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Commun. Nonlinear Anal. 1 (2023), 1–11
Research Article

Communications in Nonlinear Analysis
Publisher
Research & Scinece Group Ltd.

The Solutions of Systems of Rational Difference Equations in Terms of Fibonacci Numbers

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Abstract

In this paper, we get the form of the solutions of the following difference equation systems with non-zero real numbers initial conditions.

$$z_{n+1} = \frac{w_n(z_{n-4} + w_{n-5})}{w_{n-5} + z_{n-4} - w_n}, \qquad w_{n+1} = \frac{z_{n-3}(z_{n-3} + w_{n-4})}{2z_{n-3} + w_{n-4}},$$

$$z_{n+1} = \frac{w_n(w_{n-5} - z_{n-4})}{w_{n-5} - z_{n-4} + w_n}, \qquad w_{n+1} = \frac{z_{n-3}(w_{n-4} - z_{n-3})}{w_{n-4}}.$$

Keywords: difference equation, recursive sequences, system of difference equations
2010 MSC: 39A10

1. Introduction

Recently, there has been great interest in studying difference equation systems. In population biology, economics, probability theory, genetics, psychology, etc., there is a need for methods that can be used to investigate equations that arise in mathematical models describing real life situations. Difference equations naturally appear as discrete analogues and numerical solutions of differential and delay differential equations with applications in biology, ecology, economics, physics, etc. Although difference equations are very simple in form, it is extremely difficult to understand thoroughly the behaviors of their solution. Recent research has focused strongly on the qualitative analysis of rational difference equations and difference equation

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Received 2022-08-27

systems. There are numerous papers on the difference equations system, for instance, Cinar [5] investigated the periodicity of the system's positive solutions to the difference equations

$$z_{n+1} = \frac{1}{w_n}, \quad w_{n+1} = \frac{w_n}{z_{n-1}w_{n-1}}.$$

Ozban [28] has studied the positive solution of the system of rational difference equations

$$z_{n+1} = \frac{a}{w_{n-3}}, \quad w_{n+1} = \frac{bw_{n-3}}{z_{n-q}w_{n-q}}.$$

Kurbanli et al.[24] has examined the behavior of positive solutions of the following system

$$z_{n+1} = \frac{z_{n-1}}{1+z_{n-1}w_n}, \quad w_{n+1} = \frac{w_{n-1}}{1+w_{n-1}z_n}.$$

The periodic nature and the form of the solutions of the following nonlinear difference equations systems

$$z_{n+1} = \frac{z_n w_{n-2}}{w_{n-1(\pm 1 \pm z_n w_{n-2})}}, \quad w_{n+1} = \frac{w_n z_{n-2}}{z_{n-1(\pm 1 \pm w_n z_{n-2})}},$$

has been studied by Elsayed and El-Metwally. [18].

The boundedness, and the form of the solutions of the following systems of rational difference equations

$$z_{n+1} = \frac{z_{n-3}}{\pm 1 \pm z_{n-3}w_{n-1}}, \quad w_{n+1} = \frac{w_{n-3}}{\pm 1 \pm w_{n-3}z_{n-1}},$$

has been investigated by Touafek et al. [30] .

In [35] Zhang et al. studied the boundedness, the persistence and global asymptotic stability of the positive solutions of the system of difference equations

$$z_{n+1} = A + \frac{w_{n-m}}{z_n}, \quad w_{n+1} = A + \frac{z_{n-m}}{w_n}.$$

Our aim in this paper is to investigate the form of the solutions of the following nonlinear difference equations systems

$$z_{n+1} = \frac{w_n(z_{n-4}+w_{n-5})}{w_{n-5}+z_{n-4}-w_n}, \quad w_{n+1} = \frac{z_{n-3}(z_{n-3}+w_{n-4})}{2z_{n-3}+w_{n-4}},$$

$$z_{n+1} = \frac{w_n(w_{n-5}-z_{n-4})}{w_{n-5}-z_{n-4}+w_n}, \quad w_{n+1} = \frac{z_{n-3}(w_{n-4}-z_{n-3})}{w_{n-4}},$$

with non-zero real numbers initial conditions $z_{-4}, z_{-3}, z_{-2}, z_{-1}, z_0, w_{-5}, w_{-4}, w_{-3}, w_{-2}, w_{-1}$ *and* w_0.

