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Commun. Nonlinear Anal. 2 (2023), 1–9
Research Article

Communications in Nonlinear Analysis
Publisher
Research & Scinece Group Ltd.

On the Number of Zeros of A Polynomial

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Abstract

In this paper, we consider the problem of finding the maximum number of zeros of a polynomial in a prescribed region. Our theorems include several known results in this direction as special cases.

Keywords: Eneström Kakeya Theorem, Coefficients, Zeros of a polynomial.
2010 MSC: 30A10, 30C10, 30C15.

1. Introduction

The problem of locating the zeros of a polynomial, in particular the number of zeros of a special class of polynomials in a prescribed region, by subjecting special conditions on the coefficients is very important in the theory of polynomials. a review on location of zeros of polynomials can be foumd in ([6]-[9], [12]). Among them, the following elegant result due to Eneström Kakeya [[9],[11]] is well known in the theory of polynomials which states that:

If $\mathcal{H}(z)=\sum\limits_{\nu=0}^{n} b_\nu z^\nu$ is a polynomial of degree n with real coefficients satisfying $b_n \geq b_{n-1} \geq ... \geq b_1 \geq b_0 > 0$, then $P(z)$ has all its zeros in $|z| \leq 1$.

Several extensions and generalizations of Eneström Kakeya theorem are avaliable in literature (see [1]-[8]).

Concerning the maximum number of zeros of a polynomial in $|z| \leq \frac{1}{2}$, Q. G. Mohammad[10] proved the following result.

Theorem 1.1. *If $\mathcal{H}(z)=\sum\limits_{\nu=0}^{n} b_\nu z^\nu$ is a polynomial of degree n such that*

$$b_n \geq b_{n-1} \geq ... \geq b_1 \geq b_0 > 0,$$

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Received 2023-03-14

then the number of zeros of $\mathcal{H}(z)$ *in* $|z| \leq \frac{1}{2}$ *does not exceed*

$$1 + \frac{1}{\log 2} \log \frac{b_n}{b_0}.$$

K. K. Dewan[5] generalised Theorem 1.1 to the polynomials with complex coefficients and proved the following result.

Theorem 1.2. *Let* $\mathcal{H}(z) = \sum_{\nu=0}^{n} b_\nu z^\nu$ *be a polynomial of degree* n *with complex coefficients. If* $Re(b_\nu) = \alpha_\nu$ *and* $Im(b_\nu) = \beta_\nu, \ \ \nu = 0, 1, ..., n$ *such that*

$$\alpha_n \geq \alpha_{n-1} \geq ... \geq \alpha_1 \geq \alpha_0 > 0,$$

then the number of zeros of $\mathcal{H}(z)$ *in* $|z| \leq \frac{1}{2}$ *does not exceed*

$$1 + \frac{1}{\log 2} \log \frac{\alpha_n + \sum_{\nu=0}^{n} |\beta_\nu|}{\alpha_0}.$$

Regarding the number of zeros of a polynomial $\mathcal{H}(z)$ in $|z| \leq \delta, \ \ 0 < \delta < 1$, M. H. Gulzar[7] established the following results.

Theorem 1.3. *If* $\mathcal{H}(z) = \sum_{\nu=0}^{n} b_\nu z^\nu, \ \ b_\nu = \alpha_\nu + \iota\beta_\nu$ *is a polynomial of degree* n *with complex coefficients such that for some numbers* $k_0 \geq 0$,

$$k_0 + \alpha_n \geq \alpha_{n-1} \geq ... \geq \alpha_1 \geq \alpha_0,$$

then the number of zeros of $\mathcal{H}(z)$ *in* $\frac{|b_0|}{M_1} \leq |z| \leq \delta, \ \ 0 < \delta < 1$ *does not exceed*

$$\frac{1}{\log \frac{1}{\delta}} \log \frac{2k_0 + |\alpha_n| + \alpha_n + |\alpha_0| - \alpha_0 + 2\sum_{\nu=0}^{n} |\beta_\nu|}{|b_0|},$$

where

$$M_1 = |\alpha_n| + \alpha_n - \alpha_0 + |\beta_0| + 2k_0 + \sum_{\nu=1}^{n} |\beta_\nu|.$$

