2m+2}-1\right)B_{2m+2}}{(2m+2)!}\\ &+\mathcal{B}^1_{4,\omega}\left(\mathcal{B}^1_{2,\omega}\right)^2\sum_{m=1}^{\infty}(-1)^m\frac{m(2m-1)(2m+1)2^{2m+2}\left(2^{2m+2}-1\right)B_{2m+2}}{(2m+2)!}\\ &+\mathcal{B}^1_{6,\omega}\mathcal{B}^1_{2,\omega}\sum_{m=1}^{\infty}(-1)^m\frac{2m(2m+1)2^{2m+2}\left(2^{2m+2}-1\right)B_{2m+2}}{(2m+2)!}\\ &+\left(\mathcal{B}^1_{2,\omega}\right)^4\sum_{m=2}^{\infty}(-1)^m\binom{2m+1}{4}\frac{2^{2m+2}\left(2^{2m+2}-1\right)B_{2m+2}}{(2m+2)!}.\end{aligned}\tag{3.42}$$

With the values

$$\begin{aligned}&\sum_{m=1}^{N}(-1)^m\frac{m(2m-1)(2m+1)2^{2m+2}\left(2^{2m+2}-1\right)B_{2m+2}}{(2m+2)!}=28.35118\\ &\sum_{m=2}^{N}(-1)^m\binom{2m+1}{4}\frac{2^{2m+2}\left(2^{2m+2}-1\right)B_{2m+2}}{(2m+2)!}=16.49546\end{aligned}\tag{3.43}$$

one now gets

$$\mathsf{C}_{8,1}=\frac{33.1779\omega^4+482.83\omega^3+2446.99\omega^2+5091.47\omega+3676.4}{(\omega+1)^4(\omega+3)^2(\omega+5)(\omega+7)}.$$

Hence, we obtain

$$\mathcal{B}^1_{10,\omega}=-\frac{3.31779\omega^4+48.283\omega^3+244.699\omega^2+509.147\omega+367.64}{(\omega+1)^4(\omega+3)^2(\omega+5)(\omega+7)(\omega+9)}.\tag{3.44}$$

Therefore, the approximate analytical solution of the problem (3.25) is given by

$$\begin{aligned}y(x)=&1-\frac{0.778704}{\omega+1}x^2+\frac{0.666866}{(\omega+1)(\omega+3)}x^4-\frac{0.919892\omega+1.99822}{(\omega+1)^2(\omega+3)(\omega+5)}x^6\\ &+\frac{1.64428\omega^2+9.86746\omega+12.6856}{(\omega+1)^3(\omega+3)(\omega+5)(\omega+7)}x^8\\ &-\frac{3.31779\omega^4+48.283\omega^3+244.699\omega^2+509.147\omega+367.64}{(\omega+1)^4(\omega+3)^2(\omega+5)(\omega+7)(\omega+9)}x^{10}+\dots.\end{aligned}\tag{3.45}$$

Example 3.6. Consider the following nonlinear problem.

$$y''(x)+\frac{\omega}{x}y'(x)+\tan^2(y(x))=0,\quad y(0)=1,\ \ y'(0)=0,\quad 0<x\le 1.\tag{3.46}$$

Here we compute the first expansion coefficients $\mathcal{B}^2_{2\ell+2,\omega}$ of the series solution of (3.46). Towards this end, we use the formula

$$\mathsf{C}_{\ell,2}=\sum_{s_1=0}^{\ell}\mathsf{c}_{s_1}\mathsf{c}_{\ell-s_1},\tag{3.47}$$

where the coefficients $\mathsf{C}_{\ell,1}=\mathsf{c}_\ell,\ell=0,2,4,\dots$ are as computed in Example 3.5.

- ($\ell = 0$) Here we see that

$$\mathcal{B}^2_{2,\omega} = -\frac{\mathsf{C}_{0,2}}{2(\omega+1)} = -\frac{\tan^2(1)}{2(\omega+1)} = -\frac{1.21276}{\omega+1}.$$

- ($\ell = 1$) It is seen here that $\mathcal{B}^2_{4,\omega} = -\mathsf{C}_{2,2}/(4(\omega+3))$. Upon expanding the formula (3.47) we see that $\mathsf{C}_{2,2} = 2\mathsf{C}_{0,1}\mathsf{C}_{2,1} = -8.30866/(\omega+1)$, and as a result we have

$$\mathcal{B}^2_{4,\omega} = \frac{2.07717}{(\omega+1)(\omega+3)}. \tag{3.48}$$

- ($\ell = 2$) One sees here that $\mathcal{B}^2_{6,\omega} = -\mathsf{C}_{4,2}/(6(\omega+5))$, where, in view of the formula (3.47), it is seen that

$$\mathsf{C}_{4,2} = (\mathsf{C}_{2,1})^2 + 2\mathsf{C}_{0,1}\mathsf{C}_{4,1} = \frac{24.3071\omega + 58.6906}{(\omega+1)^2(\omega+3)};$$

  and as a result we have

$$\mathcal{B}^2_{6,\omega} = -\frac{4.05119\omega + 9.78177}{(\omega+1)^2(\omega+3)(\omega+5)}. \tag{3.49}$$

- ($\ell = 3$) It is seen that $\mathcal{B}^2_{8,\omega} = -\mathsf{C}_{6,2}/8(\omega+7)$. Formula (3.47) in this case takes the formulation

$$\mathsf{C}_{6,2} = 2\mathsf{C}_{2,1}\mathsf{C}_{4,1} + 2\mathsf{C}_{0,1}\mathsf{C}_{6,1} = -\frac{70.4184\omega^2 + 457.071\omega + 635.916}{(\omega+1)^3(\omega+3)(\omega+5)}$$

  and as a result we obtain the expansion coefficient

$$\mathcal{B}^2_{8,\omega} = \frac{8.8023\omega^2 + 57.1339\omega + 79.4896}{(\omega+1)^3(\omega+3)(\omega+5)(\omega+7)}. \tag{3.50}$$

- ($\ell = 4$) In this special case, one understands here that $\mathcal{B}^2_{10,\omega} = -\mathsf{C}_{8,2}/(10(\omega+9))$. Upon referring to the closed expression (3.47), it is understood that

$$\begin{aligned}\mathsf{C}_{8,2} &= (\mathsf{C}_{4,1})^2 + 2\mathsf{C}_{2,1}\mathsf{C}_{6,1} + 2\mathsf{C}_{0,1}\mathsf{C}_{8,1} \\ &= \frac{203.983\omega^4 + 3124.74\omega^3 + 16646.5\omega^2 + 36474.1\omega + 27852}{(\omega+1)^4(\omega+3)^2(\omega+5)(\omega+7)}\end{aligned} \tag{3.51}$$

  and as a consequence, we obtain

$$\mathcal{B}^2_{10,\omega} = -\frac{20.3983\omega^4 + 312.474\omega^3 + 1664.65\omega^2 + 3647.41\omega + 2785.2}{(\omega+1)^4(\omega+3)^2(\omega+5)(\omega+7)(\omega+9)}. \tag{3.52}$$

Hence, the approximate analytical solution of the problem (3.46) is given by

$$\begin{aligned}y(x) =& 1 - \frac{1.21276}{\omega+1}x^2 + \frac{2.07717}{(\omega+1)(\omega+3)}x^4 - \frac{4.05119\omega + 9.78177}{(\omega+1)^2(\omega+3)(\omega+5)}x^6 \\ &+ \frac{8.8023\omega^2 + 57.1339\omega + 79.4896}{(\omega+1)^3(\omega+3)(\omega+5)(\omega+7)}x^8 \\ &- \frac{20.3983\omega^4 + 312.474\omega^3 + 1664.65\omega^2 + 3647.41\omega + 2785.2}{(\omega+1)^4(\omega+3)^2(\omega+5)(\omega+7)(\omega+9)}x^{10} + \dots .\end{aligned} \tag{3.53}$$

The higher cases $m = 3$ and $m = 4$ can be computed similarly and we give the results in Examples 3.7 and 3.8 respectively.

Example 3.7. Consider the following problem.

$$y''(x) + \frac{\omega}{x} y'(x) + \tan^3(y(x)) = 0, \quad y(0) = 1, \; y'(0) = 0, \quad 0 < x \leq 1. \tag{3.54}$$

Using the closed expression

$$\mathsf{C}_{\ell,3} = \sum_{p_2=0}^{\ell} \sum_{p_1=0}^{p_2} \mathsf{c}_{p_1} \mathsf{c}_{p_2-p_1} \mathsf{c}_{\ell-p_2}, \tag{3.55}$$

where the coefficients c_ℓ are as computed in Example 3.5, we obtain the following explicit values.

$$\begin{aligned}
\mathsf{C}_{2,3} =& 3\left(\mathsf{C}_{0,1}\right)^2 \mathsf{C}_{2,1} = -\frac{19.40995}{\omega+1} \\
\mathsf{C}_{4,3} =& 3\mathsf{C}_{4,1}\left(\mathsf{C}_{0,1}\right)^2 + 3\left(\mathsf{C}_{2,1}\right)^2 \mathsf{C}_{0,1} = \frac{73.4065\omega + 186.975}{(\omega+1)^2(\omega+3)} \\
\mathsf{C}_{6,3} =& \left(\mathsf{C}_{2,1}\right)^3 + 6\mathsf{C}_{0,1}\mathsf{C}_{4,1}\mathsf{C}_{2,1} + 3\left(\mathsf{C}_{0,1}\right)^2 \mathsf{C}_{6,1} = \frac{0.10385\omega^2 + 0.794057\omega + 1.39404}{(\omega+1)^3(\omega+3)(\omega+5)} \\
\mathsf{C}_{8,3} =& 3\mathsf{C}_{8,1}\left(\mathsf{C}_{0,1}\right)^2 + 3\left(\mathsf{C}_{4,1}\right)^2 \mathsf{C}_{0,1} + 6\mathsf{C}_{2,1}\mathsf{C}_{6,1}\mathsf{C}_{0,1} + 3\left(\mathsf{C}_{2,1}\right)^2 \mathsf{C}_{4,1} \\
=& \frac{829.451\omega^4 + 13109.3\omega^3 + 72174.6\omega^2 + 163908\omega + 130251}{(\omega+1)^4(\omega+3)^2(\omega+5)(\omega+7)}.
\end{aligned}$$

As a result we have

$$\begin{aligned}
\mathcal{B}^3_{2,\omega} &= -\frac{\mathsf{C}_{0,3}}{2(\omega+1)} = -\frac{1.88876}{\omega+1} \\
\mathcal{B}^3_{4,\omega} &= -\frac{\mathsf{C}_{2,3}}{4(\omega+3)} = \frac{4.85249}{(\omega+1)(\omega+3)} \\
\mathcal{B}^3_{6,\omega} &= -\frac{\mathsf{C}_{4,3}}{6(\omega+5)} = -\frac{12.2344\omega + 31.1625}{(\omega+1)^2(\omega+3)(\omega+5)} \\
\mathcal{B}^3_{8,\omega} &= -\frac{\mathsf{C}_{6,3}}{8(\omega+7)} = \frac{31.5341\omega^2 + 214.121\omega + 314.673}{(\omega+1)^3(\omega+3)(\omega+5)(\omega+7)} \\
\mathcal{B}^3_{10,\omega} &= -\frac{\mathsf{C}_{8,3}}{10(\omega+9)} = -\frac{82.9451\omega^4 + 1310.93\omega^3 + 7217.46\omega^2 + 16390.8\omega + 13025.1}{(\omega+1)^4(\omega+3)^2(\omega+5)(\omega+7)(\omega+9)}.
\end{aligned}$$

Hence, the approximate analytical solution of the problem (3.54) is given by

$$\begin{aligned}
y(x) =& 1 - \frac{1.88876}{\omega+1}x^2 + \frac{4.85249}{(\omega+1)(\omega+3)}x^4 - \frac{12.2344\omega + 31.1625}{(\omega+1)^2(\omega+3)(\omega+5)}x^6 \\
&+ \frac{31.5341\omega^2 + 214.121\omega + 314.673}{(\omega+1)^3(\omega+3)(\omega+5)(\omega+7)}x^8 \\
&- \frac{82.9451\omega^4 + 1310.93\omega^3 + 7217.46\omega^2 + 16390.8\omega + 13025.1}{(\omega+1)^4(\omega+3)^2(\omega+5)(\omega+7)(\omega+9)}x^{10} + \dots .
\end{aligned} \tag{3.56}$$

Example 3.8. To find the analytical solution of the initial value problem

$$y''(x) + \frac{\omega}{x} y'(x) + \tan^4(y(x)) = 0, \quad y(0) = 1, \; y'(0) = 0, \quad 0 < x \leq 1, \tag{3.57}$$

one needs to expand the finite series

$$\mathsf{C}_{\ell,4}=\sum_{p_3=0}^{\ell}\sum_{p_2=0}^{p_3}\sum_{p_1=0}^{p_2}\mathsf{c}_{p_1}\mathsf{c}_{p_2-p_1}\mathsf{c}_{p_3-p_2}\mathsf{c}_{\ell-p_3}, \tag{3.58}$$

where the numbers $\mathsf{c}_\ell=\mathsf{C}_{\ell,1}$ are as calculated in Example 3.5. We have the following values.

$$\begin{aligned}
\mathsf{C}_{2,4}=&4\left(\mathsf{C}_{0,1}\right)^3\mathsf{C}_{2,1}=-\frac{40.30562}{\omega+1}\\
\mathsf{C}_{4,4}=&6\left(\mathsf{C}_{0,1}\right)^2\left(\mathsf{C}_{2,1}\right)^2+4\left(\mathsf{C}_{0,1}\right)^3\mathsf{C}_{4,1}=\frac{186.949\omega+491.812}{(\omega+1)^2(\omega+3)}\\
\mathsf{C}_{6,4}=&4\mathsf{C}_{6,1}\left(\mathsf{C}_{0,1}\right)^3+12\mathsf{C}_{2,1}\mathsf{C}_{4,1}\left(\mathsf{C}_{0,1}\right)^2+4\left(\mathsf{C}_{2,1}\right)^3\mathsf{C}_{0,1}=\frac{-745.522\omega^2-5212.15\omega-7961.26}{(\omega+1)^3(\omega+3)(\omega+5)}\\
\mathsf{C}_{8,4}=&\left(\mathsf{C}_{2,1}\right)^4+12\mathsf{C}_{0,1}\mathsf{C}_{4,1}\left(\mathsf{C}_{2,1}\right)^2+12\left(\mathsf{C}_{0,1}\right)^2\mathsf{C}_{6,1}\mathsf{C}_{2,1}+6\left(\mathsf{C}_{0,1}\right)^2\left(\mathsf{C}_{4,1}\right)^2+4\left(\mathsf{C}_{0,1}\right)^3\mathsf{C}_{8,1}\\
=&\frac{2750.53\omega^4+44398.4\omega^3+250209.\omega^2+583308.\omega+477584.}{(\omega+1)^4(\omega+3)^2(\omega+5)(\omega+7)},
\end{aligned}$$

which consequently gives the expansion coefficients

$$\begin{aligned}
\mathcal{B}^4_{2,\omega}&=-\frac{\mathsf{C}_{0,4}}{2(\omega+1)}=-\frac{2.94157}{\omega+1}\\
\mathcal{B}^4_{4,\omega}&=-\frac{\mathsf{C}_{2,4}}{4(\omega+3)}=\frac{10.0764}{(\omega+1)(\omega+3)}\\
\mathcal{B}^4_{6,\omega}&=-\frac{\mathsf{C}_{4,4}}{6(\omega+5)}=-\frac{31.1581\omega+81.9687}{(\omega+1)^2(\omega+3)(\omega+5)}\\
\mathcal{B}^4_{8,\omega}&=-\frac{\mathsf{C}_{6,4}}{8(\omega+7)}=\frac{93.1902\omega^2+651.519\omega+995.158}{(\omega+1)^3(\omega+3)(\omega+5)(\omega+7)}\\
\mathcal{B}^4_{10,\omega}&=-\frac{\mathsf{C}_{8,4}}{10(\omega+9)}=-\frac{275.053\omega^4+4439.84\omega^3+25020.9\omega^2+58330.8\omega+47758.4}{(\omega+1)^4(\omega+3)^2(\omega+5)(\omega+7)(\omega+9)}.
\end{aligned}$$

Hence, one sees that the approximate analytical solution of the problem (3.57) is given by

$$\begin{aligned}
y(x)=&1-\frac{2.94157}{\omega+1}x^2+\frac{10.0764}{(\omega+1)(\omega+3)}x^4-\frac{31.1581\omega+81.9687}{(\omega+1)^2(\omega+3)(\omega+5)}x^6\\
&+\frac{93.1902\omega^2+651.519\omega+995.158}{(\omega+1)^3(\omega+3)(\omega+5)(\omega+7)}x^8\\
&-\frac{275.053\omega^4+4439.84\omega^3+25020.9\omega^2+58330.8\omega+47758.4}{(\omega+1)^4(\omega+3)^2(\omega+5)(\omega+7)(\omega+9)}x^{10}+\dots.
\end{aligned} \tag{3.59}$$

3.2. Lane-Emden Equation Involving Secant Function $\Lambda(y(x))=\sec y(x)$

Here we consider the analytical solution of the Lane-Emden equation corresponding to the nonlinearity $\sec^m(y(x))$. We present the result as a corollary.