Definition 1.1. Let $\{F_n\}_{n\geq 0} = \{0, 1, 1, 2, 3, 5, 8, 21, ...\}$ be the Fibonacci sequence defined by

$$F_{n+2} = F_{n+1} + F_n, \quad n \in N.$$

$$F_0 = 0 \ , \ F_1 = 1.$$

2. The System $z_{n+1}=\frac{w_n(z_{n-4}+w_{n-5})}{w_{n-5}+z_{n-4}-w_n}, w_{n+1}=\frac{z_{n-3}(z_{n-3}+w_{n-4})}{2z_{n-3}+w_{n-4}}$

In this section, we study the solutions of the system of difference equations

$$z_{n+1}=\frac{w_n(z_{n-4}+w_{n-5})}{w_{n-5}+z_{n-4}-w_n},\qquad w_{n+1}=\frac{z_{n-3}(z_{n-3}+w_{n-4})}{2z_{n-3}+w_{n-4}}, \tag{2.1}$$

where the initial conditions $z_{-4}, z_{-3}, z_{-2}, z_{-1}, z_0, w_{-5}, w_{-4}, w_{-3}, w_{-2}, w_{-1}$, w_0 are arbitrary non-zero real numbers.

Theorem 2.1. *If $\{z_n, w_n\}$ are solutions of difference equation system (2.1). Then*

$$\begin{aligned} z_{5n-4} &= \frac{m(a+f)}{a+f-m},\quad n\geq 1.\\ z_{5n-3} &= b,\qquad z_{5n-2}=c,\qquad z_{5n-1}=d,\qquad z_{5n}=e,\quad n\geq 0, \end{aligned}$$

for $n \geq 0$,

$$w_{5n-4}=\frac{b(F_{2n}b+F_{2n-1}g)}{F_{2n+1}b+F_{2n}g},\qquad w_{5n-3}=\frac{c(F_{2n}c+F_{2n-1}h)}{F_{2n+1}c+F_{2n}h},$$
$$w_{5n-2}=\frac{d(F_{2n}d+F_{2n-1}k)}{F_{2n+1}d+F_{2n}k},\qquad w_{5n-1}=\frac{e(F_{2n}e+F_{2n-1}l)}{F_{2n+1}e+F_{2n}l},$$
$$w_{5n}=\frac{m(a+f)(F_{2n+1}a+F_{2n+1}f-F_{2n-1}m)}{(a+f-m)(F_{2n+2}a+F_{2n+2}f-F_{2n}m)},$$

where $z_{-4}=a, z_{-3}=b, z_{-2}=c, z_{-1}=d, z_0=e, w_{-5}=f, w_{-4}=g, w_{-3}=h, w_{-2}=k, w_{-1}=l, w_0=m$ and $\{F_n\}_{n=-1}^{\infty}=\{1,0,1,1,2,3,5,8,...\}$, $F_{n+2}=F_{n+1}+F_n$, $F_{-1}=1$.

Proof. The proof will be achieved by the Mathematical Induction. For $n=0$, the result holds. Now suppose that $n\geq 1$ and that our assumption holds for $n-1$, that is,

$$\begin{aligned} z_{5n-9} &= \frac{m(a+f)}{a+f-m},\\ z_{5n-8} &= b,\qquad z_{5n-7}=c,\qquad z_{5n-6}=d,\qquad z_{5n-5}=e, \end{aligned}$$

$$w_{5n-10}=\frac{m(a+f)(F_{2n-3}a+F_{2n-3}f-F_{2n-5}m)}{(a+f-m)(F_{2n-2}a+F_{2n-2}f-F_{2n-4}m)},$$
$$w_{5n-9}=\frac{b(F_{2n-2}b+F_{2n-3}g)}{F_{2n-1}b+F_{2n-2}g},\qquad w_{5n-8}=\frac{c(F_{2n-2}c+F_{2n-3}h)}{F_{2n-1}c+F_{2n-2}h},$$
$$w_{5n-7}=\frac{d(F_{2n-2}d+F_{2n-3}k)}{F_{2n-1}d+F_{2n-2}k},\qquad w_{5n-6}=\frac{e(F_{2n-2}e+F_{2n-3}l)}{F_{2n-1}e+F_{2n-2}l},$$
$$w_{5n-5}=\frac{m(a+f)(F_{2n-1}a+F_{2n-1}f-F_{2n-3}m)}{(a+f-m)(F_{2n}a+F_{2n}f-F_{2n-2}m)}.$$