Theorem 1.4. *If* $\mathcal{H}(z) = \sum_{\nu=0}^{n} b_\nu z^\nu$ *is a polynomial of degree* n *with complex coefficients such that for some real* $\beta, \ \ |arg b_\nu - \beta| \leq \alpha \leq \frac{\pi}{2}, \ \ \nu = 0, 1, 2, ..., n$ *and for some number* $k_0 \geq 0$,

$$|k_0 + b_n| \geq |b_{n-1}| \geq ... \geq |b_1| \geq |b_0|,$$

then the number of zeros of $\mathcal{H}(z)$ *in* $\frac{|b_0|}{M_2} \leq |z| \leq \delta, \ \ 0 < \delta < 1$ *does not exceed*

$$\frac{1}{\log \frac{1}{\delta}} \log \frac{(k_0 + |b_n|)(\cos\alpha + \sin\alpha + 1) - |b_0|(\cos\alpha - \sin\alpha - 1) + 2\sin\alpha \sum_{\nu=1}^{n-1} |b_\nu|}{|b_0|},$$

where

$$M_2 = (k_0 + |b_n|)(\cos\alpha + \sin\alpha + 1) - |b_0|(\cos\alpha - \sin\alpha) + 2\sin\alpha \sum_{\nu=1}^{n-1} |b_\nu|.$$

2. Main Results

The main purpose of this paper is to present the generalizations of the Theorems 1.3 and 1.4 which includes some well known results as a special cases. In this direction, we prove.

Theorem 2.1. *If* $\mathcal{H}(z) = \sum_{\nu=0}^{n} b_\nu z^\nu$, $b_\nu = \alpha_\nu + \iota\beta_\nu$ *is a polynomial of degree* n *with complex coefficients such that for some numbers* $k_\nu \geq 0$, $\nu = 0, 1, 2, ..., r$, $1 \leq r \leq n-1$,

$$k_0 + \alpha_n \geq k_1 + \alpha_{n-1} \geq ... \geq k_r + \alpha_{n-r} \geq \alpha_{n-r-1} \geq ... \geq \alpha_1 \geq \alpha_0,$$

then the number of zeros of $\mathcal{H}(z)$ *in* $\frac{|b_0|}{M_4} \leq |z| \leq \delta$, $0 < \delta < 1$ *does not exceed*

$$\frac{1}{\log \frac{1}{\delta}} \log \frac{M_3}{|b_0|},$$

where

$$M_3 = (|\alpha_n| + \alpha_n) + (|\alpha_0| - \alpha_0) + 2k_0 + 2\left(\sum_{\nu=1}^{r} k_\nu + \sum_{\nu=0}^{n} |\beta_\nu|\right)$$

and

$$M_4 = |\alpha_n| + \alpha_n - \alpha_0 + |\beta_0| + 2k_0 + 2\left(\sum_{\nu=1}^{r} k_\nu + \sum_{\nu=1}^{n} |\beta_\nu|\right).$$

Remark 2.2. By assuming $k_\nu = 0$, $\nu = 1, 2, ..., r$ and in Theorem 2.1, it reduces to Theorem 1.3.

Assume all the coefficients to be positive and take $\delta = \frac{1}{2}$ in Theorem 2.1, we get the following result.

Corollary 2.3. *Let* $\mathcal{H}(z) = \sum_{\nu=0}^{n} b_\nu z^\nu$ *be a polynomial of degree* n *with complex coefficients. If* $Re(b_\nu) = \alpha_\nu$ *and* $Im(b_\nu) = \beta_\nu$, $\nu = 0, 1, ..., n$ *such that for some numbers* $k_\nu \geq 0$, $\nu = 0, 1, 2, ..., r$, $1 \leq r \leq n-1$,

$$k_0 + \alpha_n \geq k_1 + \alpha_{n-1} \geq ... \geq k_r \alpha_{n-r} \geq \alpha n - r - 1 \geq ... \geq \alpha_1 \geq \alpha_0 > 0,$$

then the number of zeros of $\mathcal{H}(z)$ *in* $\frac{|b_0|}{M_4'} \leq |z| \leq \frac{1}{2}$ *does not exceed*

$$1 + \frac{1}{\log 2} \log \frac{\alpha_n + k_0 + \sum_{\nu=1}^{r} k_\nu + \sum_{\nu=0}^{n} |\beta_\nu|}{|b_0|},$$

where

$$M_4' = 2\alpha_n - \alpha_0 + |\beta_0| + 2k_0 + 2\left(\sum_{\nu=1}^{r} k_\nu + \sum_{\nu=1}^{n} |\beta_\nu|\right).$$

If all the coefficients of a polynomial $\mathcal{H}(z)$ are real, then Theorem 2.1 reduces to the following result.