Corollary 3.9. *For $m\in\mathbb{N}_0,\omega\in\mathbb{R}$, the initial value problem*

$$y''(x)+\frac{\omega}{x}y'(x)+\sec^m(y(x))=0,\quad y(0)=1,\ \ y'(0)=0,\quad 0<x\leq 1 \tag{3.60}$$

admits the analytical solution

$$y(x)=1+\sum_{\ell=0}^{\infty}\mathcal{B}^m_{2\ell+2,\omega}x^{2\ell+2}, \tag{3.61}$$

where the coefficients $\mathcal{B}^m_{2\ell+2,\omega}$ *are given by* (3.12) *with*

$$\mathsf{C}_{0,m} = (a_0 + a_2 + a_4 + a_6 + \cdots)^m = \sec^m(1) = (1.85082)^m \tag{3.62}$$

$$\mathsf{C}_{\ell,m} = \sum_{s_{m-1}=0}^{\ell} \sum_{s_{m-2}=0}^{s_{m-1}} \cdots \sum_{s_1=0}^{s_2} \mathsf{c}_{\ell-s_{m-1}} \mathsf{c}_{s_{m-1}-s_{m-2}} \cdots \mathsf{c}_{s_2-s_1} \mathsf{c}_{s_1}, \quad \ell = 2,4,6,\dots \tag{3.63}$$

$$\mathsf{C}_{\ell,1} = \mathsf{c}_\ell = a_0 \mathsf{B}_{\ell,0} + a_2 \mathsf{B}_{\ell,2} + a_4 \mathsf{B}_{\ell,4} + a_6 \mathsf{B}_{\ell,6} + \cdots \quad \ell = 2,4,6,\dots . \tag{3.64}$$

Here the (Maclaurin) coefficients $a_{2k}, k = 0,1,2,3,\dots$ *are defined by*

$$\sec y(x) = \sum_{k=0}^{\infty} a_{2k} y^{2k}(x) = \sum_{k=0}^{\infty} \frac{|E_{2k}|}{(2k)!} y^{2k}(x), \tag{3.65}$$

where E_j *is the* j*th Euler number.*

In this case, for brevity of calculation, we consider only the Lane-Emden equation involving the function $\sec y(x)$ and present explicit computation of the first coefficients $\mathsf{C}_{\ell,1}$ and $\mathcal{B}^1_{2\ell+2,\omega}$ as an example. The higher cases can be done similarly as in Subsection 3.1.

Example 3.10. Consider the nonlinear initial value problem

$$y''(x) + \frac{\omega}{x} y'(x) + \sec(y(x)) = 0, \quad y(0) = 1, \;\; y'(0) = 0, \quad 0 < x \le 1. \tag{3.66}$$

It suffices to compute the expansion coefficients $\mathcal{B}^1_{2\ell+2,\omega}$ of the series solution of (3.66) for $\ell = 0,1,2,3,4$.

- ($\ell = 0$) It is seen here upon using the formula (3.62) (with $m = 1$) that

$$\mathcal{B}^1_{2,\omega} = -\frac{\sec(1)}{2(\omega+1)} = -\frac{0.925408}{\omega+1}. \tag{3.67}$$

- ($\ell = 1$) It is understood here that $\mathcal{B}^1_{4,\omega} = -\mathsf{C}_{2,1}/(4(\omega+3))$, where

$$\mathsf{C}_{2,1} = \mathsf{c}_2 = a_2 \mathsf{B}_{2,2} + a_4 \mathsf{B}_{2,4} + a_6 \mathsf{B}_{2,6} + \cdots .$$

  Here the coefficients $\mathsf{B}_{2,m}$, $m = 2,4,6,\dots$ are given by

$$\mathsf{B}_{2,2} = 2\mathcal{B}^1_{2,\omega}, \quad \mathsf{B}_{2,4} = 4\mathcal{B}^1_{2,\omega}, \quad \mathsf{B}_{2,6} = 6\mathcal{B}^1_{2,\omega}, \quad \mathsf{B}_{2,8} = 8\mathcal{B}^1_{2,\omega}, \quad \cdots \tag{3.68}$$

  and we obtain

$$\begin{aligned} \mathsf{C}_{2,1} &= -\frac{\sec(1)}{2(\omega+1)} (2a_2 + 4a_4 + 6a_6 + 8a_8 + \cdots) \\ &= -\frac{\sec(1)}{2(\omega+1)} \sum_{m=1}^{\infty} \frac{(2m)\,|E_{2m}|}{(2m)!} = -\frac{\sec(1)}{2(\omega+1)} \sum_{m=1}^{\infty} \frac{|E_{2m}|}{(2m-1)!} = -\frac{2.66746}{\omega+1}. \end{aligned} \tag{3.69}$$

  As a result we have

$$\mathcal{B}^1_{4,\omega} = \frac{0.666866}{(\omega+1)(\omega+3)}. \tag{3.70}$$

- ($\ell = 2$) One sees here that $\mathcal{B}^1_{6,\omega} = -\mathsf{C}_{4,1}/(6(\omega+5))$, where

$$\mathsf{C}_{4,1} = \mathsf{c}_4 = a_2 \mathsf{B}_{4,2} + a_4 \mathsf{B}_{4,4} + a_6 \mathsf{B}_{4,6} + \cdots \tag{3.71}$$

with

$$\mathsf{B}_{4,2} = \left(\mathcal{B}^1_{2,\omega}\right)^2 + 2\mathcal{B}^1_{4,\omega}, \quad \mathsf{B}_{4,4} = 6\left(\mathcal{B}^1_{2,\omega}\right)^2 + 4\mathcal{B}^1_{4,\omega}, \quad \mathsf{B}_{4,6} = 15\left(\mathcal{B}^1_{2,\omega}\right)^2 + 6\mathcal{B}^1_{4,\omega}, \cdots . \tag{3.72}$$

Thus one sees that

$$\begin{aligned}\mathsf{C}_{4,1} =& \mathcal{B}^1_{4,\omega}\left(2a_2 + 4a_4 + 6a_6 + \cdots\right) + \left(\mathcal{B}^1_{2,\omega}\right)^2\left(a_2 + 6a_4 + 15a_6 + \cdots\right)\\ =& \mathcal{B}^1_{4,\omega}\sum_{m=1}^{\infty}\frac{(2m)\left|E_{2m}\right|}{(2m)!} + \left(\mathcal{B}^1_{2,\omega}\right)^2\sum_{m=1}^{\infty}\frac{m(2m-1)\left|E_{2m}\right|}{(2m)!}.\end{aligned}$$

With the value

$$\sum_{m=1}^{N}\frac{m(2m-1)\left|E_{2m}\right|}{(2m)!} = 5.41458 \tag{3.73}$$

one now gets

$$\mathsf{C}_{4,1} = \frac{6.55916\omega + 15.833}{(\omega+1)^2(\omega+3)}.$$

Consequently, we obtain

$$\mathcal{B}^1_{6,\omega} = -\frac{\mathsf{C}_{4,1}}{6(\omega+5)} = -\frac{1.09319\omega + 2.63884}{(\omega+1)^2(\omega+3)(\omega+5)}. \tag{3.74}$$

- ($\ell = 3$) It is seen that $\mathcal{B}^1_{8,\omega} = -\mathsf{C}_{6,1}/(8(\omega+7))$, where

$$\mathsf{C}_{6,1} = \mathsf{c}_6 = a_2\mathsf{B}_{6,2} + a_4\mathsf{B}_{6,4} + a_6\mathsf{B}_{6,6} + \cdots$$

with

$$\begin{aligned}\mathsf{B}_{6,2} =& 2\mathcal{B}^1_{4,\omega}\mathcal{B}^1_{2,\omega} + 2\mathcal{B}^1_{6,\omega},\ \mathsf{B}_{6,4} = 4\left(\mathcal{B}^1_{2,\omega}\right)^3 + 12\mathcal{B}^1_{4,\omega}\mathcal{B}^1_{2,\omega} + 4\mathcal{B}^1_{6,\omega}\\ \mathsf{B}_{6,6} =& 20\left(\mathcal{B}^1_{2,\omega}\right)^3 + 30\mathcal{B}^1_{4,\omega}\mathcal{B}^1_{2,\omega} + 6\mathcal{B}^1_{6,\omega},\ \cdots .\end{aligned} \tag{3.75}$$

It follows that

$$\begin{aligned}\mathsf{C}_{6,1} =& \mathcal{B}^1_{6,\omega}\left(2a_2 + 4a_4 + 6a_6 + 8a_8 + \cdots\right) + \left(\mathcal{B}^1_{2,\omega}\right)^3\left(4a_4 + 20a_6 + 56a_8 + \cdots\right)\\ &+ \mathcal{B}^1_{2,\omega}\mathcal{B}^1_{4,\omega}\left(2a_2 + 12a_4 + 30a_6 + 56a_8 + \cdots\right)\\ =& \mathcal{B}^1_{6,\omega}\left(\sum_{m=1}^{\infty}\frac{(2m)\left|E_{2m}\right|}{(2m)!}\right) + \left(\mathcal{B}^1_{2,\omega}\right)^3\sum_{m=1}^{\infty}\frac{(2m)(m+1)(2m+1)\left|E_{2m}\right|}{3(2m)!}\\ &+ \mathcal{B}^1_{2,\omega}\mathcal{B}^1_{4,\omega}\sum_{m=1}^{\infty}\frac{(2m)(2m-1)\left|E_{2m}\right|}{(2m)!}.\end{aligned}$$

Similarly using the values

$$\begin{aligned}&\sum_{m=1}^{N}\frac{(2m)(m+1)(2m+1)\left|E_{2m}\right|}{3(2m)!} = 23.105044\\ &\sum_{m=1}^{N}\frac{(2m)(2m-1)\left|E_{2m}\right|}{(2m)!}, = 10.82917\end{aligned} \tag{3.76}$$

one sees that

$$\mathsf{C}_{6,1} = -\frac{28.1448\omega^2 + 197.341\omega + 315.682}{(\omega+1)^3(\omega+3)(\omega+5)}.$$

Hence, we obtain

$$\mathcal{B}^1_{8,\omega} = \frac{3.5181\omega^2 + 24.6676\omega + 39.4603}{(\omega+1)^3(\omega+3)(\omega+5)(\omega+7)}. \tag{3.77}$$

- ($\ell = 4$) One understands here that $\mathcal{B}^1_{10,\omega} = -\mathsf{C}_{8,1}/(10(\omega+9))$, where

$$\mathsf{C}_{8,1} = \mathsf{c}_8 = a_2\mathsf{B}_{8,2} + a_4\mathsf{B}_{8,4} + a_6\mathsf{B}_{8,6} + a_8\mathsf{B}_{8,8} + \cdots \tag{3.78}$$

with

$$\begin{aligned}
\mathsf{B}_{8,2} =& 2\mathcal{B}^1_{6,\omega}\mathcal{B}^1_{2,\omega} + \left(\mathcal{B}^1_{4,\omega}\right)^2 + 2\mathcal{B}^1_{8,\omega}\\
\mathsf{B}_{8,4} =& \left(\mathcal{B}^1_{2,\omega}\right)^4 + 12\mathcal{B}^1_{4,\omega}\left(\mathcal{B}^1_{2,\omega}\right)^2 + 12\mathcal{B}^1_{6,\omega}\mathcal{B}^1_{2,\omega} + 6\left(\mathcal{B}^1_{4,\omega}\right)^2 + 4\mathcal{B}^1_{8,\omega}\\
\mathsf{B}_{8,6} =& 15\left(\mathcal{B}^1_{2,\omega}\right)^4 + 60\mathcal{B}^1_{4,\omega}\left(\mathcal{B}^1_{2,\omega}\right)^2 + 30\mathcal{B}^1_{6,\omega}\mathcal{B}^1_{2,\omega} + 15\left(\mathcal{B}^1_{4,\omega}\right)^2 + 6\mathcal{B}^1_{8,\omega}\\
\mathsf{B}_{8,8} =& 70\left(\mathcal{B}^1_{2,\omega}\right)^4 + 168\mathcal{B}^1_{4,\omega}\left(\mathcal{B}^1_{2,\omega}\right)^2 + 56\mathcal{B}^1_{6,\omega}\mathcal{B}^1_{2,\omega} + 28\left(\mathcal{B}^1_{4,\omega}\right)^2 + 8\mathcal{B}^1_{8,\omega}, \cdots .
\end{aligned} \tag{3.79}$$

Upon substituting these coefficients into (3.78) we find that

$$\begin{aligned}
\mathsf{C}_{8,1} =& \mathcal{B}^1_{8,\omega}\left(2a_2 + 4a_4 + 6a_6 + 8a_8 + \cdots\right) + \left(\mathcal{B}^1_{4,\omega}\right)^2\left(a_2 + 6a_4 + 15a_6 + 28a_8 + \cdots\right)\\
&+ \mathcal{B}^1_{4,\omega}\left(\mathcal{B}^1_{2,\omega}\right)^2\left(12a_4 + 60a_6 + 168a_8 + \cdots\right) + \mathcal{B}^1_{6,\omega}\mathcal{B}^1_{2,\omega}\left(2a_2 + 12a_4 + 30a_6 + 56a_8 + \cdots\right)\\
&+ \left(\mathcal{B}^1_{2,\omega}\right)^4\left(a_4 + 15a_6 + 70a_8 + \cdots\right)\\
=& \mathcal{B}^1_{8,\omega}\left(\sum_{m=1}^{\infty}\frac{(2m)\left|E_{2m}\right|}{(2m)!}\right) + \left(\mathcal{B}^1_{4,\omega}\right)^2\sum_{m=1}^{\infty}\frac{m(2m-1)\left|E_{2m}\right|}{(2m)!}\\
&+ \mathcal{B}^1_{4,\omega}\left(\mathcal{B}^1_{2,\omega}\right)^2\sum_{m=1}^{\infty}\frac{(2m)(m+1)(2m+1)\left|E_{2m}\right|}{(2m)!}\\
&+ \mathcal{B}^1_{6,\omega}\mathcal{B}^1_{2,\omega}\sum_{m=1}^{\infty}\frac{(2m)(2m-1)\left|E_{2m}\right|}{(2m)!} + \left(\mathcal{B}^1_{2,\omega}\right)^4\sum_{m=0}^{\infty}\binom{2m+4}{4}\frac{\left|E_{2m}\right|}{(2m)!}.
\end{aligned}$$

With the values

$$\sum_{m=1}^{N}\frac{(2m)(m+1)(2m+1)\left|E_{2m}\right|}{(2m)!} = 69.31513, \quad \sum_{m=0}^{N}\binom{2m+4}{4}\frac{\left|E_{2m}\right|}{(2m)!}, = 99.95180 \tag{3.80}$$

one has

$$\mathsf{C}_{8,1} = \frac{136.393\omega^4 + 2245.15\omega^3 + 13111.1\omega^2 + 32193.\omega + 28227.8}{(\omega+1)^4(\omega+3)^2(\omega+5)(\omega+7)}.$$

Hence, we obtain

$$\mathcal{B}^1_{10,\omega} = -\frac{13.6393\omega^4 + 224.515\omega^3 + 1311.11\omega^2 + 3219.3\omega + 2822.78}{(\omega+1)^4(\omega+3)^2(\omega+5)(\omega+7)(\omega+9)}. \tag{3.81}$$

Therefore, the approximate analytical solution of the problem (3.66) is given by

$$\begin{aligned}
y(x) =& 1 - \frac{0.925408}{\omega+1}x^2 + \frac{0.666866}{(\omega+1)(\omega+3)}x^4 - \frac{1.09319\omega + 2.63884}{(\omega+1)^2(\omega+3)(\omega+5)}x^6\\
&+ \frac{3.5181\omega^2 + 24.6676\omega + 39.4603}{(\omega+1)^3(\omega+3)(\omega+5)(\omega+7)}x^8\\
&- \frac{13.6393\omega^4 + 224.515\omega^3 + 1311.11\omega^2 + 3219.3\omega + 2822.78}{(\omega+1)^4(\omega+3)^2(\omega+5)(\omega+7)(\omega+9)}x^{10} + \dots .
\end{aligned} \tag{3.82}$$

3.3. Lane-Emden Equation Involving Hyperbolic Function $\Lambda(y(x)) = \tanh y(x)$

In this subsection, we present the result on the analytical solution of the Lane-Emden equation involving the nonlinearity $\tanh^m(y(x))$. The result is given as a corollary and the special cases $m = 1, 2, 3, 4$ are computed as in Subsection 3.1.

Corollary 3.11. *Consider the following initial value problem:*

$$y''(x) + \frac{\omega}{x}y'(x) + \tanh^m(y(x)) = 0, \quad y(0) = 1, \ y'(0) = 0, \quad 0 < x \leq 1. \tag{3.83}$$

The Lane-Emden problem (3.83) admits the analytical solution

$$y(x) = 1 + \sum_{\ell=0}^{\infty} \mathcal{B}^m_{2\ell+2,\omega} x^{2\ell}, \tag{3.84}$$

where the coefficients $\mathcal{B}^m_{2\ell+2,\omega}$ are given by (3.12) with

$$\mathsf{C}_{0,m} = (a_1 + a_3 + a_5 + a_7 + \cdots)^m = \tanh^m 1 = (0.761594)^m \tag{3.85}$$

$$\mathsf{C}_{\ell,m} = \sum_{q_{m-1}=0}^{\ell} \sum_{q_{m-2}=0}^{q_{m-1}} \cdots \sum_{q_1=0}^{q_2} \mathsf{c}_{\ell-q_{m-1}} \mathsf{c}_{q_{m-1}-q_{m-2}} \cdots \mathsf{c}_{q_2-q_1} \mathsf{c}_{q_1}, \quad \ell = 2, 4, 6, \ldots \tag{3.86}$$

$$\mathsf{C}_{\ell,1} = \mathsf{c}_\ell = a_1 \mathsf{B}_{\ell,1} + a_3 \mathsf{B}_{\ell,3} + a_5 \mathsf{B}_{\ell,5} + a_7 \mathsf{B}_{\ell,7} + \cdots \quad \ell = 2, 4, 6, \ldots . \tag{3.87}$$

Here the coefficients $a_{2k+1}, k = 0, 1, 2, 3, \ldots$ are defined by

$$\tanh y(x) = \sum_{k=0}^{\infty} a_{2k+1} y^{2k+1}(x) = \sum_{k=0}^{\infty} \frac{2^{2k+2}\left(2^{2k+2}-1\right) B_{2k+2}}{(2k+2)!} y^{2k+1}(x), \tag{3.88}$$

where B_j are the jth Bernoulli numbers.