Now it follows from (2.1) that

$$z_{5n-4}=\frac{w_{5n-5}(z_{5n-9}+w_{5n-10})}{w_{5n-10}+z_{5n-9}-w_{5n-5}}.$$

Divide the numerator and the dominator by $w_{5n-5}(z_{5n-9}+w_{5n-10})$, then

$$z_{5n-4} = \frac{1}{\frac{1}{w_{5n-5}} - \frac{1}{w_{5n-10}+z_{5n-9}}}.$$

Calculate

$$\begin{aligned}
z_{5n-9} + w_{5n-10} &= \frac{m(a+f)}{a+f-m} + \frac{m(a+f)(F_{2n-3}a + F_{2n-3}f - F_{2n-5}m)}{(a+f-m)(F_{2n-2}a + F_{2n-2}f - F_{2n-4}m)} \\
&= \frac{m(a+f)}{a+f-m}\left\{1 + \frac{F_{2n-3}a + F_{2n-3}f - F_{2n-5}m}{F_{2n-2}a + F_{2n-2}f - F_{2n-4}m}\right\} \\
&= \frac{m(a+f)}{a+f-m} \times \left\{\frac{F_{2n-2}a + F_{2n-2}f - F_{2n-4}m + F_{2n-3}a + F_{2n-3}f - F_{2n-5}m}{F_{2n-2}a + F_{2n-2}f - F_{2n-4}m}\right\} \\
&= \frac{m(a+f)\{(F_{2n-2} + F_{2n-3})a + (F_{2n-2} + F_{2n-3})f - (F_{2n-4} + F_{2n-5})m\}}{(a+f-m)(F_{2n-2}a + F_{2n-2}f - F_{2n-4}m)} \\
&= \frac{m(a+f)(F_{2n-1}a + F_{2n-1}f - F_{2n-3}m)}{(a+f-m)(F_{2n-2}a + F_{2n-2}f - F_{2n-4}m)}.
\end{aligned}$$

Therefore

$$\begin{aligned}
z_{5n-4} &= \frac{1}{\frac{1}{w_{5n-5}} - \frac{1}{w_{5n-10}+z_{5n-9}}} \\
&= \frac{1}{\frac{(a+f-m)(F_{2n}a+F_{2n}f-F_{2n-2}m)}{m(a+f)(F_{2n-1}a+F_{2n-1}f-F_{2n-3}m)} - \frac{(a+f-m)(F_{2n-2}a+F_{2n-2}f-F_{2n-4}m)}{m(a+f)(F_{2n-1}a+F_{2n-1}f-F_{2n-3}m)}} \\
&= \frac{m(a+f)(F_{2n-1}a + F_{2n-1}f - F_{2n-3}m)}{(a+f-m)(F_{2n}a + F_{2n}f - F_{2n-2}m) - (a+f-m)(F_{2n-2}a + F_{2n-2}f - F_{2n-4}m)} \\
&= \frac{m(a+f)(F_{2n-1}a + F_{2n-1}f - F_{2n-3}m)}{(a+f-m)\{(F_{2n} - F_{2n-2})a + (F_{2n} - F_{2n-2})f - (F_{2n-2} - F_{2n-4})m\}} \\
&= \frac{m(a+f)(F_{2n-1}a + F_{2n-1}f - F_{2n-3}m)}{(a+f-m)(F_{2n-1}a + F_{2n-1}f - F_{2n-3}m)} \\
&= \frac{m(a+f)}{a+f-m}.
\end{aligned}$$