Corollary 2.4. *If* $\mathcal{H}(z) = \sum_{\nu=0}^{n} b_\nu z^\nu$ *is a polynomial of degree* n *with real coefficients such that for some numbers* $k_\nu \geq 0$, $\nu = 0, 1, 2, ...,$ $1 \leq r \leq n-1$

$$k_0 + b_n \geq k_1 + b_{n-1} \geq ... \geq k_r + b_{n-r} \geq b_{n-r-1} \geq ... \geq b_1 \geq b_0,$$

then the number of zeros of $\mathcal{H}(z)$ *in* $\frac{|b_0|}{M_4''} \leq |z| \leq \delta, \quad 0 < \delta < 1$ *does not exceed*

$$\frac{1}{\log \frac{1}{\delta}} \log \frac{|b_n| + b_n + |b_0| - b_0 + 2k_0 + 2\sum_{\nu=1}^{r} k_\nu}{|b_0|},$$

where

$$M_4^{''} = |b_n| + b_n - b_0 + 2k_0 + 2\sum_{\nu=1}^{r} k_\nu.$$

Example 2.5. Consider the polynomial $\mathcal{H}(z) = 2z^3 + z^2 + 4z + 2$. Here the monotone hypothesis is violated, thus we choose $k_0 = 2, \quad k_1 = 3$ and $k_2 = 0$. Therefore, from Corollary 2.4 with $\delta = 0.5$, we get the number of zeros in $0.16 \leq |z| \leq \delta = 0.5$ does not exceed 2.83, which implies that $\mathcal{H}(z)$ has at most one zero in $0.16 \leq |z| \leq 0.5$ and of course $\mathcal{H}(z)$ has exactly one zero in $0.16 \leq |z| \leq 0.5$. This example shows that how these results are used in practice. Also the previous results of this type are not applicable for these polynomials.

Theorem 2.6. *If* $\mathcal{H}(z) = \sum_{\nu=0}^{n} b_\nu z^\nu$ *is a polynomial of degree n with complex coefficients such that for some real* $\beta, \quad |arg(k_\nu + b_\nu) - \beta| \leq \alpha \leq \frac{\pi}{2}, \quad \nu = 0, 1, 2, ..., n$ *and for some numbers* $k_\nu \geq 0, \quad \nu = 0, 1, 2, ..., r, \quad 1 \leq r \leq n - 1$

$$|k_0 + b_n| \geq |k_1 + b_{n-1}| \geq ... \geq |k_r + b_{n-r}| \geq |b_{n-r-1}| \geq ... \geq |b_1| \geq |b_0|,$$

then the number of zeros of $\mathcal{H}(z)$ *in* $\frac{|b_0|}{M_6} \leq |z| \leq \delta, \quad 0 < \delta < 1$ *does not exceed*

$$\frac{1}{\log \frac{1}{\delta}} \log \frac{M_5}{|b_0|},$$

where

$$M_5 = (k_0 + |b_n|)(\cos\alpha + \sin\alpha + 1) - |b_0|(\cos\alpha - \sin\alpha - 1) + 2\sin\alpha \left(\sum_{\nu=1}^{r} |k_\nu + b_{n-\nu}| + \sum_{\nu=r+1}^{n-1} |b_{n-\nu}| \right) + 2\sum_{\nu=1}^{r} k_\nu$$

and

$$M_6 = (k_0 + |b_n|)(\cos\alpha + \sin\alpha + 1) - |b_0|(\cos\alpha - \sin\alpha) + 2\sin\alpha \left(\sum_{\nu=1}^{r} |k_\nu + b_{n-\nu}| + \sum_{\nu=r+1}^{n-1} |b_{n-\nu}| \right) + 2\sum_{\nu=1}^{r} k_\nu.$$

Remark 2.7. Assume $k_\nu = 0, \quad \nu = 1, 2, ..., r$ in Theorem 2.6, it reduces to Theorem 1.4.

3. Lemmas

For the proofs of these Theorems, we require following lemmas.