As usual, we compute explicitly the coefficients $\mathsf{C}_{\ell,m}$ to obtain recursively the expansion coefficients $\mathcal{B}^m_{2\ell+2,\omega}$. Following a similar procedure as in Subsection 3.1, we present the values of these coefficients for $m = 1, 2, 3, 4$.

Example 3.12. Consider the following Lane-Emden initial value problem.

$$y''(x) + \frac{\omega}{x}y'(x) + \tanh(y(x)) = 0, \quad y(0) = 1, \ y'(0) = 0, \quad 0 < x \leq 1. \tag{3.89}$$

We compute the expansion coefficients $\mathcal{B}^1_{2\ell+2,\omega}$ for $\ell = 0, 1, 2, 3, 4$ as follows.

- ($\ell = 0$) It is seen here upon using the formula (3.85) (with $m = 1$) that

$$\mathcal{B}^1_{2,\omega} = -\frac{\tanh 1}{2(\omega+1)} = -\frac{0.380797}{\omega+1}. \tag{3.90}$$

- ($\ell = 1$) It is understood here that $\mathcal{B}^1_{4,\omega} = -\mathsf{C}_{2,1}/(4(\omega+3))$, where

$$\begin{aligned}\mathsf{C}_{2,1} &= -\frac{\tanh 1}{2(\omega+1)} \sum_{m=0}^{\infty} \frac{(2m+1)2^{2m+2}\left(2^{2m+2}-1\right) B_{2m+2}}{(2m+2)!} \\ &= -\frac{\tanh 1}{2(\omega+1)} \left(\operatorname{csch}^2(1) - 4\operatorname{csch}^2(2)\right) = -\frac{0.159925}{\omega+1}.\end{aligned}$$

  As a result we have

$$\mathcal{B}^1_{4,\omega} = \frac{\tanh 1\left(\operatorname{csch}^2(1) - 4\operatorname{csch}^2(2)\right)}{8(\omega+1)(\omega+3)} = \frac{0.0399813}{(\omega+1)(\omega+3)}. \tag{3.91}$$

- ($\ell = 2$) One sees here that $\mathcal{B}^1_{6,\omega} = -\mathsf{C}_{4,1}/(6(\omega+5))$, where

$$\begin{aligned}\mathsf{C}_{4,1} =& \frac{\tanh 1\left(\operatorname{csch}^2(1) - 4\operatorname{csch}^2(2)\right)^2}{8(\omega+1)(\omega+3)} + \frac{\tanh^2 1}{4(\omega+1)^2}\sum_{m=1}^{\infty}\frac{m(2m+1)2^{2m+2}\left(2^{2m+2}-1\right)B_{2m+2}}{(2m+2)!}\\ =& -\frac{0.0295888\omega + 0.122349}{(\omega+1)^2(\omega+3)}.\end{aligned}$$

Consequently, we have

$$\mathcal{B}^1_{6,\omega} = \frac{0.00493146\omega + 0.0203914}{(\omega+1)^2(\omega+3)(\omega+5)}. \tag{3.92}$$

- ($\ell = 3$) It is seen that $\mathcal{B}^1_{8,\omega} = -\mathsf{C}_{6,1}/(8(\omega+7))$, where

$$\begin{aligned}\mathsf{C}_{6,1} =& \mathcal{B}^1_{6,\omega}\left(\operatorname{csch}^2(1) - 4\operatorname{csch}^2(2)\right) + \left(\mathcal{B}^1_{2,\omega}\right)^3\sum_{m=1}^{\infty}\frac{m\left(4m^2-1\right)2^{2m+2}\left(2^{2m+2}-1\right)B_{2m+2}}{3(2m+2)!}\\ &+ \mathcal{B}^1_{2,\omega}\mathcal{B}^1_{4,\omega}\sum_{m=1}^{\infty}\frac{2m(2m+1)2^{2m+2}\left(2^{2m+2}-1\right)B_{2m+2}}{(2m+2)!}\\ =& \frac{0.00608725\omega^2 + 0.0232859\omega - 0.0285854}{(\omega+1)^3(\omega+3)(\omega+5)}.\end{aligned}$$

One now obtains

$$\mathcal{B}^1_{8,\omega} = -\frac{0.000760907\omega^2 + 0.00291074\omega - 0.00357318}{(\omega+1)^3(\omega+3)(\omega+5)(\omega+7)}. \tag{3.93}$$

- ($\ell = 4$) One understands here that $\mathcal{B}^1_{10,\omega} = -\mathsf{C}_{8,1}/(10(\omega+9))$, where

$$\begin{aligned}\mathsf{C}_{8,1} =& \mathcal{B}^1_{8,\omega}\left(\operatorname{csch}^2(1) - 4\operatorname{csch}^2(2)\right) + \left(\mathcal{B}^1_{4,\omega}\right)^2\sum_{m=1}^{\infty}\frac{m(2m+1)2^{2m+2}\left(2^{2m+2}-1\right)B_{2m+2}}{(2m+2)!}\\ &+ \mathcal{B}^1_{4,\omega}\left(\mathcal{B}^1_{2,\omega}\right)^2\sum_{m=1}^{\infty}\frac{m(2m-1)(2m+1)2^{2m+2}\left(2^{2m+2}-1\right)B_{2m+2}}{(2m+2)!}\\ &+ \mathcal{B}^1_{6,\omega}\mathcal{B}^1_{2,\omega}\sum_{m=1}^{\infty}\frac{2m(2m+1)2^{2m+2}\left(2^{2m+2}-1\right)B_{2m+2}}{(2m+2)!}\\ &+ \left(\mathcal{B}^1_{2,\omega}\right)^4\sum_{m=2}^{\infty}\binom{2m+1}{4}\frac{2^{2m+2}\left(2^{2m+2}-1\right)B_{2m+2}}{(2m+2)!}\\ =& \frac{0.00276395\omega^4 + 0.0480006\omega^3 + 0.280423\omega^2 + 0.644784\omega + 0.466321}{(\omega+1)^4(\omega+3)^2(\omega+5)(\omega+7)}.\end{aligned}$$

It follows that

$$\mathcal{B}^1_{10,\omega} = -\frac{0.000276395\omega^4 + 0.00480006\omega^3 + 0.0280423\omega^2 + 0.0644784\omega + 0.0466321}{(\omega+1)^4(\omega+3)^2(\omega+5)(\omega+7)(\omega+9)}. \tag{3.94}$$

Hence, the approximate analytical solution of the problem (3.89) is given by

$$\begin{aligned}y(x) =& 1 - \frac{0.380797}{\omega+1}x^2 + \frac{0.0399813}{(\omega+1)(\omega+3)}x^4 + \frac{0.00493146\omega + 0.0203914}{(\omega+1)^2(\omega+3)(\omega+5)}x^6\\ &- \frac{0.000760907\omega^2 + 0.00291074\omega - 0.00357318}{(\omega+1)^3(\omega+3)(\omega+5)(\omega+7)}x^8\\ &- \frac{0.000276395\omega^4 + 0.00480006\omega^3 + 0.0280423\omega^2 + 0.0644784\omega + 0.0466321}{(\omega+1)^4(\omega+3)^2(\omega+5)(\omega+7)(\omega+9)}x^{10} + \cdots.\end{aligned} \tag{3.95}$$

For the higher cases $m = 2, 3, 4$ we follow the procedure as in Subsection 3.1 and we present the results as follows.

Example 3.13. The Lane-Emden initial value problem

$$y''(x) + \frac{\omega}{x}y'(x) + \tanh^2(y(x)) = 0, \quad y(0) = 1, \ y'(0) = 0, \quad 0 < x \le 1 \tag{3.96}$$

has the series solution

$$\begin{aligned} y(x) =& 1 - \frac{0.290013}{\omega+1}x^2 + \frac{0.060899}{(\omega+1)(\omega+3)}x^4 + \frac{0.00324888\omega + 0.018272}{(\omega+1)^2(\omega+3)(\omega+5)}x^6 \\ & - \frac{0.002342\omega^2 + 0.0152402\omega + 0.0190156}{(\omega+1)^3(\omega+3)(\omega+5)(\omega+7)}x^8 \\ & - \frac{0.00031385\omega^4 + 0.00639422\omega^3 + 0.0453408\omega^2 + 0.135019\omega + 0.142622}{(\omega+1)^4(\omega+3)^2(\omega+5)(\omega+7)(\omega+9)}x^{10} + \cdots. \end{aligned} \tag{3.97}$$

Example 3.14. The Lane-Emden problem

$$y''(x) + \frac{\omega}{x}y'(x) + \tanh^3(y(x)) = 0, \quad y(0) = 1, \ y'(0) = 0, \quad 0 < x \le 1 \tag{3.98}$$

admits the approximate solution

$$\begin{aligned} y(x) =& 1 - \frac{0.220872}{\omega+1}x^2 + \frac{0.0695705}{(\omega+1)(\omega+3)}x^4 + \frac{0.00626484 - 0.00115814n}{(\omega+1)^2(\omega+3)(\omega+5)}x^6 \\ & - \frac{0.0281252\omega^2 + 0.205324\omega + 0.335959}{8(\omega+1)^3(\omega+3)(\omega+5)(\omega+7)}x^8 \\ & - \frac{9.10242 \times 10^{-6}\omega^4 + 0.00191273\omega^3 + 0.0245977\omega^2 + 0.105802\omega + 0.146147}{(\omega+1)^4(\omega+3)^2(\omega+5)(\omega+7)(\omega+9)}x^{10} + \cdots. \end{aligned} \tag{3.99}$$

Example 3.15. The initial value problem

$$y''(x) + \frac{\omega}{x}y'(x) + \tanh^4(y(x)) = 0, \quad y(0) = 1, \ y'(0) = 0, \quad 0 < x \le 1 \tag{3.100}$$

has the approximate analytical solution

$$\begin{aligned} y(x) =& 1 - \frac{0.168215}{\omega+1}x^2 + \frac{0.0706459}{(\omega+1)(\omega+3)}x^4 - \frac{0.00612096\omega + 0.00847305}{(\omega+1)^2(\omega+3)(\omega+5)}x^6 \\ & - \frac{0.00390396\omega^2 + 0.0302915\omega + 0.0554414}{(\omega+1)^3(\omega+3)(\omega+5)(\omega+7)}x^8 \\ & + \frac{0.000510721\omega^4 + 0.00676694\omega^3 + 0.0257211\omega^2 + 0.0128412\omega - 0.0518764}{(\omega+1)^4(\omega+3)^2(\omega+5)(\omega+7)(\omega+9)}x^{10} + \cdots. \end{aligned} \tag{3.101}$$

3.4. Lane-Emden Equation Involving Hyperbolic Function $\Lambda(y(x)) = \operatorname{sech} y(x)$

Here we consider the analytical solution of the Lane-Emden equation associated with the nonlinearity $\operatorname{sech}^m(y(x))$. The result is given as a corollary.

Corollary 3.16. *For $m \in \mathbb{N}_0, \omega \in \mathbb{R}$, the initial value problem*

$$y''(x) + \frac{\omega}{x}y'(x) + \operatorname{sech}^m(y(x)) = 0, \quad y(0) = 1, \ y'(0) = 0, \quad 0 < x \le 1. \tag{3.102}$$

admits the analytical solution

$$y(x) = 1 + \sum_{\ell=0}^{\infty} \mathcal{B}^m_{2\ell+2,\omega} x^{2\ell+2}, \tag{3.103}$$

where the coefficients $\mathcal{B}^m_{2\ell+2,\omega}$ *are given by* (3.12) *with*

$$\mathsf{C}_{0,m} = (a_0 + a_2 + a_4 + a_6 + \cdots)^m = \operatorname{sech}^m(1) = (0.648054)^m \tag{3.104}$$

$$\mathsf{C}_{\ell,m} = \sum_{s_{m-1}=0}^{\ell} \sum_{s_{m-2}=0}^{s_{m-1}} \cdots \sum_{s_1=0}^{s_2} \mathsf{c}_{\ell-s_{m-1}} \mathsf{c}_{s_{m-1}-s_{m-2}} \cdots \mathsf{c}_{s_2-s_1} \mathsf{c}_{s_1}, \quad \ell = 2,4,6,\ldots \tag{3.105}$$

$$\mathsf{C}_{\ell,1} = \mathsf{c}_\ell = a_0 \mathsf{B}_{\ell,0} + a_2 \mathsf{B}_{\ell,2} + a_4 \mathsf{B}_{\ell,4} + a_6 \mathsf{B}_{\ell,6} + \cdots \quad \ell = 2,4,6,\ldots . \tag{3.106}$$

Here the (*Maclaurin*) *coefficients* $a_{2k}, k = 0,1,2,3,\ldots$ *are defined by*

$$\operatorname{sech} y(x) = \sum_{k=0}^{\infty} a_{2k} y^{2k}(x) = \sum_{k=0}^{\infty} \frac{E_{2k}}{(2k)!} y^{2k}(x), \tag{3.107}$$

where E_j *is the* j*th Euler number.*

We now compute explicitly the coefficients $\mathsf{C}_{\ell,1}$ to obtain recursively the expansion coefficients $\mathcal{B}^1_{2\ell+2,\omega}$. The higher cases $m = 2,3,4$ can be done similarly as in Subsection 3.1 or Subsection 3.3.

Example 3.17. Consider the following problem.

$$y''(x) + \frac{\omega}{x} y'(x) + \operatorname{sech}(y(x)) = 0, \quad y(0) = 1, \ y'(0) = 0, \quad 0 < x \le 1. \tag{3.108}$$

We compute the expansion coefficients $\mathcal{B}^1_{2\ell+2,\omega}$ of the series solution of (3.108) for the first values $\ell = 0,1,2,3,4$ as follows.

- ($\ell = 0$) It is seen here upon using the formula (3.104) (with $m = 1$) that

$$\mathcal{B}^1_{2,\omega} = -\frac{\operatorname{sech}(1)}{2(\omega+1)} = -\frac{0.324027}{\omega+1}. \tag{3.109}$$

- ($\ell = 1$) It is understood here that $\mathcal{B}^1_{4,\omega} = -\mathsf{C}_{2,1}/(4(\omega+3))$, where

$$\mathsf{C}_{2,1} = -\frac{\operatorname{sech}(1)}{2(\omega+1)} \sum_{m=1}^{\infty} \frac{(2m)E_{2m}}{(2m)!} = -\frac{\operatorname{sech}(1)}{2(\omega+1)} \sum_{m=1}^{\infty} \frac{E_{2m}}{(2m-1)!} = \frac{0.159925}{\omega+1}.$$

  As a result we have

$$\mathcal{B}^1_{4,\omega} = -\frac{0.0399812}{(\omega+1)(\omega+3)}. \tag{3.110}$$

- ($\ell = 2$) One sees here that $\mathcal{B}^1_{6,\omega} = -\mathsf{C}_{4,1}/(6(\omega+5))$, where

$$\begin{aligned}\mathsf{C}_{4,1} =& \mathcal{B}^1_{4,\omega} \sum_{m=1}^{\infty} \frac{(2m)E_{2m}}{(2m)!} + \left(\mathcal{B}^1_{2,n}\right)^2 \sum_{m=1}^{\infty} \frac{m(2m-1)E_{2m}}{(2m)!} \\ =& \frac{0.0251784\omega + 0.0360695}{(\omega+1)^2(\omega+3)}\end{aligned}$$

  As a result we have the expansion coefficient

$$\mathcal{B}^1_{6,\omega} = -\frac{0.00419641\omega + 0.00601158}{(\omega+1)^2(\omega+3)(\omega+5)}. \tag{3.111}$$

- ($\ell = 3$) It is seen that $\mathcal{B}^1_{8,\omega} = -\mathsf{C}_{6,1}/(8(\omega+7))$, where

$$\begin{aligned}\mathsf{C}_{6,1} =& \mathcal{B}^1_{6,\omega}\left(\sum_{m=1}^{\infty}\frac{(2m)E_{2m}}{(2m)!}\right) + \left(\mathcal{B}^1_{2,\omega}\right)^3\sum_{m=1}^{\infty}\frac{(2m)(m+1)(2m+1)E_{2m}}{3(2m)!}\\ &+ \mathcal{B}^1_{2,\omega}\mathcal{B}^1_{4,\omega}\sum_{m=1}^{\infty}\frac{(2m)(2m-1)E_{2m}}{(2m)!}\\ =&\frac{0.0124218\omega^2+0.0851557\omega+0.144788}{(\omega+1)^3(\omega+3)(\omega+5)}.\end{aligned}$$