And

$$\begin{aligned}
w_{5n-4} &= \frac{z_{5n-8}(z_{5n-8} + w_{5n-9})}{2z_{5n-8} + w_{5n-9}} \\
&= \frac{b(b + \frac{b(F_{2n-2}b+F_{2n-3}g)}{F_{2n-1}b+F_{2n-2}g}}{2b + \frac{b(F_{2n-2}b+F_{2n-3}g)}{F_{2n-1}b+F_{2n-2}g}} \\
&= \frac{b[b(F_{2n-1}b + F_{2n-2}g) + b(F_{2n-2}b + F_{2n-3}g)]}{2b(F_{2n-1}b + F_{2n-2}g) + b(F_{2n-2}b + F_{2n-3}g)} \\
&= \frac{b[b(F_{2n-1} + F_{2n-2}) + g(F_{2n-2} + F_{2n-3})]}{b(2F_{2n-1} + F_{2n-2}) + g(2F_{2n-2} + F_{2n-3})} \\
&= \frac{b[b(F_{2n-1} + F_{2n-2}) + g(F_{2n-2} + F_{2n-3})]}{b[F_{2n-1} + (F_{2n-1} + F_{2n-2})] + g[F_{2n-2} + (F_{2n-2} + F_{2n-3})]} \\
&= \frac{b(F_{2n}b + F_{2n-1}g)}{b(F_{2n-1} + F_{2n}) + g(F_{2n-2} + F_{2n-1})} \\
&= \frac{b(F_{2n}b + F_{2n-1}g)}{F_{2n+1}b + F_{2n}g}.
\end{aligned}$$

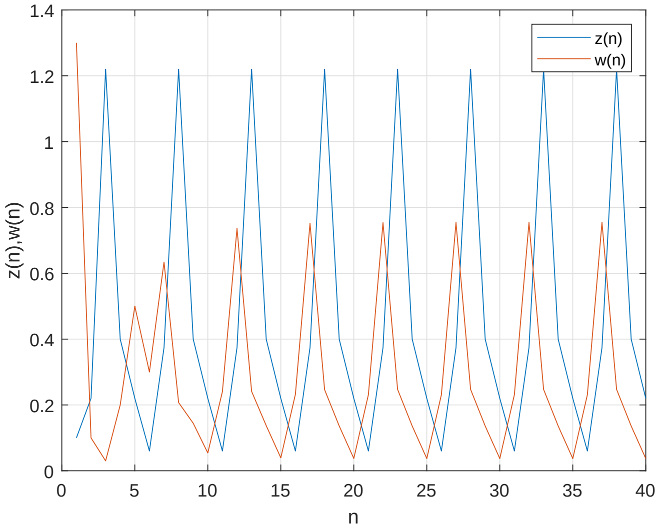

Figure 1: $z_0 = 0.1, z_{-1} = 0.22, z_{-2} = 1.22, z_{-3} = 0.4, z_{-4} = 0.22, z_{-5} = 0,06, w_0 = 1.3, w_{-1} = 0.1, w_{-2} = 0.03, w_{-3} = 0.2, w_{-4} = 0.5$ and $w_{-5} = 0.3$.

$$\begin{aligned} w_{5n-3} &= \frac{z_{5n-7}(z_{5n-7}+w_{5n-8})}{2z_{5n-7}+w_{5n-8}} \\ &= \frac{c(c+\frac{c(F_{2n-2}c+F_{2n-3}h)}{F_{2n-1}c+F_{2n-2}h}}{2c+\frac{c(F_{2n-2}c+F_{2n-3}h)}{F_{2n-1}c+F_{2n-2}h}} \\ &= \frac{c[c(F_{2n-1}c+F_{2n-2}h)+c(F_{2n-2}c+F_{2n-3}h)]}{2c(F_{2n-1}c+F_{2n-2}h)+c(F_{2n-2}c+F_{2n-3}h)} \\ &= \frac{c[c(F_{2n-1}+F_{2n-2})+h(F_{2n-2}+F_{2n-3})]}{c(2F_{2n-1}+F_{2n-2})+h(2F_{2n-2}+F_{2n-3})} \\ &= \frac{c(F_{2n}c+F_{2n-1}h)}{c(F_{2n-1}+F_{2n})+h(F_{2n-2}+F_{2n-1})} \\ &= \frac{c(F_{2n}c+F_{2n-1}h)}{F_{2n+1}c+F_{2n}h}. \end{aligned}$$

Similarly, we can obtain the other relations. Thus, the proof is completed. □

2.1. Numerical Examples

From Eq.(2.1) we assume the initial conditions

3. The System $z_{n+1} = \frac{w_n(w_{n-5}-z_{n-4})}{w_{n-5}-z_{n-4}+w_n}, w_{n+1} = \frac{z_{n-3}(w_{n-4}-z_{n-3})}{w_{n-4}}$