Lemma 3.1. *If for some numbers k_ν and $k_{\nu+1}$, $|k_\nu + b_{n-\nu}| \geq |k_{\nu+1} + b_{n-\nu-1}|$ and $|arg(k_\nu + b_{n-\nu}) - \beta| \leq \frac{\pi}{2}$, for some real β, then*

$$|(k_\nu + b_{n-\nu}) - (k_{\nu+1} + b_{n-\nu-1})| \leq (|k_\nu + b_{n-\nu}| - |k_{\nu+1} + b_{n-\nu-1}|)\cos\alpha + (|k_\nu + b_{n-\nu}| + |k_{\nu+1} + b_{n-\nu-1}|)\sin\alpha.$$

This Lemma is due to Govil and Rahman[8].

Lemma 3.2. *Let $f(z)$ be analytic for $|z| \leq 1$, $f(0) \neq 0$ and $|f(z)| \leq M$ for $|z| \leq 1$. Then the number of zeros of $f(z)$ in $|z| \leq \delta$, $0 < \delta < 1$ does not exceed*

$$\frac{1}{\log\frac{1}{\delta}}\log\frac{M}{|f(0)|}, \qquad (\text{ see[13]}).$$

4. Proof of Main Results

Proof of Theorem 2.1. Consider the polynomial

$$\begin{aligned}
\mathcal{T}(z) = (1-z)\mathcal{H}(z) &= -b_n z^{n+1} + (b_n - b_{n-1})z^n + (b_{n-1} - b_{n-2})z^{n-1} + ... + (b_{n-r+2} - b_{n-r+1})z^{n-r+2} \\
&+ (b_{n-r+1} - b_{n-r})z^{n-r+1} + (b_{n-r} - b_{n-r-1})z^{n-r} + ... + (b_2 - b_1)z^2 + (b_1 - b_0)z + b_0 \\
&= -\alpha_n z^{n+1} + (\alpha_n - \alpha_{n-1})z^n + (\alpha_{n-1} - \alpha_{n-2})z^{n-1} + ... + (\alpha_{n-r+2} - \alpha_{n-r+1})z^{n-r+2} \\
&+ (\alpha_{n-r+1} - \alpha_{n-r})z^{n-r+1} + (\alpha_{n-r} - \alpha_{n-r-1})z^{n-r} + ... + (\alpha_2 - \alpha_1)z^2 + (\alpha_1 - \alpha_0) + \alpha_0 \\
&+ \iota\left[-\beta_n z^{n+1} + \sum_{\nu=1}^{n}(\beta_\nu - \beta_{\nu-1})z^\nu + \beta_0\right] \\
&= -\alpha_n z^{n+1} + [(k_0 + \alpha_n) - (k_1 + \alpha_{n-1})]z^n + [(k_1 + \alpha_{n-1}) - (k_2 + \alpha_{n-2})]z^{n-1} + ... + \\
&[(k_{r-2} + \alpha_{n-r+2}) - (k_{r-1} + \alpha_{n-r+1})]z^{n-r+1} + [(k_{r-1} + \alpha_{n-r+1}) - (k_r + \alpha_{n-r})]z^{n-r+1} \\
&+ [(k_r + \alpha_{n-r}) - \alpha_{n-r-1}]z^{n-r} + ... + (\alpha_2 - \alpha_1)z^2 + (\alpha_1 - \alpha_0)z + \alpha_0 - \sum_{\nu=0}^{r-1}(k_\nu - k_{\nu+1})z^{n-\nu} - k_r z^{n-r} \\
&+ \iota\left[-\beta_n z^{n+1} + \sum_{\nu=1}^{n}(\beta_\nu - \beta_{\nu-1})z^\nu + \beta_0\right].
\end{aligned}$$