Consequently, we have

$$\mathcal{B}^1_{8,\omega} = -\frac{0.00155273\omega^2+0.0106445\omega+0.0180986}{(\omega+1)^3(\omega+3)(\omega+5)(\omega+7)}. \tag{3.112}$$

- ($\ell = 4$) One understands here that $\mathcal{B}^1_{10,\omega} = -\mathsf{C}_{8,1}/(10(\omega+9))$, where

$$\begin{aligned}\mathsf{C}_{8,1} =& \mathcal{B}^1_{8,\omega}\left(\sum_{m=1}^{\infty}\frac{(2m)E_{2m}}{(2m)!}\right) + \left(\mathcal{B}^1_{4,\omega}\right)^2\sum_{m=1}^{\infty}\frac{m(2m-1)E_{2m}}{(2m)!}\\ &+ \mathcal{B}^1_{4,\omega}\left(\mathcal{B}^1_{2,\omega}\right)^2\sum_{m=1}^{\infty}\frac{(2m)(m+1)(2m+1)E_{2m}}{(2m)!}\\ &+ \mathcal{B}^1_{6,\omega}\mathcal{B}^1_{2,\omega}\sum_{m=1}^{\infty}\frac{(2m)(2m-1)E_{2m}}{(2m)!} + \left(\mathcal{B}^1_{2,\omega}\right)^4\sum_{m=0}^{\infty}\binom{2m+4}{4}\frac{E_{2m}}{(2m)!}\\ =& -\frac{0.00228548\omega^4+0.0543989\omega^3+0.436196\omega^2+1.44762\omega+1.69807}{(\omega+1)^4(\omega+3)^2(\omega+5)(\omega+7)}.\end{aligned}$$

Hence, we obtain

$$\mathcal{B}^1_{10,\omega} = \frac{0.000228548\omega^4+0.00543989\omega^3+0.0436196\omega^2+0.144762\omega+0.169807}{(\omega+1)^4(\omega+3)^2(\omega+5)(\omega+7)(\omega+9)}. \tag{3.113}$$

Therefore, the approximate analytical solution of the problem (3.108) is given by

$$\begin{aligned}y(x) =& 1 - \frac{0.324027}{\omega+1}x^2 - \frac{0.0399812}{(\omega+1)(\omega+3)}x^4 - \frac{0.00419641\omega+0.00601158}{(\omega+1)^2(\omega+3)(\omega+5)}x^6\\ &- \frac{0.00155273\omega^2+0.0106445\omega+0.0180986}{(\omega+1)^3(\omega+3)(\omega+5)(\omega+7)}x^8\\ &+ \frac{0.000228548\omega^4+0.00543989\omega^3+0.0436196\omega^2+0.144762\omega+0.169807}{(\omega+1)^4(\omega+3)^2(\omega+5)(\omega+7)(\omega+9)}x^{10} + \cdots.\end{aligned} \tag{3.114}$$

4. The Adomian Decomposition Method

This section presents applications of the Adomian decomposition method ([2, 3, 4]) to obtain highly accurate analytical solutions to the strongly nonlinear Lane-Emden initial value problem (2.1). The Adomian decomposition method expresses the solution of a given problem as a series and is capable of solving analytically nonlinear differential equations ([47, 48]).

With respect to the problem under consideration, the starting point is the construction of the Adomian polynomials for the nonlinear functions $f(y) = \Lambda^m(y)$ appearing in the strongly nonlinear Lane-Emden equation (2.1). Consequently, the method expresses the solution $y(x)$ as an infinite series of components:

$$y(x) = \sum_{k=0}^{\infty} y_k(x). \tag{4.1}$$

Let the nonlinear term $f(y)$ admit an infinite series

$$f(y(x)) = \sum_{k=0}^{\infty} \mu_k(Y), \tag{4.2}$$

where

$$\mu_k(Y) := \mu_k\left(y_0, y_1, y_2, \ldots, y_k\right) \tag{4.3}$$

are the Adomian polynomials. The functions μ_k are the Adomian polynomials and are given by the formula

$$\mu_k = \frac{1}{k!}\frac{d^k}{d\lambda^k}\left[f\left(\sum_{j=0}^{k}\lambda^j y_j\right)\right]_{\lambda=0}, \quad k = 0, 1, 2, 3, \ldots . \tag{4.4}$$

Upon evaluating the right hand side of (4.4) we find that the first these polynomials are given by

$$\begin{aligned}
&\mu_0\left(y_0\right) = f\left(y_0\right)\\
&\mu_1\left(y_0, y_1\right) = y_1 f'\left(y_0\right)\\
&\mu_2\left(y_0, y_1, y_2\right) = y_2 f'\left(y_0\right) + \frac{1}{2!}\left(y_1\right)^2 f''\left(y_0\right)\\
&\mu_3\left(y_0, y_1, y_2, y_3\right) = y_3 f'\left(y_0\right) + y_1 y_2 f''\left(y_0\right) + \frac{1}{3!}\left(y_1\right)^3 f'''\left(y_0\right)\\
&\mu_4\left(y_0, y_1, y_2, y_3, y_4\right) = y_4 f'\left(y_0\right) + \left(\frac{\left(y_2\right)^2}{2!} + y_1 y_3\right) f''\left(y_0\right) + \frac{\left(y_1\right)^2}{2!} y_2 f'''\left(y_0\right) + \frac{\left(y_1\right)^4}{4!} f^{(4)}\left(y_0\right).
\end{aligned} \tag{4.5}$$

[49, 47, 48] for further discussions on Adomian polynomials and their explicit computations for various classes of nonlinear functions. Following [48], one has the solution of the problem (2.1) given by

$$y(x) = \sum_{k=0}^{\infty} y_k(x) = A - L^{-1} f(y), \qquad y(0) = y_0, \quad y'(0) = 0, \tag{4.6}$$

where

$$L^{-1} f(y) = \int_0^x \frac{1}{x^\omega} \int_0^x x^\omega f(y)\, dx dx \tag{4.7}$$

satisfying the recurrence relation

$$\begin{aligned}
&y_0(x) = y_0\\
&y_{k+1}(x) = -L^{-1}\mu_k\left(y_0, y_1, y_2, \ldots, y_k\right), \qquad k = 0, 1, 2, 3, \ldots .
\end{aligned} \tag{4.8}$$

Indeed with $f(y) = \Lambda^m(y)$, we make the following notations. Let

$$\begin{aligned}
A_1 =& p_1 a_0^{p_1-1} a_1,\ A_2 = p_1 a_0^{p_1-1} a_2 + p_2 a_0^{p_1-2} a_1^2,\ A_3 = p_1 a_0^{p_1-1} a_3 + 3p_2 a_0^{p_1-2} a_1 a_2 + p_3 a_0^{p_1-3} a_1^3\\
A_4 =& p_1 a_0^{p_1-1} a_4 + 3p_2 a_0^{p_1-2} a_2^2 + 4p_2 a_0^{p_1-2} a_1 a_3 + 6p_3 a_0^{p_1-3} a_1^2 a_2 + p_4 a_0^{p_1-4} a_1^4,
\end{aligned} \tag{4.9}$$

where we have used the notation $\Lambda\left(y_0\right) = a_0, \Lambda'\left(y_0\right) = a_1, \Lambda''\left(y_0\right) = a_2, \Lambda'''\left(y_0\right) = a_3, \Lambda^{(4)}\left(y_0\right) = a_4$; and $p_1 = m, p_2 = m(m-1), p_3 = m(m-1)(m-2), p_4 = m(m-1)(m-2)(m-3)$. Thus we have the following first Adomian polynomials.

$$\begin{aligned}
&\mu_0\left(y_0\right) = \Lambda^m\left(y_0\right)\\
&\mu_1\left(y_0, y_1\right) = A_1 y_1\\
&\mu_2\left(y_0, y_1, y_2\right) = A_1 y_2 + \frac{A_2}{2}\left(y_1\right)^2\\
&A_3\left(y_0, y_1, y_2, y_3\right) = A_1 y_3 + A_2 y_1 y_2 + \frac{A_3}{6}\left(y_1\right)^3\\
&\mu_4\left(y_0, y_1, y_2, y_3, y_4\right) = A_1 y_4 + A_2\left(\frac{\left(y_2\right)^2}{2} + y_1 y_3\right) + \frac{A_3}{2}\left(y_1\right)^2 y_2 + \frac{A_4}{24}\left(y_1\right)^4.
\end{aligned} \tag{4.10}$$

Upon evaluating the recurrence relation (4.8) gives the following result.

Theorem 4.1 ([10])**.** *Let $0 < x \leq 1$, $m \in \mathbb{N}_0$, real $\omega \geq 0$. The Lane-Emden problem*

$$\begin{aligned} & y''(x) + \frac{\omega}{x} y'(x) + \Lambda^m(y(x)) = 0 \\ & y(0) = y_0, \quad y'(0) = 0, \quad y_0 = 0 \text{ or } 1, \end{aligned} \tag{4.11}$$

admits the analytical solution

$$y(x) = y_0 - \mathscr{P}_1^\omega x^2 + \mathscr{P}_2^\omega x^4 - \mathscr{P}_3^\omega x^6 + \mathscr{P}_4^\omega x^8 - \mathscr{P}_5^\omega x^{10} + \cdots, \tag{4.12}$$

where the first coefficients $\mathscr{P}_\ell^\omega, \ell = 1, 2, 3, 4, 5$ are given by

$$\mathscr{P}_1^\omega = \frac{\Lambda^m(y_0)}{2(\omega+1)}, \quad \mathscr{P}_2^\omega = \frac{m\Lambda'(y_0)\Lambda^{2m-1}(y_0)}{8(\omega+1)(\omega+3)} \tag{4.13}$$

$$\mathscr{P}_3^\omega = \frac{(\Lambda'(y_0))^2 \Lambda^{3p_1-2}(y_0)\left[\omega\left(p_1^2+p_2\right)+p_1^2+3p_2\right] + (\omega+3)p_1\Lambda^{3p_1-1}\Lambda''(y_0)}{48(\omega+1)^2(\omega+3)(\omega+5)} \tag{4.14}$$

$$\begin{aligned} \mathscr{P}_4^\omega = & \frac{\left[\omega^2\left(p_1^3+4p_2p_1+p_3\right)+2\omega\left(p_1^3+11p_2p_1+4p_3\right)+p_1^3+18p_2p_1+15p_3\right] a_0^{4p_1-3}a_1^3}{384(\omega+1)^3(\omega+3)(\omega+5)(\omega+7)} \\ & + \frac{\left[\omega^2\left(4p_1^2+3p_2\right)+2\omega\left(11p_1^2+12p_2\right)+18p_1^2+45p_2\right] a_0^{4p_1-2}a_1a_2}{384(\omega+1)^3(\omega+3)(\omega+5)(\omega+7)} \\ & + \frac{(\omega+3)(\omega+5)p_1a_0^{4p_1-1}a_3}{384(\omega+1)^3(\omega+3)(\omega+5)(\omega+7)} \end{aligned} \tag{4.15}$$

$$\mathscr{P}_5^\omega = \frac{B_1a_0^{5p_1-4}a_1^4 + B_2a_0^{5p_1-3}a_1^2a_2 + B_3a_0^{5p_1-2}a_2^2 + B_4a_0^{5p_1-1}a_4 + B_5a_0^{5p_1-2}a_1a_3}{3840(\omega+1)^4(\omega+3)^2(\omega+5)(\omega+7)(\omega+9)} \tag{4.16}$$

with

$$\begin{aligned} B_1 = & \omega^4\left(p_1^4+11p_2p_1^2+7p_3p_1+4p_2^2+p_4\right)+2\omega^3\left(3p_1^4+64p_2p_1^2+54p_3p_1+28p_2^2+9p_4\right) \\ & + 2\omega^2\left(6p_1^4+233p_2p_1^2+283p_3p_1+128p_2^2+58p_4\right) \\ & + 2\omega\left(5p_1^4+296p_2p_1^2+570p_3p_1+228p_2^2+159p_4\right) \\ & + 3\left(p_1^4+81p_2p_1^2+225p_3p_1+84p_2^2+105p_4\right) \end{aligned} \tag{4.17}$$

$$\begin{aligned} B_2 = & \omega^4\left(11p_1^3+29p_2p_1+6p_3\right)+4\omega^3\left(32p_1^3+109p_2p_1+27p_3\right) \\ & + 2\omega^2\left(233p_1^3+1105p_1p_2+348p_3\right)+4\omega\left(148p_1^3+1083p_2p_1+477p_3\right) \\ & + 9\left(27p_1^3+281p_2p_1+210p_3\right) \end{aligned} \tag{4.18}$$

$$\begin{aligned} B_3 = & (\omega+3)^2(\omega+7)\left(4\omega p_1^2+3\omega p_2+4p_1^2+15p_2\right), \quad B_4 = (\omega+3)^2(\omega+5)(\omega+7)p_1 \\ B_5 = & (\omega+3)(\omega+5)\left(\omega^2\left(7p_1^2+4p_2\right)+4\omega(13p_1^2+10p_2)+45p_1^2+84p_2\right). \end{aligned} \tag{4.19}$$

Here $p_1 = m$, $p_2 = m(m-1)$, $p_3 = m(m-1)(m-2)$, $p_4 = m(m-1)(m-2)(m-3)$; and $\Lambda(y_0) = a_0$, $\Lambda'(y_0) = a_1$, $\Lambda''(y_0) = a_2, \Lambda'''(y_0) = a_3$, $\Lambda^{(4)}(y_0) = a_4$.

In particular, for $m = 0$, one has the solution

$$y(x) = y_0 - \frac{x^2}{2(\omega+1)}. \tag{4.20}$$

4.1. Lane-Emden Equation Involving Nonlinearity $\Lambda(y(x)) = \tan y(x)$

We consider the case of the tangent function $\tan y$. This case is given as a corollary and is presented as follows with a set of examples.

Corollary 4.2. *For $0 < x \leq 1$, $m \in \mathbb{N}_0$, real $\omega \geq 0$, the Lane-Emden type problem*

$$y''(x) + \frac{\omega}{x}y'(x) + \tan^m(y(x)) = 0, \quad y(0) = 1, \ \ y'(0) = 0, \quad 0 < x \leq 1, \tag{4.21}$$

admits the analytical solution given by the series

$$y(x) = 1 - \frac{k_1^m x^2}{2(\omega+1)} + \frac{m k_2^2 k_1^{2m-1} x^4}{8(\omega+1)(\omega+3)} - \mathscr{P}_3^\omega x^6 + \mathscr{P}_4^\omega x^8 - \mathscr{P}_5^\omega x^{10} + \cdots, \tag{4.22}$$

where $\mathscr{P}_3^\omega, \mathscr{P}_4^\omega, \mathscr{P}_5^\omega$ are given respectively by

$$\mathscr{P}_3^\omega = \frac{\left[\omega\left(p_1^2 + p_2\right) + p_1^2 + 3p_2\right] k_2^4 k_1^{3p_1-2} + 2p_1(\omega+3)k_1^{3p_1}k_2^2}{48(\omega+1)^2(\omega+3)(\omega+5)} \tag{4.23}$$

$$\begin{aligned}
\mathscr{P}_4^\omega =& \frac{\left[\omega^2\left(4p_1p_2 + p_3\right) + 2\omega\left(11p_1p_2 + 4p_3\right) + 18p_1p_2 + 15p_3\right] k_2^6 k_1^{4p_1-3}}{384(\omega+1)^3(\omega+3)(\omega+5)(\omega+7)} \\
&+ \frac{2p_1\left[\omega^2\left(4p_1+1\right) + 2\omega\left(11p_1+4\right) + 18p_1 + 15\right] k_2^4 k_1^{4p_1-1} + p_1^3(\omega+1)^2 k_1^{4p_1-3}k_2^6}{384(\omega+1)^3(\omega+3)(\omega+5)(\omega+7)} \\
&+ \frac{6p_2(\omega+3)(\omega+5)k_2^4k_1^{4p_1-1} + 4p_1(\omega+3)(\omega+5)k_2^2k_1^{4p_1+1}}{384(\omega+1)^3(\omega+3)(\omega+5)(\omega+7)}
\end{aligned} \tag{4.24}$$

$$\begin{aligned}
\mathscr{P}_5^\omega =& \frac{B_1k_2^4k_1^{5p_1} + B_2k_2^8k_1^{5p_1-4} + B_3k_2^6k_1^{5p_1-2} + B_4k_2^2k_1^{5p_1+2}}{3840(\omega+1)^4(\omega+3)^2(\omega+5)(\omega+7)(\omega+9)} \\
B_1 =& 4(\omega+3)\left[\omega^3\left(11p_1^2 + 4p_1 + 7p_2\right) + \omega^2\left(131p_1^2 + 60p_1 + 105p_2\right)\right. \\
&\left. + \omega\left(429p_1^2 + 284p_1 + 497p_2\right) + 309p_1^2 + 420p_1 + 735p_2\right] \\
B_2 =& \left[\omega^4\left(p_1^4 + 11p_2p_1^2 + 7p_3p_1 + 4p_2^2 + p_4\right) + 2\omega^3\left(3p_1^4 + 64p_2p_1^2 + 54p_3p_1 + 28p_2^2 + 9p_4\right)\right. \\
& + 2\omega^2\left(6p_1^4 + 233p_2p_1^2 + 283p_3p_1 + 128p_2^2 + 58p_4\right) \\
& + 2\omega\left(5p_1^4 + 296p_2p_1^2 + 570p_3p_1 + 228p_2^2 + 159p_4\right) \\
& \left. + 3p_1^4 + 252p_2^2 + 243p_1^2p_2 + 675p_1p_3 + 315p_4\right] \\
B_3 =& 2\left[\omega^4\left(11p_1^3 + 7p_1^2 + 29p_2p_1 + 4p_2 + 6p_3\right) + 4\omega^3\left(32p_1^3 + 27p_1^2 + 109p_2p_1 + 18p_2 + 27p_3\right)\right. \\
& + 2\omega^2\left(233p_1^3 + 283p_1^2 + 1105p_2p_1 + 232p_2 + 348p_3\right) \\
& + 4\omega\left(148p_1^3 + 285p_1^2 + 1083p_2p_1 + 318p_2 + 477p_3\right) \\
& \left. + 243p_1^3 + 675p_1^2 + 2529p_1p_2 + 1260p_2 + 1890p_3\right] \\
B_4 =& 8p_1(\omega+3)^2(\omega+5)(\omega+7).
\end{aligned} \tag{4.25}$$

Proof. One sees in this case that $\Lambda(y) = \tan(y)$ with $y_0 = 1$. Now, setting $a_0 = \Lambda(1) = \tan(1) = k_1$, $a_1 = \Lambda'(1) = \sec^2(1) = k_2^2$, $a_2 = \Lambda''(1) = 2\tan(1)\sec^2(1) = 2k_1k_2^2$, $a_3 = \Lambda'''(1) = 2\sec^4(1) + 4\tan^2(1)\sec^2(1) = 2k_2^4 + 4k_1^2k_2^2$, $a_4 = \Lambda^{(4)}(1) = 16\tan(1)\sec^4(1) + 8\tan^3(1)\sec^2(1) = 16k_1k_2^4 + 8k_1^3k_2^2$; with $p_1 = m$, $p_2 = m(m-1)$, $p_3 = m(m-1)(m-2)$, $p_4 = m(m-1)(m-2)(m-3)$ in Theorem 4.1, we obtain the required result. □

For clearer and more explicit expressions of the expansion coefficients, we give some special cases of Corollary 4.2 as examples.