In this section, we study the solutions of the system of difference equations

$$z_{n+1} = \frac{w_n(w_{n-5}-z_{n-4})}{w_{n-5}-z_{n-4}+w_n}, \qquad w_{n+1} = \frac{z_{n-3}(w_{n-4}-z_{n-3})}{w_{n-4}} \tag{3.1}$$

where the initial conditions $z_{-4}, z_{-3}, z_{-2}, z_{-1}, z_0, w_{-5}, w_{-4}, w_{-3}, w_{-2}, w_{-1}$, w_0 are arbitrary non-zero real numbers.

Theorem 3.1. *If $\{z_n, w_n\}$ are solutions of difference equation system (3.1). Then for $n = 0, 1, 2, ...,$*

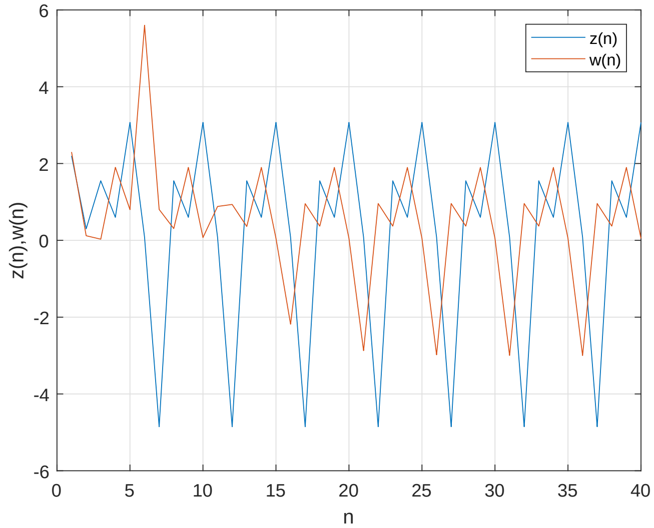

Figure 2: It shows the solution of Eq.(2.1) when we consider that $z_0 = 2.2, z_{-1} = 0.3, z_{-2} = 1.55, z_{-3} = 0.6, z_{-4} = 3.07, z_{-5} = 0,08, w_0 = 2.3, w_{-1} = 0.12, w_{-2} = 0.03, w_{-3} = 1.9,\ w_{-4} = 0.8$ and $w_{-5} = 5.6$.

$$z_{5n-4} = \frac{-m^2(a-f)^2}{(F_{n-1}a - F_{n-1}f - F_{n-2}m)(F_na - F_nf - F_{n-1}m)(F_{n+1}a - F_{n+1}f - F_nm)}, \quad n \geq 1,$$

for $n \geq 0$,

$$z_{5n-3} = \frac{-b^2g(b-g)}{(F_{n-2}b + F_{n-1}g)(F_{n-1}b + F_ng)(F_nb + F_{n+1}g)},$$

$$z_{5n-2} = \frac{-c^2(c-h)}{(F_{n-2}c + F_{n-1}h)(F_{n-1}c + F_nh)(F_nc + F_{n+1}h)},$$

$$z_{5n-1} = \frac{-d^2k(d-k)}{(F_{n-2}d + F_{n-1}k)(F_{n-1}d + F_nk)(F_nd + F_{n+1}k)},$$

$$z_{5n} = \frac{-e^2l(e-l)}{(F_{n-2}e + F_{n-1}l)(F_{n-1}e + F_nl)(F_ne + F_{n+1}l)},$$

$$w_{5n-4} = \frac{-b^2g(b-g)}{(F_{n-2}b + F_{n-1}g)(F_{n-1}b + F_ng)^2}, \qquad w_{5n-3} = \frac{-c^2h(c-h)}{(F_{n-2}c + F_{n-1}h)(F_{n-1}c + F_nh)^2},$$

$$w_{5n-2} = \frac{-d^2k(d-k)}{(F_{n-2}d + F_{n-1}k)(F_{n-1}d + F_nk)^2}, \qquad w_{5n-1} = \frac{-e^2l(e-l)}{(F_{n-2}e + F_{n-1}l)(F_{n-1}e + F_nl)^2},$$