Now for $|z| \leq 1$, we have by using hypothesis

$$\begin{aligned}
|\mathcal{T}(z)| &\leq |\alpha_n| + |(k_0 + \alpha_n) - (k_1 + \alpha_{n-1})| + |(k_1 + \alpha_{n-1}) - (k_2 + \alpha_{n-2})| + ... + |(k_{r-2} + \alpha_{n-r+2}) \\
&- (k_{r-1} + \alpha_{n-r+1})| + |(k_{r-1} + \alpha_{n-r+1}) - (k_r + \alpha_{n-r})| + |(k_r + \alpha_{n-r}) - \alpha_{n-r-1}| + ... + |(\alpha_2 - \alpha_1)| \\
&+ |(\alpha_1 - \alpha_0)| + |\alpha_0| + \sum_{\nu=0}^{r-1}(|k_\nu| + |k_{\nu+1}|) + |k_r| + |\beta_n| + \sum_{\nu=1}^{n}(|\beta_\nu| + |\beta_{\nu-1}|) + |\beta_0| \\
&= |\alpha_n| + \alpha_n - \alpha_0 + |\alpha_0| + (k_0 + |k_0|) + 2\sum_{\nu=1}^{r}|k_\nu| + 2\sum_{\nu=0}^{n}|\beta_\nu| \\
&= (|\alpha_n| + \alpha_n) + (|\alpha_0| - \alpha_0) + 2k_0 + 2\left(\sum_{\nu=1}^{r}k_\nu + \sum_{\nu=0}^{n}|\beta_\nu|\right) = M_3 \quad (\text{say}).
\end{aligned}$$

Now $\mathcal{T}(z)$ is analytic in $|z| \leq 1$ and $|\mathcal{T}(z)| \leq 1$ for $|z| \leq 1$. Applying Lemma 3.2 to $\mathcal{T}(z)$, we get the number of zeros of $\mathcal{T}(z)$ and hence of $\mathcal{H}(z)$ in $|z| \leq \delta, \quad 0 < \delta < 1$ does not exceed

$$\frac{1}{\log \frac{1}{\delta}} \log \frac{M_3}{|b_0|}.$$

Now to show that $\mathcal{H}(z)$ has no zeros in $|z| \leq \frac{|a_0|}{M_4}$, let

$$\begin{aligned}
\mathcal{G}(z) &= -b_n z^{n+1} + (b_n - b_{n-1})z^n + (b_{n-1} - b_{n-2})z^{n-1} + ... + (b_{n-r+2} - b_{n-r+1})z^{n-r+2} \\
&\quad + (b_{n-r+1} - b_{n-r})z^{n-r+1} + (b_{n-r} - b_{n-r-1})z^{n-r} + ... + (b_2 - b_1)z^2 + (b_1 - b_0)z \\
&= -\alpha_n z^{n+1} + [(k_0 + \alpha_n) - (k_1 + \alpha_{n-1})]z^n + [(k_1 + \alpha_{n-1}) - (k_2 + \alpha_{n-2})]z^{n-1} + ...+ \\
&\quad [(k_{r-2} + \alpha_{n-r+2}) - (k_{r-1} + \alpha_{n-r+1})]z^{n-r+1} + [(k_{r-1} + \alpha_{n-r+1}) - (k_r + \alpha_{n-r})]z^{n-r+1} \\
&\quad + [(k_r + \alpha_{n-r}) - \alpha_{n-r-1}]z^{n-r} + ... + (\alpha_2 - \alpha_1)z^2 + (\alpha_1 - \alpha_0)z - \sum_{\nu=0}^{r-1}(k_\nu - k_{\nu+1})z^{n-\nu} - k_r z^{n-r} \\
&\quad + \iota\Big[- \beta_n z^{n+1} + \sum_{\nu=1}^{n}(\beta_\nu - \beta_{\nu-1})z^\nu \Big].
\end{aligned}$$

This gives by using hypoythesis for $|z| \leq 1$,

$$\begin{aligned}
|\mathcal{G}(z)| &\leq |\alpha_n| + |(k_0 + \alpha_n) - (k_1 + \alpha_{n-1})| + |(k_1 + \alpha_{n-1}) - (k_2 + \alpha_{n-2})| + ... + |(k_{r-2} + \alpha_{n-r+2}) \\
&\quad - (k_{r-1} + \alpha_{n-r+1})| + |(k_{r-1} + \alpha_{n-r+1}) - (k_r + \alpha_{n-r})| + |(k_r + \alpha_{n-r}) - \alpha_{n-r-1}| + ... + |(\alpha_2 - \alpha_1)| \\
&\quad + |(\alpha_1 - \alpha_0)| + \sum_{\nu=0}^{r-1}(|k_\nu| + |k_{\nu+1}|) + |k_r| + |\beta_n| + \sum_{\nu=1}^{n}(|\beta_\nu| + |\beta_{\nu-1}|) \\
&= |\alpha_n| + \alpha_n - \alpha_0 + (k_0 + |k_0|) + |\beta_0| + 2\sum_{\nu=1}^{r}|k_\nu| + 2\sum_{\nu=1}^{n}|\beta_\nu| \\
&= |\alpha_n| + \alpha_n - \alpha_0 + |\beta_0| + 2k_0 + 2\left(\sum_{\nu=1}^{r}k_\nu + \sum_{\nu=1}^{n}|\beta_\nu|\right) \\
&= M_4.
\end{aligned}$$