Example 4.3 ($m = 1$)**.** The analytical solution of the Lane-Emden problem

$$y''(x) + \frac{\omega}{x}y'(x) + \tan(y(x)) = 0, \quad y(0) = 1, \ \ y'(0) = 0, \quad 0 < x \leq 1, \tag{4.26}$$

has the series expansion given by

$$\begin{aligned} y(x) =&1-\frac{k_1x^2}{2(\omega+1)}+\frac{k_1k_2^2x^4}{8(\omega+1)(\omega+3)}-\frac{(\omega+1)k_1k_2^4+2(\omega+3)k_1^3k_2^2}{48(\omega+1)^2(\omega+3)(\omega+5)}x^6\\ &+\frac{2\left(5\omega^2+30\omega+33\right)k_1^3k_2^4+(\omega+1)^2k_1k_2^6+4(\omega+3)(\omega+5)k_1^5k_2^2}{384(\omega+1)^3(\omega+3)(\omega+5)(\omega+7)}x^8\\ &-\frac{4(\omega+1)\left(9\omega^3+109\omega^2+407\omega+459\right)k_1^3k_2^6}{3840(\omega+1)^4(\omega+3)^2(\omega+5)(\omega+7)(\omega+9)}x^{10}\\ &-\frac{4(\omega+3)\left(15\omega^3+191\omega^2+713\omega+729\right)k_1^5k_2^4}{3840(\omega+1)^4(\omega+3)^2(\omega+5)(\omega+7)(\omega+9)}x^{10}\\ &-\frac{(\omega+1)^3(\omega+3)k_1k_2^8+8(\omega+3)^2(\omega+5)(\omega+7)k_1^7k_2^2}{3840(\omega+1)^4(\omega+3)^2(\omega+5)(\omega+7)(\omega+9)}x^{10}+\cdots. \end{aligned} \tag{4.27}$$

Example 4.4 ($m=2$)**.** Consider the Lane-Emden initial value problem

$$y''(x)+\frac{\omega}{x}y'(x)+\tan^2(y(x))=0,\quad y(0)=1,\ \ y'(0)=0,\quad 0<x\le 1. \tag{4.28}$$

We have the solution

$$\begin{aligned} y(x) =&1-\frac{k_1^2x^2}{2(\omega+1)}+\frac{k_1^3k_2^2x^4}{4(\omega+1)(\omega+3)}-\frac{4(\omega+3)k_2^2k_1^6+(6\omega+10)k_2^4k_1^4}{48(\omega+1)^2(\omega+3)(\omega+5)}x^6\\ &+\frac{16\left(3\omega^2+19\omega+24\right)k_2^4k_1^7+8(\omega+3)(\omega+5)k_2^2k_1^9+8(\omega+1)(3\omega+10)k_2^6k_1^5}{384(\omega+1)^3(\omega+3)(\omega+5)(\omega+7)}x^8\\ &-\frac{24(\omega+1)(\omega+5)\left(5\omega^2+26\omega+25\right)k_2^8k_1^6}{3840(\omega+1)^4(\omega+3)^2(\omega+5)(\omega+7)(\omega+9)}x^{10}\\ &-\frac{8(\omega+3)\left(33\omega^3+427\omega^2+1639\omega+1773\right)k_2^4k_1^{10}}{3840(\omega+1)^4(\omega+3)^2(\omega+5)(\omega+7)(\omega+9)}x^{10}\\ &-\frac{32\left(15\omega^4+209\omega^3+985\omega^2+1823\omega+1080\right)k_2^6k_1^8}{3840(\omega+1)^4(\omega+3)^2(\omega+5)(\omega+7)(\omega+9)}x^{10}\\ &-\frac{16(\omega+3)^2(\omega+5)(\omega+7)k_2^2k_1^{12}}{3840(\omega+1)^4(\omega+3)^2(\omega+5)(\omega+7)(\omega+9)}x^{10}+\cdots. \end{aligned} \tag{4.29}$$

Example 4.5 ($m=3$)**.** The solution of the Lane-Emden problem

$$y''(x)+\frac{\omega}{x}y'(x)+\tan^3(y(x))=0,\quad y(0)=1,\ \ y'(0)=0,\quad 0<x\le 1, \tag{4.30}$$

is given by

$$\begin{aligned} y(x) =&1-\frac{k_1^3x^2}{2(\omega+1)}+\frac{3k_1^5k_2^2x^4}{8(\omega+1)(\omega+3)}-\frac{6(\omega+3)k_2^2k_1^9+(15\omega+27)k_2^4k_1^7}{48(\omega+1)^2(\omega+3)(\omega+5)}x^6\\ &+\frac{6\left(19\omega^2+122\omega+159\right)k_2^4k_1^{11}+3\left(35\omega^2+166\omega+147\right)k_2^6k_1^9+12(\omega+3)(\omega+5)k_2^2k_1^{13}}{384(\omega+1)^3(\omega+3)(\omega+5)(\omega+7)}x^8\\ &-\frac{36(\omega+3)\left(17\omega^3+221\omega^2+855\omega+939\right)k_2^4k_1^{15}}{3840(\omega+1)^4(\omega+3)^2(\omega+5)(\omega+7)(\omega+9)}x^{10}\\ &-\frac{9(\omega+1)\left(105\omega^3+1157\omega^2+3903\omega+3843\right)k_2^8k_1^{11}}{3840(\omega+1)^4(\omega+3)^2(\omega+5)(\omega+7)(\omega+9)}x^{10}\\ &-\frac{12\left(157\omega^4+2226\omega^3+10736\omega^2+20550\omega+12843\right)k_2^6k_1^{13}}{3840(\omega+1)^4(\omega+3)^2(\omega+5)(\omega+7)(\omega+9)}x^{10}\\ &-\frac{24(\omega+3)^2(\omega+5)(\omega+7)k_2^2k_1^{17}}{3840(\omega+1)^4(\omega+3)^2(\omega+5)(\omega+7)(\omega+9)}x^{10}+\cdots. \end{aligned} \tag{4.31}$$

Example 4.6 $(m = 4)$**.** The solution of the initial value problem

$$y''(x) + \frac{\omega}{x}y'(x) + \tan^4(y(x)) = 0, \quad y(0) = 1, \;\; y'(0) = 0, \quad 0 < x \leq 1, \tag{4.32}$$

admits the series expansion

$$\begin{aligned} y(x) =& 1 - \frac{k_1^4 x^2}{2(\omega+1)} + \frac{k_1^7 k_2^2 x^4}{2(\omega+1)(\omega+3)} - \frac{8(\omega+3)k_2^2k_1^{12} + (28\omega+52)k_2^4k_1^{10}}{48(\omega+1)^2(\omega+3)(\omega+5)}x^6 \\ &+ \frac{16\left(13\omega^2+84\omega+111\right)k_2^4k_1^{15} + 8\left(35\omega^2+172\omega+161\right)k_2^6k_1^{13} + 16(\omega+3)(\omega+5)k_2^2k_1^{17}}{384(\omega+1)^3(\omega+3)(\omega+5)(\omega+7)}x^8 \\ &- \frac{16(\omega+3)(3\omega+13)\left(23\omega^2+200\omega+297\right)k_2^4k_1^{20}}{3840(\omega+1)^4(\omega+3)^2(\omega+5)(\omega+7)(\omega+9)}x^{10} \\ &- \frac{64\left(75\omega^4+1072\omega^3+5226\omega^2+10160\omega+6507\right)k_2^6k_1^{18}}{3840(\omega+1)^4(\omega+3)^2(\omega+5)(\omega+7)(\omega+9)}x^{10} \\ &- \frac{8\left(455\omega^4+5622\omega^3+23316\omega^2+37370\omega+19509\right)k_2^8k_1^{16}}{3840(\omega+1)^4(\omega+3)^2(\omega+5)(\omega+7)(\omega+9)}x^{10} \\ &- \frac{32(\omega+3)^2(\omega+5)(\omega+7)k_2^2k_1^{22}}{3840(\omega+1)^4(\omega+3)^2(\omega+5)(\omega+7)(\omega+9)}x^{10} + \cdots . \end{aligned} \tag{4.33}$$

4.2. Lane-Emden Equation Involving Nonlinearity $\Lambda(y(x)) = \sec y(x)$

We consider the analytical solution of the Lane-Emden equation corresponding to the nonlinearity $\sec^m(y(x))$. Here also the special cases $m = 1, 2, 3, 4$ are computed in a similar way.

Corollary 4.7. *For $0 < x \leq 1$, $m \in \mathbb{N}_0$, real $\omega \geq 0$, the Lane-Emden type problem*

$$y''(x) + \frac{\omega}{x}y'(x) + \sec^m(y(x)) = 0, \quad y(0) = 1, \;\; y'(0) = 0, \quad 0 < x \leq 1, \tag{4.34}$$

admits the analytical solution given by the series

$$y(x) = 1 - \frac{k_2^m x^2}{2(\omega+1)} + \frac{m k_1 k_2^{2m} x^4}{8(\omega+1)(\omega+3)} - \mathscr{P}_3^\omega x^6 + \mathscr{P}_4^\omega x^8 - \mathscr{P}_5^\omega x^{10} + \cdots , \tag{4.35}$$

where $\mathscr{P}_3^\omega, \mathscr{P}_4^\omega, \mathscr{P}_5^\omega$ are given respectively by

$$\mathscr{P}_3^\omega = \frac{\left[\omega\left(p_1^2+p_1+p_2\right)+p_1^2+3p_1+3p_2\right]k_1^2k_2^{3p_1} + (\omega+3)p_1k_2^{3p_1+2}}{48(\omega+1)^2(\omega+3)(\omega+5)} \tag{4.36}$$

$$\begin{aligned} \mathscr{P}_4^\omega =& \frac{\left[\omega^2\left(p_1^3+4p_1^2+4p_2p_1+p_1+3p_2+p_3\right)\right]k_1^3k_2^{4p_1}}{384(\omega+1)^3(\omega+3)(\omega+5)(\omega+7)} \\ &+ \frac{\left[2\omega\left(p_1^3+11p_1^2+11p_2p_1+4p_1+12p_2+4p_3\right)\right]k_1^3k_2^{4p_1}}{384(\omega+1)^3(\omega+3)(\omega+5)(\omega+7)} \\ &+ \frac{\left(p_1^3+18p_1^2+18p_2p_1+15p_1+45p_2+15p_3\right)k_1^3k_2^{4p_1}}{384(\omega+1)^3(\omega+3)(\omega+5)(\omega+7)} \\ &+ \frac{\left[\omega^2\left(4p_1^2+5p_1+3p_2\right)+2\omega\left(11p_1^2+20p_1+12p_2\right)+3\left(6p_1^2+25p_1+15p_2\right)\right]k_1k_2^{4p_1+2}}{384(\omega+1)^3(\omega+3)(\omega+5)(\omega+7)} \end{aligned} \tag{4.37}$$

$$\begin{aligned}
\mathscr{P}_5^\omega =&\frac{B_1 k_2^{5p_1+4}+B_2 k_1^2 k_2^{5p_1+2}+B_3 k_1^4 k_2^{5p_1}}{3840(\omega+1)^4(\omega+3)^2(\omega+5)(\omega+7)(\omega+9)}\\
B_1 =&(\omega+3)^2(\omega+7)\left[\omega\left(4p_1^2+5p_1+3p_2\right)+4p_1^2+25p_1+15p_2\right]\\
B_2 =&\omega^4\left(11p_1^3+43p_1^2+29p_2p_1+18p_1+26p_2+6p_3\right)\\
&+4\omega^3\left(32p_1^3+163p_1^2+109p_2p_1+81p_1+117p_2+27p_3\right)\\
&+2\omega^2\left(233p_1^3+1671p_1^2+1105p_2p_1+1044p_1+1508p_2+348p_3\right)\\
&+4\omega\left(148p_1^3+1653p_1^2+1083p_2p_1+1431p_1+2067p_2+477p_3\right)\\
&+9\left(27p_1^3+431p_1^2+281p_2p_1+630p_1+910p_2+210p_3\right)\\
B_3 =&\omega^4\left(p_1^4+11p_1^3+11p_2p_1^2+11p_1^2+29p_2p_1+7p_3p_1+p_1+4p_2^2+7p_2+6p_3+p_4\right)\\
&+2\omega^3\left(3p_1^4+64p_1^3+64p_2p_1^2+82p_1^2+218p_2p_1+54p_3p_1+9p_1+28p_2^2+63p_2\right.\\
&\left.+54p_3+9p_4\right)+2\omega^2\left(6p_1^4+233p_1^3+233p_2p_1^2+411p_1^2+1105p_2p_1+283p_3p_1+58p_1\right.\\
&\left.+128p_2^2+406p_2+348p_3+58p_4\right)+2\omega\left(5p_1^4+296p_1^3+296p_2p_1^2+798p_1^2+2166p_2p_1\right.\\
&\left.+570p_3p_1+159p_1+228p_2^2+1113p_2+954p_3+159p_4\right)+3\left(p_1^4+81p_1^3+81p_2p_1^2\right.\\
&\left.+309p_1^2+843p_2p_1+225p_3p_1+105p_1+84p_2^2+735p_2+630p_3+105p_4\right).
\end{aligned} \tag{4.38}$$

Proof. It is seen here that $\Lambda(y)=\sec(y(x))$ with $y_0=1$. Now, putting $a_0=\Lambda(1)=\sec(1)=k_2$, $a_1=\Lambda'(1)=\tan(1)\sec(1)=k_1k_2$, $a_2=\Lambda''(1)=\sec^3(1)+\tan^2(1)\sec(1)=k_2^3+k_1^2k_2$, $a_3=\Lambda'''(1)=5\tan(1)\sec^3(1)+\tan^3(1)\sec(1)=k_2k_1^3+5k_2^3k_1$, $a_4=\Lambda^{(4)}(1)=5\sec^5(1)+18\tan^2(1)\sec^3(1)+\tan^4(1)\sec(1)=5k_2^5+18k_1^2k_2^3+k_1^4k_2$; with $p_1=m$, $p_2=m(m-1)$, $p_3=m(m-1)(m-2)$, $p_4=m(m-1)(m-2)(m-3)$ in Theorem 4.1, we obtain the result as required.

□

Next we give some special cases of Corollary 4.7 for clearer and more explicit expressions of these expansion coefficients. These special cases are given as examples.