$$w_{5n} = \frac{-m^2(a-f)^2}{(F_na - F_nf - F_{n-1}m)(F_{n+1}a - F_{n+1}f - F_nm)^2},$$

where where $z_{-4} = a, z_{-3} = b, z_{-2} = c, z_{-1} = d, z_0 = e, w_{-5} = f, w_{-4} = g, w_{-3} = h, w_{-2} = k, w_{-1} = l, w_0 = m,\ \{F_n\}_{n=-2}^{\infty} = \{-1, 1, 0, 1, 1, 2, 3, 5, 8, ...\},\ F_{n+2} = F_{n+1} + F_n$ and $F_{-2} = -1, F_{-1} = 1$.

Proof. The proof will be achieved by the Mathematical Induction. For $n = 0$, the result holds. Now suppose that $n \geq 1$and that our assumption holds for $n - 1$, that is,

$$z_{5n-9} = \frac{-m^2(a-f)^2}{(F_{n-2}a - F_{n-2}f - F_{n-3}m)(F_{n-1}a - F_{n-1}f - F_{n-2}m)(F_na - F_nf - F_{n-1}m)}$$

$$z_{5n-8} = \frac{-b^2g(b-g)}{(F_{n-3}b+F_{n-2}g)(F_{n-2}b+F_{n-1}g)(F_{n-1}b+F_ng)},$$

$$z_{5n-7} = \frac{-c^2(c-h)}{(F_{n-3}c+F_{n-2}h)(F_{n-2}c+F_{n-1}h)(F_{n-1}c+F_nh)},$$

$$z_{5n-6} = \frac{-d^2k(d-k)}{(F_{n-3}d+F_{n-2}k)(F_{n-2}d+F_{n-1}k)(F_{n-1}d+F_nk)},$$

$$z_{5n-5} = \frac{-e^2l(e-l)}{(F_{n-3}e+F_{n-2}l)(F_{n-2}e+F_{n-1}l)(F_{n-1}e+F_nl)},$$

$$w_{5n-10} = \frac{-m^2(a-f)^2}{(F_{n-2}a-F_{n-2}f-F_{n-3}m)(F_{n-1}a-F_{n-1}f-F_{n-2}m)^2}$$

$$w_{5n-9} = \frac{-b^2g(b-g)}{(F_{n-3}b+F_{n-2}g)(F_{n-2}b+F_{n-1}g)^2}, \quad w_{5n-8} = \frac{-c^2h(c-h)}{(F_{n-3}c+F_{n-2}h)(F_{n-2}c+F_{n-1}h)^2},$$

$$w_{5n-7} = \frac{-d^2k(d-k)}{(F_{n-3}d+F_{n-2}k)(F_{n-2}d+F_{n-1}k)^2}, \quad w_{5n-6} = \frac{-e^2l(e-l)}{(F_{n-3}e+F_{n-2}l)(F_{n-2}e+F_{n-1}l)^2},$$

$$w_{5n-5} = \frac{-m^2(a-f)^2}{(F_{n-1}a-F_{n-1}f-F_{n-2}m)(F_na-F_nf-F_{n-1}m)^2}.$$

Now it follows from (3.1) that

$$\begin{aligned} z_{5n-4} &= \frac{w_{5n-5}(w_{5n-10}-z_{5n-9})}{w_{5n-10}-z_{5n-9}+w_{5n-5}} \\ &= \frac{1}{\frac{1}{w_{5n-5}}+\frac{1}{w_{5n-10}-z_{5n-9}}}. \end{aligned}$$