Since $\mathcal{G}(z)$ is analytic in $|z| \leq 1$ and $\mathcal{G}(0) = 0$, it follows by Schwarz Lemma that

$$|\mathcal{G}(z)| \leq M_4|z|.$$

Hence for $|z| \leq 1$, we have

$$\begin{aligned}
|\mathcal{T}(z)| &= |b_0 + \mathcal{G}(z)| \\
&\geq |b_0| - |\mathcal{G}(z)| \\
&\geq |b_0| - M_4|z| \\
&> 0,
\end{aligned}$$

if

$$|z| < \frac{|b_0|}{M_4}.$$

This shows that $\mathcal{T}(z)$ has no zeros in $|z| < \frac{|b_0|}{M_4}$. Since the zeros of $\mathcal{T}(z)$ are same as the zeros of $\mathcal{H}(z)$, it follows that $\mathcal{H}(z)$ has no zeros in $|z| < \frac{|b_0|}{M_4}$. Consequently it follows that the number of zeros of $\mathcal{H}(z)$ in $\frac{|b_0|}{M_4} \leq |z| \leq \delta, \ 0 < \delta < 1$ does not exceed $\frac{1}{\log \frac{1}{\delta}} \log \frac{M_3}{|b_0|}$. This completes the proof of Theorem 2.1. □

Proof of Theorem 2.6. Consider the polynomial

$$\begin{aligned}\mathcal{M}(z) &= (1-z)\mathcal{H}(z)\\ &= -b_n z^{n+1} + (b_n - b_{n-1})z^n + (b_{n-1} - b_{n-2})z^{n-1} + \ldots + (b_{n-r+2} - b_{n-r+1})z^{n-r+2}\\ &+ (b_{n-r+1} - b_{n-r})z^{n-r+1} + (b_{n-r} - b_{n-r-1})z^{n-r} + \ldots + (b_2 - b_1)z^2 + (b_1 - b_0)z + b_0\\ &= -b_n z^{n+1} + [(k_0 + b_n) - (k_1 + b_{n-1})]z^n + [(k_1 + b_{n-1}) - (k_2 + b_{n-2})]z^{n-1} + \ldots +\\ &[(k_{r-2} + b_{n-r+2}) - (k_{r-1} + b_{n-r+1})]z^{n-r+1} + [(k_{r-1} + b_{n-r+1}) - (k_r + b_{n-r})]z^{n-r+1}\\ &+ [(k_r + b_{n-r}) - b_{n-r-1}]z^{n-r} + \ldots + (b_2 - b_1)z^2 + (b_1 - b_0)z + b_0 - \sum_{\nu=0}^{r-1}(k_\nu - k_{\nu+1})z^{n-\nu} - k_r z^{n-r}.\end{aligned}$$