Example 4.8 ($m=1$)**.** The analytical solution of the Lane-Emden type problem

$$y''(x)+\frac{\omega}{x}y'(x)+\sec(y(x))=0,\quad y(0)=1,\ \ y'(0)=0,\quad 0<x\le 1, \tag{4.39}$$

admits the series expansion

$$\begin{aligned}
y(x)=&1-\frac{k_2x^2}{2(\omega+1)}+\frac{k_1k_2^2x^4}{8(\omega+1)(\omega+3)}-\frac{(\omega+3)k_2^5+2(\omega+2)k_1^2k_2^3}{48(\omega+1)^2(\omega+3)(\omega+5)}x^6\\
&+\frac{\left(9\omega^2+62\omega+93\right)k_1k_2^6+2\left(3\omega^2+16\omega+17\right)k_1^3k_2^4}{384(\omega+1)^3(\omega+3)(\omega+5)(\omega+7)}x^8\\
&-\frac{4(\omega+4)\left(6\omega^3+55\omega^2+134\omega+93\right)k_1^4k_2^5}{3840(\omega+1)^4(\omega+3)^2(\omega+5)(\omega+7)(\omega+9)}x^{10}\\
&-\frac{8\left(9\omega^4+138\omega^3+737\omega^2+1616\omega+1224\right)k_1^2k_2^7}{3840(\omega+1)^4(\omega+3)^2(\omega+5)(\omega+7)(\omega+9)}x^{10}\\
&-\frac{(\omega+3)^2(\omega+7)(9\omega+29)k_2^9}{3840(\omega+1)^4(\omega+3)^2(\omega+5)(\omega+7)(\omega+9)}x^{10}+\cdots.
\end{aligned} \tag{4.40}$$

Example 4.9 ($m=2$)**.** Consider the Lane-Emden type problem

$$y''(x)+\frac{\omega}{x}y'(x)+\sec^2(y(x))=0,\quad y(0)=1,\ \ y'(0)=0,\quad 0<x\le 1. \tag{4.41}$$

The solution of this problem is given by

$$\begin{aligned}y(x) =&1-\frac{k_2^2x^2}{2(\omega+1)}+\frac{k_1k_2^4x^4}{4(\omega+1)(\omega+3)}-\frac{2(\omega+3)k_2^8+8(\omega+2)k_1^2k_2^6}{48(\omega+1)^2(\omega+3)(\omega+5)}x^6\\&+\frac{8\left(4\omega^2+27\omega+39\right)k_1k_2^{10}+16\left(3\omega^2+16\omega+17\right)k_1^3k_2^8}{384(\omega+1)^3(\omega+3)(\omega+5)(\omega+7)}x^8\\&-\frac{64(\omega+4)\left(6\omega^3+55\omega^2+134\omega+93\right)k_1^4k_2^{10}}{3840(\omega+1)^4(\omega+3)^2(\omega+5)(\omega+7)(\omega+9)}x^{10}\\&-\frac{16\left(29\omega^4+435\omega^3+2259\omega^2+4781\omega+3456\right)k_1^2k_2^{12}+32(\omega+3)^3(\omega+7)k_2^{14}}{3840(\omega+1)^4(\omega+3)^2(\omega+5)(\omega+7)(\omega+9)}x^{10}+\cdots.\end{aligned}\tag{4.42}$$

Example 4.10 ($m=3$)**.** The solution of the Lane-Emden type problem

$$y''(x)+\frac{\omega}{x}y'(x)+\sec^3(y(x))=0,\quad y(0)=1,\ \ y'(0)=0,\quad 0<x\le 1,\tag{4.43}$$

is given by

$$\begin{aligned}y(x) =&1-\frac{k_2^3x^2}{2(\omega+1)}+\frac{3k_1k_2^6x^4}{8(\omega+1)(\omega+3)}-\frac{3(\omega+3)k_2^{11}+18(\omega+2)k_1^2k_2^9}{48(\omega+1)^2(\omega+3)(\omega+5)}x^6\\&+\frac{3\left(23\omega^2+154\omega+219\right)k_1k_2^{14}+54\left(3\omega^2+16\omega+17\right)k_1^3k_2^{12}}{384(\omega+1)^3(\omega+3)(\omega+5)(\omega+7)}x^8\\&-\frac{324(\omega+4)\left(6\omega^3+55\omega^2+134\omega+93\right)k_1^4k_2^{15}}{3840(\omega+1)^4(\omega+3)^2(\omega+5)(\omega+7)(\omega+9)}x^{10}\\&-\frac{12\left(121\omega^4+1800\omega^3+9248\omega^2+19308\omega+13707\right)k_1^2k_2^{17}}{3840(\omega+1)^4(\omega+3)^2(\omega+5)(\omega+7)(\omega+9)}x^{10}\\&-\frac{3(\omega+3)^2(\omega+7)(23\omega+67)k_2^{19}}{3840(\omega+1)^4(\omega+3)^2(\omega+5)(\omega+7)(\omega+9)}x^{10}+\cdots.\end{aligned}\tag{4.44}$$

Example 4.11 ($m=4$)**.** The analytical solution of the Lane-Emden type problem

$$y''(x)+\frac{\omega}{x}y'(x)+\sec^4(y(x))=0,\quad y(0)=1,\ \ y'(0)=0,\quad 0<x\le 1,\tag{4.45}$$

is given by

$$\begin{aligned}y(x) =&1-\frac{k_2^4x^2}{2(\omega+1)}+\frac{k_1k_2^8x^4}{2(\omega+1)(\omega+3)}-\frac{4(\omega+3)k_2^{14}+32(\omega+2)k_1^2k_2^{12}}{48(\omega+1)^2(\omega+3)(\omega+5)}x^6\\&+\frac{8\left(15\omega^2+100\omega+141\right)k_1k_2^{18}+128\left(3\omega^2+16\omega+17\right)k_1^3k_2^{16}}{384(\omega+1)^3(\omega+3)(\omega+5)(\omega+7)}x^8\\&-\frac{1024(\omega+4)\left(6\omega^3+55\omega^2+134\omega+93\right)k_1^4k_2^{20}}{3840(\omega+1)^4(\omega+3)^2(\omega+5)(\omega+7)(\omega+9)}x^{10}\\&-\frac{16\left(207\omega^4+3066\omega^3+15664\omega^2+32470\omega+22833\right)k_1^2k_2^{22}}{3840(\omega+1)^4(\omega+3)^2(\omega+5)(\omega+7)(\omega+9)}x^{10}\\&-\frac{8(\omega+3)^2(\omega+7)(15\omega+43)k_2^{24}}{3840(\omega+1)^4(\omega+3)^2(\omega+5)(\omega+7)(\omega+9)}x^{10}+\cdots.\end{aligned}\tag{4.46}$$

,

Table 1: Comparison between PSM and ADM for the nonlinearities $f(y) = \sec y, \operatorname{sech} y$ and $f(y) = \tan^m y, \tanh^m y$ with the special values $m = 1, 2, 3, 4; \omega = 2$. We use the notation PSM, ADM

$x/f(y)$	0.1	0.2	0.5	1.0
$\tan y$	$0.997409, 0.997409$	$0.989688, 0.989688$	$0.937712, 0.937712$	$0.775293, 0.775293$
$\tan^2 y$	$0.995971, 0.995979$	$0.984048, 0.984165$	$0.906796, 0.91028$	$0.690429, 0.640153$
$\tan^3 y$	$0.993736, 0.993781$	$0.975323, 0.975992$	$0.860401, 0.87531$	$0.557853, -3.49457$
$\tan^4 y$	$0.990261, 0.99044$	$0.961825, 0.964346$	$0.790712, 0.761073$	$0.330305, -127.887$
$x/f(y)$	0.1	0.2	0.5	1.0
$\sec y$	$0.99692, 0.99692$	$0.987731, 0.987731$	$0.925458, 0.925444$	$0.720993, 0.724$
$x/f(y)$	0.1	0.2	0.5	1.0
$\tanh y$	$0.998731, 0.998731$	$0.994927, 0.994927$	$0.968435, 0.968435$	$0.875828, 0.875828$
$\tanh^2 y$	$0.999034, 0.999034$	$0.99614, 0.996138$	$0.976087, 0.976026$	$0.90746, 0.906439$
$\tanh^3 y$	$0.999264, 0.999264$	$0.997062, 0.997059$	$0.981884, 0.981762$	$0.931014, 0.929037$
$\tanh^4 y$	$0.99944, 0.999439$	$0.997765, 0.99776$	$0.986275, 0.986111$	$0.948557, 0.945968$
$x/f(y)$	0.1	0.2	0.5	1.0
$\operatorname{sech} y$	$0.99892, 0.99892$	$0.995675, 0.995675$	$0.97283, 0.97283$	$0.889275, 0.889281$

4.3. Lane-Emden Equation Involving Nonlinearity $\Lambda(y(x)) = \tanh y(x)$

In this subsection we present the result on the analytical solution of the Lane-Emden equation corresponding to the nonlinearity $\tanh^m(y(x))$. The result is given as a corollary and the special cases $m = 1, 2, 3, 4$ are computed similarly as in Subsection 3.1.

Corollary 4.12. *For $0 < x \leq 1$, $m \in \mathbb{N}_0$, real $\omega \geq 0$, the Lane-Emden type problem*

$$y''(x) + \frac{\omega}{x} y'(x) + \tanh^m(y(x)) = 0, \quad y(0) = 1, \ \ y'(0) = 0, \quad 0 < x \leq 1, \tag{4.47}$$

admits the analytical solution given by the series

$$y(x) = 1 - \frac{h_1^m x^2}{2(\omega+1)} + \frac{m h_2^2 h_1^{2m-1} x^4}{8(\omega+1)(\omega+3)} - \mathscr{P}_3^\omega x^6 + \mathscr{P}_4^\omega x^8 - \mathscr{P}_5^\omega x^{10} + \cdots, \tag{4.48}$$

where $\mathscr{P}_3^\omega, \mathscr{P}_4^\omega, \mathscr{P}_5^\omega$ are given respectively by

$$\mathscr{P}_3^\omega = \frac{\left[\omega\left(p_1^2 + p_2\right) + p_1^2 + 3p_2\right] h_2^4 h_1^{3p_1-2} - 2(\omega+3) p_1 h_1^{3p_1} h_2^2}{48(\omega+1)^2(\omega+3)(\omega+5)} \tag{4.49}$$

$$\begin{aligned}\mathscr{P}_4^\omega =& \frac{\left[\omega^2\left(p_1^3 + 4p_2p_1 + p_3\right) + 2\omega\left(p_1^3 + 11p_2p_1 + 4p_3\right) + p_1^3 + 18p_2p_1 + 15p_3\right] h_2^6 h_1^{4p_1-3}}{384(\omega+1)^3(\omega+3)(\omega+5)(\omega+7)} \\ &- \frac{2\left[\omega^2\left(4p_1^2 + p_1 + 3p_2\right) + 2\omega\left(11p_1^2 + 4p_1 + 12p_2\right) + 18p_1^2 + 15p_1 + 45p_2\right] h_2^4 h_1^{4p_1-1}}{384(\omega+1)^3(\omega+3)(\omega+5)(\omega+7)} \\ &+ \frac{4p_1(\omega+3)(\omega+5) h_1^{4p_1+1} h_2^2}{384(\omega+1)^3(\omega+3)(\omega+5)(\omega+7)}\end{aligned} \tag{4.50}$$

$$
\begin{aligned}
\mathscr{P}_5^{\omega} =&\frac{B_1 h_2^4 h_1^{5p_1} + B_2 h_2^8 h_1^{5p_1-4} + B_3 h_2^6 h_1^{5p_1-2} + B_4 h_1^{5p_1+2} h_2^2}{3840(\omega+1)^4(\omega+3)^2(\omega+5)(\omega+7)(\omega+9)}\\
B_1 =&4(\omega+3)\left[\omega^3\left(11p_1^2+4p_1+7p_2\right)+\omega^2\left(131p_1^2+60p_1+105p_2\right)\right.\\
&\left.+\omega\left(429p_1^2+284p_1+497p_2\right)+309p_1^2+420p_1+735p_2\right]\\
B_2 =&\left[\omega^4\left(p_1^4+11p_2p_1^2+7p_3p_1+4p_2^2+p_4\right)+2\omega^3\left(3p_1^4+64p_2p_1^2+54p_3p_1+28p_2^2+9p_4\right)\right.\\
&+2\omega^2\left(6p_1^4+233p_2p_1^2+283p_3p_1+128p_2^2+58p_4\right)\\
&+2\omega\left(5p_1^4+296p_2p_1^2+570p_3p_1+228p_2^2+159p_4\right)\\
&\left.+3p_1^4+252p_2^2+243p_1^2p_2+675p_1p_3+315p_4\right]\\
B_3 =&-2\left[\omega^4\left(11p_1^3+7p_1^2+29p_2p_1+4p_2+6p_3\right)+4\omega^3\left(32p_1^3+27p_1^2+109p_2p_1+18p_2+27p_3\right)\right.\\
&+2\omega^2\left(233p_1^3+283p_1^2+1105p_2p_1+232p_2+348p_3\right)\\
&+4\omega\left(148p_1^3+285p_1^2+1083p_2p_1+318p_2+477p_3\right)\\
&\left.+243p_1^3+675p_1^2+2529p_1p_2+1260p_2+1890p_3\right],\ B_4=-8p_1(\omega+3)^2(\omega+5)(\omega+7).
\end{aligned}
\tag{4.51}
$$

Proof. With $\Lambda(y)=\tanh(y)$, $y_0=1$, we make the following substitutions in Theorem 4.1: $a_0=\Lambda(1)=\tanh(1)=h_1$, $a_1=\Lambda'(1)=\operatorname{sech}^2(1)=h_2^2$, $a_2=\Lambda''(1)=-2\tanh(1)\operatorname{sech}^2(1)=-2h_1h_2^2$, $a_3=\Lambda'''(1)=-2\operatorname{sech}^4(1)+4\tanh^2(1)\sec^2(1)=-2h_2^4+4h_1^2h_2^2$, $a_4=\Lambda^{(4)}(1)=16\tanh(1)\operatorname{sech}^4(1)-8\tanh^3(1)\operatorname{sech}^2(1)=16h_1h_2^4-8h_1^3h_2^2$; with $p_1=m$, $p_2=m(m-1)$, $p_3=m(m-1)(m-2)$, $p_4=m(m-1)(m-2)(m-3)$. The result follows immediately. □

Next we consider some special cases of Corollary 4.12, namely, the particular cases $m=1,2,3,4$. These special cases are given as examples.

Example 4.13 ($m=1$)**.** The solution of the Lane-Emden type problem

$$
y''(x)+\frac{\omega}{x}y'(x)+\tanh(y(x))=0,\quad y(0)=1,\ \ y'(0)=0,\quad 0<x\leq 1,
\tag{4.52}
$$

is given by

$$
\begin{aligned}
y(x)=&1-\frac{h_1x^2}{2(\omega+1)}+\frac{h_1h_2^2x^4}{8(\omega+1)(\omega+3)}-\frac{h_1h_2^4(\omega+1)-2h_1^3h_2^2(\omega+3)}{48(\omega+1)^2(\omega+3)(\omega+5)}x^6\\
&+\frac{-2\left(5\omega^2+30\omega+33\right)h_1^3h_2^4+(\omega+1)^2h_1h_2^6+4(\omega+3)(\omega+5)h_1^5h_2^2}{384(\omega+1)^3(\omega+3)(\omega+5)(\omega+7)}x^8\\
&-\frac{-4(\omega+1)\left(9\omega^3+109\omega^2+407\omega+459\right)h_1^3h_2^6}{3840(\omega+1)^4(\omega+3)^2(\omega+5)(\omega+7)(\omega+9)}x^{10}\\
&-\frac{4(\omega+3)\left(15\omega^3+191\omega^2+713\omega+729\right)h_1^5h_2^4}{3840(\omega+1)^4(\omega+3)^2(\omega+5)(\omega+7)(\omega+9)}x^{10}\\
&-\frac{(\omega+1)^3(\omega+3)h_1h_2^8-8(\omega+3)^2(\omega+5)(\omega+7)h_1^7h_2^2}{3840(\omega+1)^4(\omega+3)^2(\omega+5)(\omega+7)(\omega+9)}x^{10}+\cdots.
\end{aligned}
\tag{4.53}
$$

Example 4.14 ($m=2$)**.** Consider the Lane-Emden type problem

$$
y''(x)+\frac{\omega}{x}y'(x)+\tanh^2(y(x))=0,\quad y(0)=1,\ \ y'(0)=0,\quad 0<x\leq 1.
\tag{4.54}
$$

The solution of this problem is given by

$$\begin{aligned}
y(x) =& 1-\frac{h_1^2x^2}{2(\omega+1)}+\frac{h_1^3h_2^2x^4}{4(\omega+1)(\omega+3)}-\frac{(6\omega+10)h_1^4h_2^4-4(\omega+3)h_1^6h_2^2}{48(\omega+1)^2(\omega+3)(\omega+5)}x^6\\
&+\frac{-16\left(3\omega^2+19\omega+24\right)h_2^4h_1^7+8(\omega+3)(\omega+5)h_2^2h_1^9+8(\omega+1)(3\omega+10)h_2^6h_1^5}{384(\omega+1)^3(\omega+3)(\omega+5)(\omega+7)}x^8\\
&-\frac{24(\omega+1)(\omega+5)\left(5\omega^2+26\omega+25\right)h_2^8h_1^6}{3840(\omega+1)^4(\omega+3)^2(\omega+5)(\omega+7)(\omega+9)}x^{10}\\
&-\frac{+8(\omega+3)\left(33\omega^3+427\omega^2+1639\omega+1773\right)h_2^4h_1^{10}}{3840(\omega+1)^4(\omega+3)^2(\omega+5)(\omega+7)(\omega+9)}x^{10}\\
&-\frac{-32\left(15\omega^4+209\omega^3+985\omega^2+1823\omega+1080\right)h_2^6h_1^8}{3840(\omega+1)^4(\omega+3)^2(\omega+5)(\omega+7)(\omega+9)}x^{10}\\
&-\frac{-16(\omega+3)^2(\omega+5)(\omega+7)h_2^2h_1^{12}}{3840(\omega+1)^4(\omega+3)^2(\omega+5)(\omega+7)(\omega+9)}x^{10}+\cdots.
\end{aligned} \tag{4.55}$$