Calculate

$$\begin{aligned} w_{5n-10}-z_{5n-9} &= \frac{-m^2(a-f)^2}{(F_{n-2}a-F_{n-2}f-F_{n-3}m)(F_{n-1}a-F_{n-1}f-F_{n-2}m)^2} \\ &\quad + \frac{m^2(a-f)^2}{(F_{n-2}a-F_{n-2}f-F_{n-3}m)(F_{n-1}a-F_{n-1}f-F_{n-2}m)(F_na-F_nf-F_{n-1}m)} \\ &= \frac{-m^2(a-f)^2}{(F_{n-2}a-F_{n-2}f-F_{n-3}m)(F_{n-1}a-F_{n-1}f-F_{n-2}m)} \\ &\quad \times\left(\frac{1}{F_{n-1}a-F_{n-1}f-F_{n-2}m}-\frac{1}{F_na-F_nf-F_{n-1}m}\right) \\ &= \frac{-m^2(a-f)^2}{(F_{n-2}a-F_{n-2}f-F_{n-3}m)(F_{n-1}a-F_{n-1}f-F_{n-2}m)} \\ &\quad \times\left(\frac{F_na-F_nf-F_{n-1}m-(F_{n-1}a-F_{n-1}f-F_{n-2}m)}{(F_{n-1}a-F_{n-1}f-F_{n-2}m)(F_na-F_nf-F_{n-1}m)}\right) \\ &= \frac{-m^2(a-f)^2[(F_n-F_{n-1})a-(F_n-F_{n-1})f-(F_{n-1}-F_{n-2})m]}{(F_{n-2}a-F_{n-2}f-F_{n-3}m)(F_{n-1}a-F_{n-1}f-F_{n-2}m)^2(F_na-F_nf-F_{n-1}m)} \\ &= \frac{-m^2(a-f)^2(F_{n-2}a-F_{n-2}f-F_{n-3}m)}{(F_{n-2}a-F_{n-2}f-F_{n-3}m)(F_{n-1}a-F_{n-1}f-F_{n-2}m)^2(F_na-F_nf-F_{n-1}m)} \\ &= \frac{-m^2(a-f)^2}{(F_{n-1}a-F_{n-1}f-F_{n-2}m)^2(F_na-F_nf-F_{n-1}m)} \end{aligned}$$

Also

$$w_{5n-4} = \frac{z_{5n-8}(w_{5n-9}-z_{5n-8})}{w_{5n-9}}.$$

Calculate

$$\begin{aligned} w_{5n-9} - z_{5n-8} &= \frac{-b^2 g(b-g)}{(F_{n-3}b + F_{n-2}g)(F_{n-2}b + F_{n-1}g)^2} \\ &\quad + \frac{b^2 g(b-g)}{(F_{n-3}b + F_{n-2}g)(F_{n-2}b + F_{n-1}g)(F_{n-1}b + F_n g)} \\ &= \frac{-b^2 g(b-g)}{(F_{n-3}b + F_{n-2}g)(F_{n-2}b + F_{n-1}g)}\left(\frac{1}{F_{n-2}b + F_{n-1}g} - \frac{1}{F_{n-1}b + F_n g}\right). \end{aligned}$$

After some calculations we get

$$w_{5n-9} - z_{5n-8} = \frac{b^4 g^2 (b-g)^2}{(F_{n-3}b + F_{n-2}g)(F_{n-2}b + F_{n-1}g)^3 (F_{n-1}b + F_n g)^2}.$$

Then

$$\begin{aligned} w_{5n-4} &= \frac{\frac{b^4 g^2 (b-g)^2}{(F_{n-3}b+F_{n-2}g)(F_{n-2}b+F_{n-1}g)^3(F_{n-1}b+F_n g)^2}}{\frac{-b^2 g(b-g)}{(F_{n-3}b+F_{n-2}g)(F_{n-2}b+F_{n-1}g)^2}} \\ &= \frac{-b^2 g(b-g)}{(F_{n-2}b + F_{n-1}g)(F_{n-1}b + F_n g)^2}. \end{aligned}$$

$$w_{5n-4} = \frac{z_{5n-7}(w_{5n-8} - z_{5n-7})}{w_{5n-8}}.$$

Calculate

$$\begin{aligned} w_{5n-8} - z_{5n-7} &= \frac{-c^2 h(c-h)}{(F_{n-3}c + F_{n-2}h)(F_{n-2}c + F_{n-1}h)^2} \\ &\quad + \frac{c^2 h(c-h)}{(F_{n-3}c + F_{n-2}h)(F_{n-2}c + F_{n-1}h)(F_{n-1}c + F_n h)} \\ &= \frac{-c^2 h(c-h)}{(F_{n-3}c + F_{n-2}h)(F_{n-2}c + F_{n-1}h)}\left(\frac{1}{F_{n-2}c + F_{n-1}h} - \frac{1}{F_{n-1}c + F_n h}\right). \end{aligned}$$