Now for $|z| \leq 1$, we have by using hypothesis and Lemma 3.1

$$\begin{aligned}|\mathcal{M}(z)| &\leq |b_n| + |(k_0 + b_n) - (k_1 + b_{n-1})| + |(k_1 + b_{n-1}) - (k_2 + b_{n-2})| + \ldots + |(k_{r-2} + b_{n-r+2})\\ &- (k_{r-1} + b_{n-r+1})| + |(k_{r-1} + b_{n-r+1}) - (k_r + b_{n-r})| + |(k_r + b_{n-r}) - b_{n-r-1}| + \ldots + |(b_2 - b_1)|\\ &+ |(b_1 - b_0)| + |b_0| + \sum_{\nu=0}^{r-1}(|k_\nu| + |k_{\nu+1}|) + |k_r|\\ &\leq |b_n| + \Big[|k_0 + b_n| - |k_1 + b_{n-1}| + |k_1 + b_{n-1}| - |k_2 + b_{n-2}| + \ldots + |k_{r-2} - b_{n-r+2}|\\ &- |k_{r-1} + b_{n-r+1}| + |k_{r-1} + b_{n-r+1}| - |k_r + b_{n-r}| + |k_r + b_{n-r}| - |b_{n-r-1}| + \ldots|b_2| - |b_1| + |b_1|\\ &- |b_0|\Big]\cos\alpha + \Big[|k_0 + b_n| + |k_1 + b_{n-1}| + |k_1 + b_{n-1}| + |k_2 + b_{n-2}| + \ldots + |k_{r-2} - b_{n-r+2}|\\ &+ |k_{r-1} + b_{n-r+1}| + |k_{r-1} + b_{n-r+1}| + |k_r + b_{n-r}| + |k_r + b_{n-r}| + |b_{n-r-1}| + \ldots|b_2| + |b_1| + |b_1|\\ &+ |b_0|\Big]\sin\alpha + |b_0| + |k_0| + 2\sum_{\nu=1}^{r}|k_\nu|\\ &= |b_n| + \{|k_0 + b_n| - |b_0|\}\cos\alpha + \left\{|k_0 + b_n| + 2\sum_{\nu=1}^{r}|k_\nu + b_{n-\nu}| + 2\sum_{\nu=r+1}^{n-1}|b_{n-\nu}| + |b_0|\right\}\sin\alpha\\ &+ |b_0| + |k_0| + 2\sum_{\nu=1}^{r}|k_\nu|\\ &= |b_n| + k_0 + (|k_0 + b_n|)(\cos\alpha + \sin\alpha) + 2\sin\alpha\left(\sum_{\nu=1}^{r}|k_\nu + b_{n-\nu}| + \sum_{\nu=r+1}^{n-1}|b_{n-\nu}|\right) + 2\sum_{\nu=1}^{r}k_\nu\\ &- |b_0|(\cos\alpha - \sin\alpha - 1)\\ &\leq |b_n| + k_0 + (|k_0| + |b_n|)(\cos\alpha + \sin\alpha) - |b_0|(\cos\alpha - \sin\alpha - 1) + 2\sin\alpha\left(\sum_{\nu=1}^{r}|k_\nu + b_{n-\nu}|\right.\\ &\left.+ \sum_{\nu=r+1}^{n-1}|b_{n-\nu}|\right) + 2\sum_{\nu=1}^{r}k_\nu\end{aligned}$$

or,

$$\begin{aligned}|\mathcal{M}(z)| &= (k_0 + |b_n|)(\cos\alpha + \sin\alpha + 1) - |b_0|(\cos\alpha - \sin\alpha - 1) + 2\sin\alpha\left(\sum_{\nu=1}^{r}|k_\nu + b_{n-\nu}| + \sum_{\nu=r+1}^{n-1}|b_{n-\nu}|\right)\\ &+ 2\sum_{\nu=1}^{r}k_\nu = M_5 \quad \text{(say)}.\end{aligned}$$

Now $\mathcal{M}(z)$ is analytic in $|z| \leq 1$ and $|\mathcal{M}(z)| \leq 1$ for $|z| \leq 1$. Applying Lemma 3.2 to $\mathcal{M}(z)$, we get the number of zeros of $\mathcal{M}(z)$ and hence of $\mathcal{H}(z)$ in $|z| \leq \delta, \quad 0 < \delta < 1$ does not exceed

$$\frac{1}{\log \frac{1}{\delta}} \log \frac{M_5}{|b_0|}.$$

Now let

$$\begin{aligned}
\mathcal{N}(z) &= -b_n z^{n+1} + (b_n - b_{n-1})z^n + (b_{n-1} - b_{n-2})z^{n-1} + ... + (b_{n-r+2} - b_{n-r+1})z^{n-r+2} \\
&+ (b_{n-r+1} - b_{n-r})z^{n-r+1} + (b_{n-r} - b_{n-r-1})z^{n-r} + ... + (b_2 - b_1)z^2 + (b_1 - b_0)z \\
&= -b_n z^{n+1} + [(k_0 + b_n) - (k_1 + b_{n-1})]z^n + [(k_1 + b_{n-1}) - (k_2 + b_{n-2})]z^{n-1} + ...+ \\
&[(k_{r-2} + b_{n-r+2}) - (k_{r-1} + b_{n-r+1})]z^{n-r+1} + [(k_{r-1} + b_{n-r+1}) - (k_r + b_{n-r})]z^{n-r+1} \\
&+ [(k_r + b_{n-r}) - b_{n-r-1}]z^{n-r} + ... + (b_2 - b_1)z^2 + (b_1 - b_0)z - \sum_{\nu=0}^{r-1}(k_\nu - k_{\nu+1})z^{n-\nu} - k_r z^{n-r}.
\end{aligned}$$