Example 4.15 $(m=3)$**.** The approximate solution of the Lane-Emden type problem

$$y''(x)+\frac{\omega}{x}y'(x)+\tanh^3(y(x))=0,\quad y(0)=1,\ y'(0)=0,\quad 0<x\leq 1, \tag{4.56}$$

is given by

$$\begin{aligned}
y(x) =& 1-\frac{h_1^3x^2}{2(\omega 1)}+\frac{3h_1^5h_2^2x^4}{8(\omega+1)(\omega+3)}-\frac{(15\omega+27)h_1^7h_2^4-6(\omega+3)h_1^9h_2^2}{48(\omega+1)^2(\omega+3)(\omega+5)}x^6\\
&+\frac{-6\left(19\omega^2+122\omega+159\right)h_2^4h_1^{11}+3\left(35\omega^2+166\omega+147\right)h_2^6h_1^9+12(\omega+3)(\omega+5)h_2^2h_1^{13}}{384(\omega+1)^3(\omega+3)(\omega+5)(\omega+7)}x^8\\
&-\frac{36(\omega+3)\left(17\omega^3+221\omega^2+855\omega+939\right)h_2^4h_1^{15}}{3840(\omega+1)^4(\omega+3)^2(\omega+5)(\omega+7)(\omega+9)}x^{10}\\
&-\frac{9(\omega+1)\left(105\omega^3+1157\omega^2+3903\omega+3843\right)h_2^8h_1^{11}}{3840(\omega+1)^4(\omega+3)^2(\omega+5)(\omega+7)(\omega+9)}\\
&-\frac{-12\left(157\omega^4+2226\omega^3+10736\omega^2+20550\omega+12843\right)h_2^6h_1^{13}}{3840(\omega+1)^4(\omega+3)^2(\omega+5)(\omega+7)(\omega+9)}x^{10}\\
&-\frac{-24(\omega+3)^2(\omega+5)(\omega+7)h_2^2h_1^{17}}{3840(\omega+1)^4(\omega+3)^2(\omega+5)(\omega+7)(\omega+9)}x^{10}+\cdots.
\end{aligned} \tag{4.57}$$

Example 4.16 $(m=4)$**.** The solution of the Lane-Emden type problem

$$y''(x)+\frac{\omega}{x}y'(x)+\tanh^4(y(x))=0,\quad y(0)=1,\ y'(0)=0,\quad 0<x\leq 1, \tag{4.58}$$

admits the series expansion

$$
\begin{aligned}
y(x) =&1-\frac{h_1^4x^2}{2(\omega+1)}+\frac{h_1^7h_2^2x^4}{2(\omega+1)(\omega+3)}-\frac{(28\omega+52)h_1^{10}h_2^4-8(\omega+3)h_1^{12}h_2^2}{48(\omega+1)^2(\omega+3)(\omega+5)}x^6\\
&+\frac{-16\left(13\omega^2+84\omega+111\right)h_2^4h_1^{15}+8\left(35\omega^2+172\omega+161\right)h_2^6h_1^{13}+16(\omega+3)(\omega+5)h_2^2h_1^{17}}{384(\omega+1)^3(\omega+3)(\omega+5)(\omega+7)}x^8\\
&-\frac{16(\omega+3)(3\omega+13)\left(23\omega^2+200\omega+297\right)h_2^4h_1^{20}}{3840(\omega+1)^4(\omega+3)^2(\omega+5)(\omega+7)(\omega+9)}x^{10}\\
&-\frac{-64\left(75\omega^4+1072\omega^3+5226\omega^2+10160\omega+6507\right)h_2^6h_1^{18}}{3840(\omega+1)^4(\omega+3)^2(\omega+5)(\omega+7)(\omega+9)}x^{10}\\
&-\frac{8\left(455\omega^4+5622\omega^3+23316\omega^2+37370\omega+19509\right)h_2^8h_1^{16}}{3840(\omega+1)^4(\omega+3)^2(\omega+5)(\omega+7)(\omega+9)}x^{10}\\
&-\frac{-32(\omega+3)^2(\omega+5)(\omega+7)h_2^2h_1^{22}}{3840(\omega+1)^4(\omega+3)^2(\omega+5)(\omega+7)(\omega+9)}x^{10}+\cdots . \qquad (4.59)
\end{aligned}
$$

Table 2: Comparison between PSM and ADM for the nonlinearities $f(y)=\sec y, \operatorname{sech} y$ and $f(y)=\tan^m y, \tanh^m y$ with the special values $m=1,2,3,4; \omega=19$. We use the notation PSM, ADM

$x/f(y)$	0.1	0.2	0.5	1.0
$\tan y$	0.999611, 0.999611	0.998445, 0.998445	0.99036, 0.99036	0.962495, 0.962495
$\tan^2 y$	0.999394, 0.999394	0.997582, 0.997586	0.985129, 0.985281	0.943708, 0.945646
$\tan^3 y$	0.999057, 0.999058	0.99624, 0.996265	0.977061, 0.977915	0.915469, 0.9243
$\tan^4 y$	0.998532, 0.998538	0.994153, 0.994251	0.964613, 0.967745	0.873003, 0.872822
$x/f(y)$	0.1	0.2	0.5	1.0
$\sec y$	0.999537, 0.999537	0.998152, 0.998152	0.988525, 0.988525	0.955148, 0.955143
$x/f(y)$	0.1	0.2	0.5	1.0
$\tanh y$	0.99981, 0.99981	0.999239, 0.999239	0.995246, 0.995246	0.981052, 0.981052
$\tanh^2 y$	0.999855, 0.999855	0.99942, 0.99942	0.996383, 0.996381	0.985638, 0.985605
$\tanh^3 y$	0.99989, 0.99989	0.999559, 0.999558	0.997249, 0.997245	0.989114, 0.989048
$\tanh^4 y$	0.999916, 0.999916	0.999664, 0.999664	0.997907, 0.997902	0.991749, 0.99166
$x/f(y)$	0.1	0.2	0.5	1.0
$\operatorname{sech} y$	0.999838, 0.999838	0.999352, 0.999352	0.995944, 0.995944	0.983707, 0.983707

4.4. Lane-Emden Equation with Nonlinear Function $\Lambda(y(x))=\operatorname{sech} y(x)$

Here the solution of the Lane-Emden equation whose nonlinear term is given by $f(y)=\operatorname{sech}^m(y(x))$. The result is given as a corollary and the special cases $m=1,2,3,4$ are considered.

Corollary 4.17. *For $0<x\leq 1$, $m\in\mathbb{N}_0$, real $\omega\geq 0$, the Lane-Emden type problem*

$$y''(x)+\frac{\omega}{x}y'(x)+\operatorname{sech}^m(y(x))=0, \quad y(0)=1, \;\; y'(0)=0, \quad 0<x\leq 1, \qquad (4.60)$$

admits the analytical solution given by the series

$$y(x) =1-\frac{h_2^m x^2}{2(\omega+1)}-\frac{mh_1h_2^{2m}x^4}{8(\omega+1)(\omega+3)}-\mathscr{P}_3^\omega x^6+\mathscr{P}_4^\omega x^8-\mathscr{P}_5^\omega x^{10}+\cdots , \qquad (4.61)$$

where $\mathscr{P}_3^\omega, \mathscr{P}_4^\omega, \mathscr{P}_5^\omega$ *are given respectively by*

$$\mathscr{P}_3^\omega = \frac{\left[\omega\left(p_1^2+p_1+p_2\right)+p_1^2+3p_1+3p_2\right]h_1^2h_2^{3p_1}-(\omega+3)p_1h_2^{3p_1+2}}{48(\omega+1)^2(\omega+3)(\omega+5)} \tag{4.62}$$

$$\begin{aligned}\mathscr{P}_4^\omega =& -\frac{\left[\omega^2\left(p_1^3+4p_1^2+4p_2p_1+p_1+3p_2+p_3\right)\right]h_1^3h_2^{4p_1}}{384(\omega+1)^3(\omega+3)(\omega+5)(\omega+7)}\\ & -\frac{\left[2\omega\left(p_1^3+11p_1^2+11p_2p_1+4p_1+12p_2+4p_3\right)\right]h_1^3h_2^{4p_1}}{384(\omega+1)^3(\omega+3)(\omega+5)(\omega+7)}\\ & -\frac{\left[p_1^3+18p_1^2+18p_2p_1+15p_1+45p_2+15p_3\right]h_1^3h_2^{4p_1}}{384(\omega+1)^3(\omega+3)(\omega+5)(\omega+7)}\\ & +\frac{\left[\omega^2\left(4p_1^2+5p_1+3p_2\right)+2\omega\left(11p_1^2+20p_1+12p_2\right)+3\left(6p_1^2+25p_1+15p_2\right)\right]h_1h_2^{4p_1+2}}{384(\omega+1)^3(\omega+3)(\omega+5)(\omega+7)}\end{aligned} \tag{4.63}$$

$$\begin{aligned}\mathscr{P}_5^\omega =& \frac{B_1h_2^{5p_1+4}+B_2h_1^2h_2^{5p_1+2}+B_3h_1^4h_2^{5p_1}}{3840(\omega+1)^4(\omega+3)^2(\omega+5)(\omega+7)(\omega+9)}\\ B_1 =& (\omega+3)^2(\omega+7)\left[\omega\left(4p_1^2+5p_1+3p_2\right)+4p_1^2+25p_1+15p_2\right]\\ B_2 =& -\omega^4\left(11p_1^3+43p_1^2+29p_2p_1+18p_1+26p_2+6p_3\right)\\ & -4\omega^3\left(32p_1^3+163p_1^2+109p_2p_1+81p_1+117p_2+27p_3\right)\\ & -2\omega^2\left(233p_1^3+1671p_1^2+1105p_2p_1+1044p_1+1508p_2+348p_3\right)\\ & -4\omega\left(148p_1^3+1653p_1^2+1083p_2p_1+1431p_1+2067p_2+477p_3\right)\\ & -9\left(27p_1^3+431p_1^2+281p_2p_1+630p_1+910p_2+210p_3\right)\\ B_3 =& \omega^4\left(p_1^4+11p_1^3+11p_2p_1^2+11p_1^2+29p_2p_1+7p_3p_1+p_1+4p_2^2+7p_2+6p_3+p_4\right)\\ & +2\omega^3\left(3p_1^4+64p_1^3+64p_2p_1^2+82p_1^2+218p_2p_1+54p_3p_1+9p_1+28p_2^2+63p_2\right.\\ & \left.+54p_3+9p_4\right)+2\omega^2\left(6p_1^4+233p_1^3+233p_2p_1^2+411p_1^2+1105p_2p_1+283p_3p_1+58p_1\right.\\ & \left.+128p_2^2+406p_2+348p_3+58p_4\right)+2\omega\left(5p_1^4+296p_1^3+296p_2p_1^2+798p_1^2+2166p_2p_1\right.\\ & \left.+570p_3p_1+159p_1+228p_2^2+1113p_2+954p_3+159p_4\right)+3\left(p_1^4+81p_1^3+81p_2p_1^2\right.\\ & \left.+309p_1^2+843p_2p_1+225p_3p_1+105p_1+84p_2^2+735p_2+630p_3+105p_4\right).\end{aligned} \tag{4.64}$$

Proof. Upon noting that $\Lambda(y) = \operatorname{sech}(y(x))$ with $y_0 = 1$ and setting $a_0 = \Lambda(1) = \operatorname{sech}(1) = h_2$, $a_1 = \Lambda'(1) = -\tanh(1)\operatorname{sech}(1) = -h_1h_2$, $a_2 = \Lambda''(1) = \tanh^2(1)\operatorname{sech}(1) - \operatorname{sech}^3(1) = h_1^2h_2 - h_2^3$, $a_3 = \Lambda'''(1) = 5\tanh(1)\operatorname{sech}^3(1) - \tanh^3(1)\operatorname{sech}(1) = 5h_1h_2^3 - h_1^3h_2$, $a_4 = \Lambda^{(4)}(1) = 5\operatorname{sech}^5(1) - 18\tanh^2(1)\operatorname{sech}^3(1) + \tanh^4(1)\operatorname{sech}(1) = 5h_2^5 - 18h_1^2h_2^3 + h_1^4h_2$; with $p_1 = m$, $p_2 = m(m-1)$, $p_3 = m(m-1)(m-2)$, $p_4 = m(m-1)(m-2)(m-3)$, we obtain the result as required. □

We consider some particular cases of Corollary 4.17 for a more explicit expression of the expansion coefficients of the series solution. These special cases are given as examples.

Example 4.18 ($m = 1$)**.** The solution of the Lane-Emden type problem

$$y''(x) + \frac{\omega}{x}y'(x) + \operatorname{sech}(y(x)) = 0, \quad y(0) = 1, \;\; y'(0) = 0, \quad 0 < x \leq 1, \tag{4.65}$$

is given by

$$\begin{aligned} y(x) =& 1 - \frac{h_2 x^2}{2(\omega+1)} - \frac{h_1 h_2^2 x^4}{8(\omega+1)(\omega+3)} - \frac{2(\omega+2)h_1^2 h_2^3 - (\omega+3)h_2^5}{48(\omega+1)^2(\omega+3)(\omega+5)} x^6 \\ &+ \frac{\left(9\omega^2+62\omega+93\right) h_1 h_2^6 - 2\left(3\omega^2+16\omega+17\right) h_1^3 h_2^4}{384(\omega+1)^3(\omega+3)(\omega+5)(\omega+7)} x^8 \\ &- \frac{4(\omega+4)\left(6\omega^3+55\omega^2+134\omega+93\right) h_1^4 h_2^5}{3840(\omega+1)^4(\omega+3)^2(\omega+5)(\omega+7)(\omega+9)} x^{10} \\ &+ \frac{8\left(9\omega^4+138\omega^3+737\omega^2+1616\omega+1224\right) h_1^2 h_2^7}{3840(\omega+1)^4(\omega+3)^2(\omega+5)(\omega+7)(\omega+9)} x^{10} \\ &- \frac{(\omega+3)^2(\omega+7)(9\omega+29)h_2^9}{3840(\omega+1)^4(\omega+3)^2(\omega+5)(\omega+7)(\omega+9)} x^{10} + \cdots . \end{aligned} \tag{4.66}$$

Example 4.19 ($m = 2$). Consider the Lane-Emden type problem

$$y''(x) + \frac{\omega}{x} y'(x) + \operatorname{sech}^2(y(x)) = 0, \quad y(0) = 1, \ y'(0) = 0, \quad 0 < x \leq 1. \tag{4.67}$$

The solution of this problem is given by

$$\begin{aligned} y(x) =& 1 - \frac{h_2^2 x^2}{2(\omega+1)} - \frac{h_1 h_2^4 x^4}{4(\omega+1)(\omega+3)} - \frac{8(\omega+2)h_1^2 h_2^6 - 2(\omega+3)h_2^8}{48(\omega+1)^2(\omega+3)(\omega+5)} x^6 \\ &+ \frac{8\left(4\omega^2+27\omega+39\right) h_1 h_2^{10} - 16\left(3\omega^2+16\omega+17\right) h_1^3 h_2^8}{384(\omega+1)^3(\omega+3)(\omega+5)(\omega+7)} x^8 \\ &- \frac{64(\omega+4)\left(6\omega^3+55\omega^2+134\omega+93\right) h_1^4 h_2^{10}}{3840(\omega+1)^4(\omega+3)^2(\omega+5)(\omega+7)(\omega+9)} x^{10} \\ &- \frac{-16\left(29\omega^4+435\omega^3+2259\omega^2+4781\omega+3456\right) h_1^2 h_2^{12} + 32(\omega+3)^3(\omega+7)h_2^{14}}{3840(\omega+1)^4(\omega+3)^2(\omega+5)(\omega+7)(\omega+9)} x^{10} + \cdots . \end{aligned} \tag{4.68}$$