After some calculations we get

$$w_{5n-8} - z_{5n-7} = \frac{c^4 h^2 (c-h)^2}{(F_{n-3}c + F_{n-2}h)(F_{n-2}c + F_{n-1}h)^3 (F_{n-1}c + F_n h)^2}.$$

Then

$$\begin{aligned} w_{5n-4} &= \frac{\frac{c^4 h^2 (c-h)^2}{(F_{n-3}c+F_{n-2}h)(F_{n-2}c+F_{n-1}h)^3(F_{n-1}c+F_n h)^2}}{\frac{-c^2 h(c-h)}{(F_{n-3}c+F_{n-2}h)(F_{n-2}c+F_{n-1}h)^2}} \\ &= \frac{-c^2 h(c-h)}{(F_{n-2}c + F_{n-1}h)(F_{n-1}c + F_n h)^2}. \end{aligned}$$

Similarly, we can obtain the other relations. Thus, the proof is completed. □

3.1. Numerical Examples

Consider the difference system equation (3.1) with the initial conditions

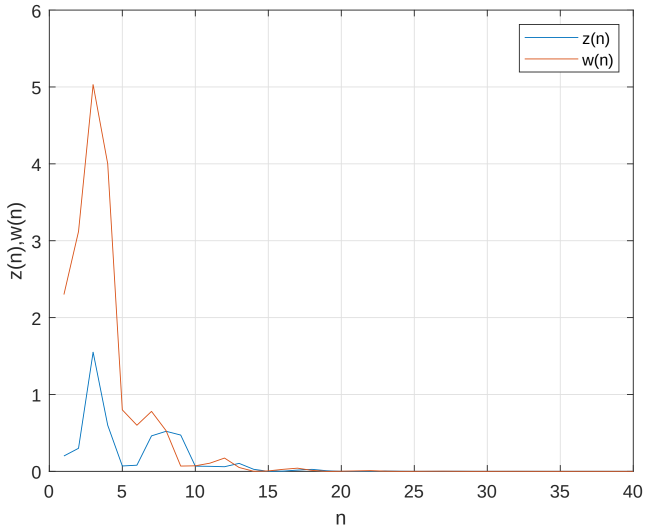

Figure 3: $z_0 = 0.2, z_{-1} = 0.3, z_{-2} = 1.55, z_{-3} = 0.6, z_{-4} = 0.07, z_{-5} = 0, 08, w_0 = 2.3, w_{-1} = 3.12, w_{-2} = 5.03, w_{-3} = 4, w_{-4} = 0.8$ and $w_{-5} = 0.6$.

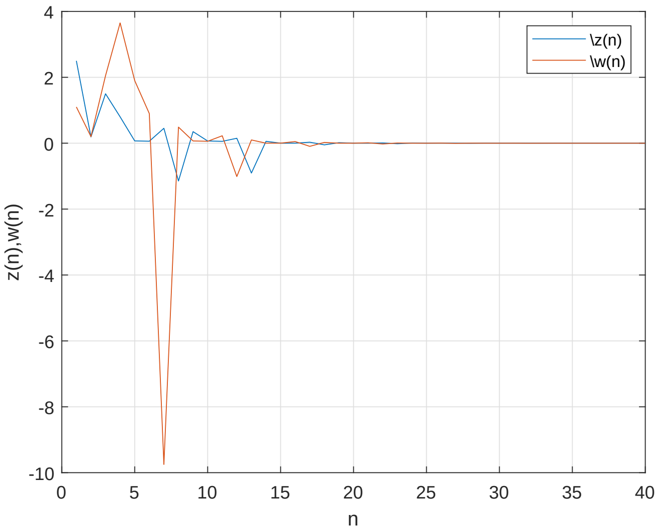

Figure 4: It shows the solution of Eq.(3.1) when we assume that $z_0 = 2.5, z_{-1} = 0.19, z_{-2} = 1.50, z_{-3} = 0.8, z_{-4} = 0.07, z_{-5} = 0, 06, w_0 = 1.1, w_{-1} = 0.20, w_{-2} = 2.05, w_{-3} = 3.65, w_{-4} = 1.9$ and $w_{-5} = 0.9$.

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