This gives for $|z| \leq 1$ by using hypothesis and Lemma 3.1,

$$\begin{aligned}
|\mathcal{N}(z)| \leq |b_n| &+ \Big[|k_0 + b_n| - |k_1 + b_{n-1}| + |k_1 + b_{n-1}| - |k_2 + b_{n-2}| + ... + |k_{r-2} - b_{n-r+2}| \\
&- |k_{r-1} + b_{n-r+1}| + |k_{r-1} + b_{n-r+1}| - |k_r + b_{n-r}| + |k_r + b_{n-r}| - |b_{n-r-1}| + ...|b_2| - |b_1| + |b_1| \\
&- |b_0|\Big] \cos\alpha + \Big[|k_0 + b_n| + |k_1 + b_{n-1}| + |k_1 + b_{n-1}| + |k_2 + b_{n-2}| + ... + |k_{r-2} - b_{n-r+2}| \\
&+ |k_{r-1} + b_{n-r+1}| + |k_{r-1} + b_{n-r+1}| + |k_r + b_{n-r}| + |k_r + b_{n-r}| + |b_{n-r-1}| + ...|b_2| + |b_1| + |b_1| \\
&+ |b_0|\Big] \sin\alpha + |k_0| + 2\sum_{\nu=1}^{r}|k_\nu|,
\end{aligned}$$

which implies

$$\begin{aligned}
|\mathcal{N}(z)| &= |b_n| + k_0 + (|k_0 + b_n|)(\cos\alpha + \sin\alpha) + 2\sin\alpha\left(\sum_{\nu=1}^{r}|k_\nu + b_{n-\nu}| + \sum_{\nu=r+1}^{n-1}|b_{n-\nu}|\right) + 2\sum_{\nu=1}^{r}k_\nu \\
&- |b_0|(\cos\alpha - \sin\alpha) \\
&\leq |b_n| + k_0 + (|k_0| + |b_n|)(\cos\alpha + \sin\alpha) - |b_0|(\cos\alpha - \sin\alpha) + 2\sin\alpha\left(\sum_{\nu=1}^{r}|k_\nu + b_{n-\nu}|\right. \\
&+ \left.\sum_{\nu=r+1}^{n-1}|b_{n-\nu}|\right) + 2\sum_{\nu=1}^{r}k_\nu \\
&= (k_0 + |b_n|)(\cos\alpha + \sin\alpha + 1) - |b_0|(\cos\alpha - \sin\alpha) + 2\sin\alpha\left(\sum_{\nu=1}^{r}|k_\nu + b_{n-\nu}| + \sum_{\nu=r+1}^{n-1}|b_{n-\nu}|\right) \\
&+ 2\sum_{\nu=1}^{r}k_\nu = M_6 \quad \text{(say)}.
\end{aligned}$$

Since $\mathcal{N}(z)$ is analytic in $|z| \leq 1$ and $\mathcal{N}(0) = 0$, it follows by Schwarz Lemma that

$$|\mathcal{N}(z)| \leq M_6|z|.$$

Hence for $|z| \leq 1$, we have

$$\begin{aligned}|\mathcal{M}(z)| &= |b_0 + \mathcal{N}(z)| \\ &\geq |b_0| - |\mathcal{N}(z)| \\ &\geq |b_0| - M_6|z| \\ &> 0,\end{aligned}$$

if

$$|z| < \frac{|b_0|}{M_6}.$$

This shows that $\mathcal{M}(z)$ has no zeros in $|z| < \frac{|b_0|}{M_6}$. Since the zeros of $\mathcal{M}(z)$ are same as the zeros of $\mathcal{H}(z)$, it follows that $\mathcal{H}(z)$ has no zeros in $|z| < \frac{|b_0|}{M_6}$. Consequently it follows that the number of zeros of $\mathcal{H}(z)$ in $\frac{|b_0|}{M_6} \leq |z| \leq \delta,\ 0 < \delta < 1$ does not exceed $\frac{1}{\log \frac{1}{\delta}} \log \frac{M_5}{|b_0|}$. This completes the proof of Theorem 2.6.

□

Conflict of Interest

The authors have no conflict of interest regarding the publication of this article.

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