Example 4.20 ($m = 3$). The analytical solution of the Lane-Emden type problem

$$y''(x) + \frac{\omega}{x} y'(x) + \operatorname{sech}^3(y(x)) = 0, \quad y(0) = 1, \ y'(0) = 0, \quad 0 < x \leq 1, \tag{4.69}$$

is given by

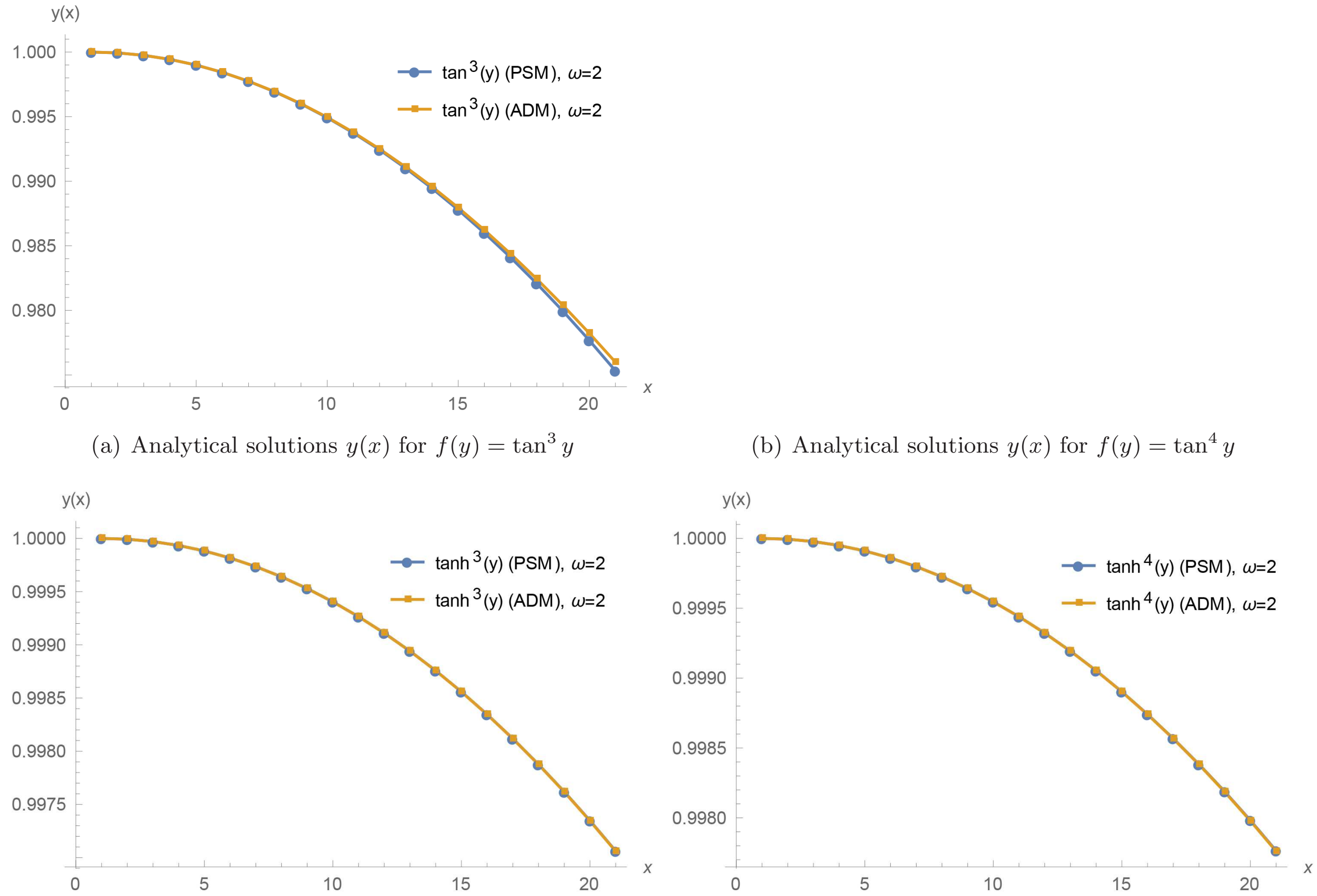
$$\begin{aligned} y(x) =& 1 - \frac{h_2^3 x^2}{2(\omega+1)} - \frac{3h_1 h_2^6 x^4}{8(\omega+1)(\omega+3)} - \frac{18(\omega+2)h_1^2 h_2^9 - 3(\omega+3)h_2^{11}}{48(\omega+1)^2(\omega+3)(\omega+5)} x^6 \\ &+ \frac{3\left(23\omega^2+154\omega+219\right) h_1 h_2^{14} - 54\left(3\omega^2+16\omega+17\right) h_1^3 h_2^{12}}{384(\omega+1)^3(\omega+3)(\omega+5)(\omega+7)} x^8 \\ &- \frac{324(\omega+4)\left(6\omega^3+55\omega^2+134\omega+93\right) h_1^4 h_2^{15}}{3840(\omega+1)^4(\omega+3)^2(\omega+5)(\omega+7)(\omega+9)} x^{10} \\ &+ \frac{12\left(121\omega^4+1800\omega^3+9248\omega^2+19308\omega+13707\right) h_1^2 h_2^{17}}{3840(\omega+1)^4(\omega+3)^2(\omega+5)(\omega+7)(\omega+9)} x^{10} \\ &- \frac{3(\omega+3)^2(\omega+7)(23\omega+67)h_2^{19}}{3840(\omega+1)^4(\omega+3)^2(\omega+5)(\omega+7)(\omega+9)} x^{10} + \cdots . \end{aligned} \tag{4.70}$$

Example 4.21 ($m = 4$). The Lane-Emden type problem

$$y''(x) + \frac{\omega}{x} y'(x) + \operatorname{sech}^4(y(x)) = 0, \quad y(0) = 1, \ y'(0) = 0, \quad 0 < x \leq 1, \tag{4.71}$$

has the analytical solution given by

$$\begin{aligned} y(x) =& 1 - \frac{h_2^4 x^2}{2(\omega+1)} - \frac{h_1 h_2^8 x^4}{2(\omega+1)(\omega+3)} - \frac{32(\omega+2)h_1^2 h_2^{12} - 4(n+3)h_2^{14}}{48(n+1)^2(\omega+3)(\omega+5)} x^6 \\ &+ \frac{8\left(15\omega^2+100\omega+141\right) h_1 h_2^{18} - 128\left(3\omega^2+16\omega+17\right) h_1^3 h_2^{16}}{384(\omega+1)^3(\omega+3)(\omega+5)(\omega+7)} x^8 \\ &- \frac{1024(\omega+4)\left(6\omega^3+55\omega^2+134\omega+93\right) h_1^4 h_2^{20}}{3840(\omega+1)^4(\omega+3)^2(\omega+5)(\omega+7)(\omega+9)} x^{10} \\ &+ \frac{16\left(207\omega^4+3066\omega^3+15664\omega^2+32470\omega+22833\right) h_1^2 h_2^{22}}{3840(\omega+1)^4(\omega+3)^2(\omega+5)(\omega+7)(\omega+9)} x^{10} \\ &- \frac{8(\omega+3)^2(\omega+7)(15\omega+43)h_2^{24}}{3840(\omega+1)^4(\omega+3)^2(\omega+5)(\omega+7)(\omega+9)} x^{10} + \cdots . \end{aligned} \tag{4.72}$$



(a) Analytical solutions $y(x)$ for $f(y) = \tan^3 y$

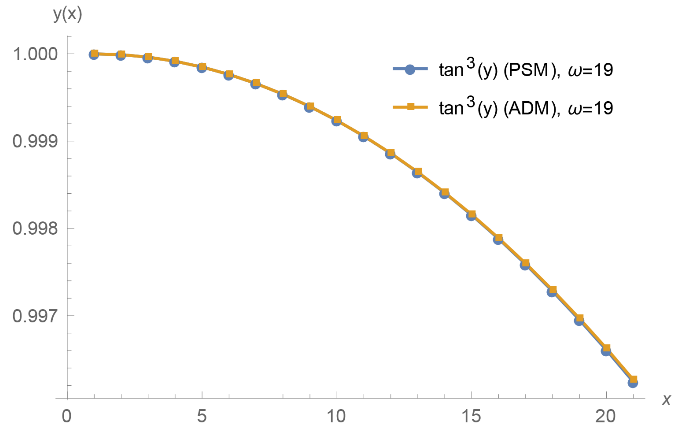
(b) Analytical solutions $y(x)$ for $f(y) = \tan^4 y$

(c) Analytical solutions $y(x)$ for $f(y) = \tanh^3 y$

(d) Analytical solutions $y(x)$ for $f(y) = \tanh^4 y$

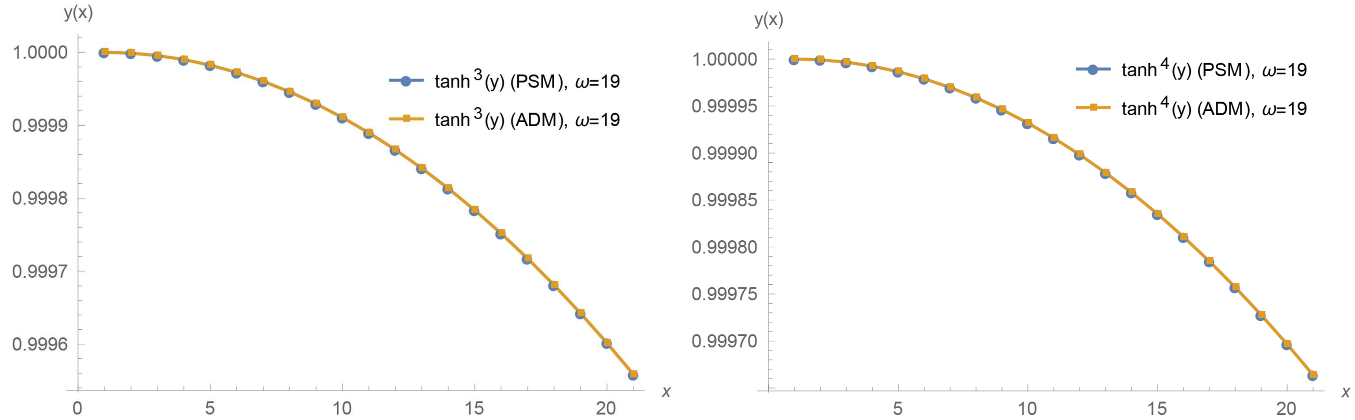
Figure 1: Comparisons of the approximate analytical solutions $y(x)$ using the present methods for the nonlinear functions $f(y) = \tan^m y, \tanh^m y, m = 3, 4; \omega = 2$.

The numerical comparison of results is given in Tables 1, 2 and 3; while the graphical illustration of results are provided in Figures 1, 2 and 3.



(a) Analytical solutions $y(x)$ for $f(y) = \tan^3 y$

(b) Analytical solutions $y(x)$ for $f(y) = \tan^4 y$



(c) Analytical solutions $y(x)$ for $f(y) = \tanh^3 y$

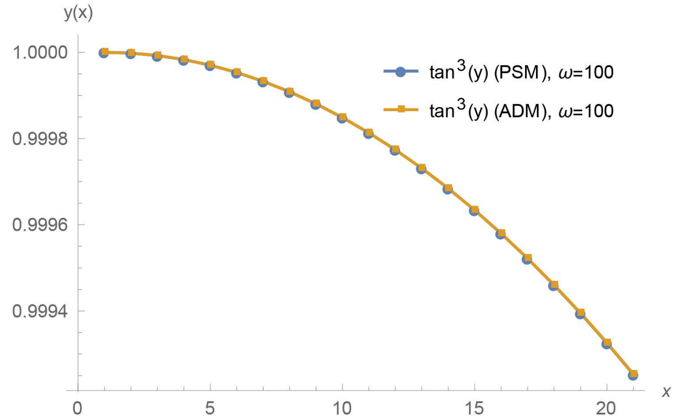
(d) Analytical solutions $y(x)$ for $f(y) = \tanh^4 y$

Figure 2: Comparisons of the approximate analytical solutions $y(x)$ using the present methods for the nonlinear functions $f(y) = \tan^m y, \tanh^m y, m = 3, 4; \omega = 19$.

5. Comparison of Results and Discussion

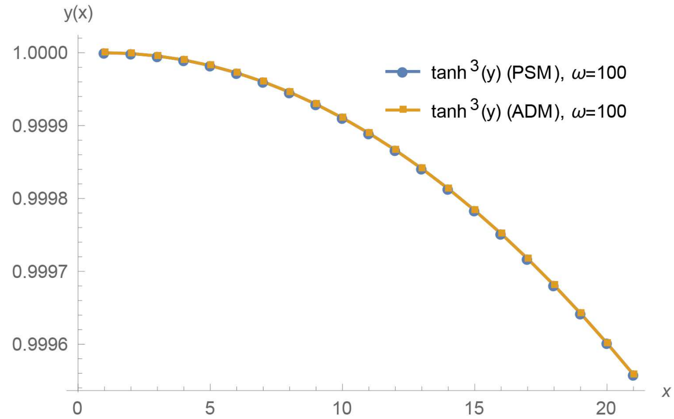
This section illustrates the accuracy and reliability of the present methods. Comparisons of the analytical solutions of the Lane-Emden equations associated with the nonlinear terms $f(y) = \tan^m y, \sec y, \tanh^m y, \operatorname{sech} y$ with the special values $m = 1, 2, 3, 4$ using the present methods are presented in Tables 1, 2 and 3 respectively for the values $\omega = 2, 19, 100$. The graphical comparison of solutions for the strong nonlinearities $f(y) = \tan^m y, \tanh^m y$, with the special values $m = 3, 4$ using the present methods is shown in Figures 1, 2 and 3 respectively for the values $\omega = 2, 19, 100$. One observes in these tables excellent agreements between PSM and ADM in the cases of the nonlinear functions $f(y) = \tan y, \sec y$; $\tanh y, \operatorname{sech} y$ for all the values of ω considered. These excellent comparisons also extend to the higher cases $f(y) = \tan^m y, \tanh^m y; m = 1, 2, 3, 4; \omega = 2, 19, 100$ as shown in Tables 1, 2, 3.

Table 1 shows the numerical comparison of approximate analytical solutions $y(x)$ using the PSM and the ADM for the Lane-Emden problem associated with the nonlinear terms $f(y) = \tan^m y, \sec y, \tanh^m y, \operatorname{sech} y$, $m = 1, 2, 3, 4; \omega = 2$. By increasing the values of ω, the numerical values of the solutions get closer as can be seen in Tables 1, 2 and 3 (see, for instance, the cases of $\tan^3 y, \tan^4 y$ in these tables); as reflected in Figures 1, 2 and 3 (see, for instance, Figures 1(a), 2(a), 3(a); 1(b), 2(b), 3(b)). In all these tables and figures, it is observed and clear that the solutions using the present methods are highly and excellently accurate, even for relatively large values of x; and for arbitrarily large values of ω. This excellent comparison of the results shows that the present methods are reliable, accurate, convenient and efficient in solving any strongly

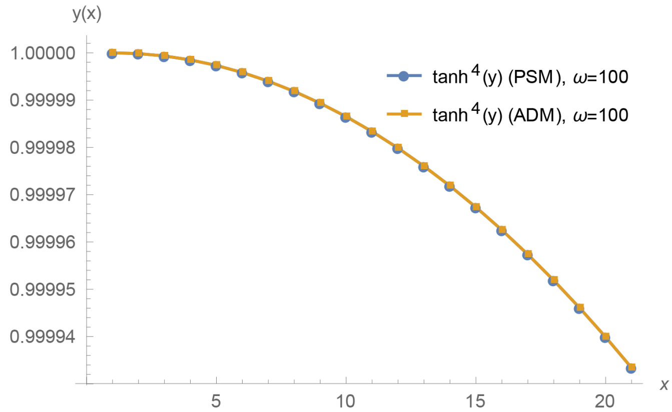


(a) Analytical solutions $y(x)$ for $f(y) = \tan^3 y$

(b) Analytical solutions $y(x)$ for $f(y) = \tan^4 y$



(c) Analytical solutions $y(x)$ for $f(y) = \tanh^3 y$



(d) Analytical solutions $y(x)$ for $f(y) = \tanh^4 y$

Figure 3: Comparisons of the approximate analytical solutions $y(x)$ using the present methods for the nonlinear functions $f(y) = \tan^m y, \tanh^m y, m = 3, 4; \omega = 100$.

nonlinear Lane-Emden equations with arbitrarily large values of real $\omega > 0$.

6. Concluding Remarks

In this paper, we have used a power series expansion method and the Adomian decomposition method to find highly accurate, reliable, effective and convenient approximate analytical solutions of a class of strongly nonlinear Lane-Emden equations whose nonlinear terms $f(y)$ were given explicitly by the mth powers of trigonometric and hyperbolic functions. In all the cases of the nonlinear terms considered, the solutions of the Lane-Emden equations for the special values $m = 1, 2, 3, 4$, were explicitly calculated and the numerical and graphical comparisons of the associated results were tabulated and graphed in Tables 1 - 3 and Figures 1 - 3 respectively for different values of ω. Interestingly, these comparisons of results show excellent agreements between the two methods even for large values of ω. In the case where the functions $f(y)$ are $\tan y, \sec y; \tanh y, \operatorname{sech} y$, the present methods yielded same results, an indication that PSM and ADM are accurate, reliable and efficient. It is also worth remarking that the results are reliable and accurate for arbitrarily large values of ω.

The results therefore show that the present methods can be accurately and reliably used to compute approximate analytical solutions of several other strongly nonlinear ordinary differential equations arising in applied sciences, mechanical and engineering applications.

Table 3: Comparison between PSM and ADM for the nonlinearities $f(y) = \sec y, \operatorname{sech} y$ and $f(y) = \tan^m y, \tanh^m y$ with the special values $m = 1, 2, 3, 4; \omega = 100$. We use the notation PSM, ADM

$x/f(y)$	0.1	0.2	0.5	1.0
$\tan y$	0.999923, 0.999923	0.999692, 0.999692	0.998077, 0.998077	0.992353, 0.992353
$\tan^2 y$	0.99988, 0.99988	0.99952, 0.99952	0.997011, 0.997017	0.988188, 0.988292
$\tan^3 y$	0.999813, 0.999813	0.999253, 0.999254	0.995354, 0.995394	0.981755, 0.982343
$\tan^4 y$	0.999709, 0.999709	0.998837, 0.998841	0.992779, 0.992938	0.971816, 0.973996
$x/f(y)$	0.1	0.2	0.5	1.0
$\sec y$	0.999908, 0.999908	0.999634, 0.999634	0.997713, 0.997713	0.990901, 0.990901
$x/f(y)$	0.1	0.2	0.5	1.0
$\tanh y$	0.999962, 0.999962	0.999849, 0.999849	0.999058, 0.999058	0.996234, 0.996234
$\tanh^2 y$	0.999971, 0.999971	0.999885, 0.999885	0.999283, 0.999282	0.997134, 0.997133
$\tanh^3 y$	0.999978, 0.999978	0.999913, 0.999913	0.999454, 0.999454	0.99782, 0.997817
$\tanh^4 y$	0.999983, 0.999983	0.999933, 0.999933	0.999584, 0.999584	0.998341, 0.998338
$x/f(y)$	0.1	0.2	0.5	1.0
$\operatorname{sech} y$	0.999968, 0.999968	0.999872, 0.999872	0.999198, 0.999198	0.996788, 0.996788

Conflict of Interest

The authors have no conflict of interest regarding the publication of this article. ——————————————————-

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