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Strong Convergence of Monotone Hybrid Algorithms for Maximal Monotone Mappings and Generalized Nonexpansive Mappings

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Abstract

A class of generalized nonexpansive mappings in Banach spaces is considered and a new monotone hybrid algorithm is presented for finding a common element of the zero point set of a maximal monotone operator and the fixed point set of a generalized nonexpansive mapping. Under certain conditions on the associated parameters, a strong convergence result is established. Moreover, the obtained result is applied to prove a strong convergence theorem for finding a common element of the zero point set of a maximal monotone operator and the fixed point set of a generalized nonexpansive mapping in a Hilbert space.

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1. Introduction

Let E be Banach space and its dual will be denoted by E^*. Let $A \subset E \times E^*$ be a monotone mapping. If $0 \in Ax$, then x is called a zero of A. A fundamental problem in nonlinear analysis and optimization problem is finding such a point. For instance, let $\psi : E \to \mathbb{R} \cup \{\infty\}$ be a proper lower semicontinuous (lsc) and convex function. The subdifferential of ψ, $\partial\psi \subset E \times E^*$, is defined at $x \in E$ by

$$\partial\psi(x) := \{x^* \in E^* : \psi y - \psi x \geq \langle y - x, x^* \rangle \,, \ \forall \ y \in E\} \,.$$

The monotonicity nature of $\partial\psi$ is well known and that $0 \in \partial\psi(x)$ if and only if x^* is a minimizer of ψ. Stipulating $\partial\psi := A$, it implies that in this case, finding a solution of the inclusion $0 \in Ax$ corresponds to finding a minimizer of ψ. Moreover, any maximal monotone mapping $A \subset \mathbb{R} \times \mathbb{R}$ is known to be a

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subdifferential of a proper, convex, and lsc function (See, e.g., [8] Corollary 4.5, p. 170). A prominent method for solving such fundamental problem in general, is known as Proximal Point Algorithm (PPA). Let J_{r_n} denote the resolvent of A, for $x_1 \in E$, PPA is given by

$$x_{n+1} = J_{r_n} x_n, \ n \in \mathbb{N},$$

where $\{r_n\} \subset \{0, \infty\}$. PPA has been studied widely in the literature, both in a Hilbert space (See, e.g., [12, 23, 24]) and Banach spaces (See, e.g., [13, 20, 26]). Typically, most paramount nonlinear problems of mathematics often reduce to finding the fixed points of a certain operator with contractive type conditions. Consequently, how to find the fixed points of such mappings are of great interest to many mathematicians. Therefore, various modifications of Mann iteration (see e.g., [17]) and Ishikawa iteration (see e.g., [11]) have been introduced for the study of nonlinear equations of nonexpansive type (see e.g., [9, 19, 6, 1, 2, 3]).

The hybrid iteration method with generalized projection was introduced for finding the fixed point of relatively nonexpansive mapping T in a uniformly convex and uniformly smooth Banach space E, where T is a self mapping of K, while K is a nonempty closed convex subset of E (See e.g., [18]). For $x_1 = x \in K$ and $n \in \mathbb{N}$, the iteration is given by

$$\begin{cases} u_n = J^{-1}\left(\lambda_n J x_n + (1-\lambda_n) J T x_n\right), \\ K_n = \{u \in K_{n-1} \cap Q_{n-1} : \varphi(u, u_n) \le \varphi(u, x_n)\} \\ Q_n = \{u \in K_{n-1} \cap Q_{n-1} : \langle x_n - u, Jx - Jx_n \rangle \ge 0\} \\ x_{n+1} = \Pi_{K_n \cap Q_n} x, \end{cases}$$

where J is the duality mapping on E and $\{\lambda_n\} \subset [0,1]$. Under the condition that $\limsup_{n\to\infty} \lambda_n < 1$, the sequence of iteration converges strongly to a fixed point of T.

An important generalization of the class of relatively nonexpansive mappings is the class of hemi-relatively nonexpansive mappings. The above hybrid method iteration method is applicable to relatively nonexpansive mapping, but it is not suitable for hemi-relatively nonexpansive mapping (See e.g., [14]). A modification of hybrid iteration method, which is known as monotone hybrid method was introduced for finding a fixed point of a closed hemi-relatively nonexpansive mapping in a uniformly convex and uniformly smooth Banach space E (See, e.g., [27]). It is defined as follow: $x_1 = x \in K$ chosen arbitrarily, then

$$\begin{cases} x_1 = x \in K, K_0 = Q_0 = K, \\ u_n = J^{-1}\left(\lambda_n J x_n + (1-\lambda_n) J T x_n\right), \\ K_n = \{u \in K_{n-1} \cap Q_{n-1} : \varphi(u, u_n) \le \varphi(u, x_n)\} \\ Q_n = \{u \in K_{n-1} \cap Q_{n-1} : \langle x_n - u, Jx - Jx_n \rangle \ge 0\} \\ x_{n+1} = \Pi_{K_n \cap Q_n} x, n \in \mathbb{N}. \end{cases}$$

Given $\limsup_{n\to\infty} \lambda_n < 1$, the sequence of iteration converges strongly to a fixed point of T.

For finding a common element of the zero point set of a maximal monotone operator and the fixed point set of a relatively nonexpansive mapping in a Banach space, the following iteration, which is being referred to as hybrid method was considered.

Let K be a nonempty closed convex subset of a uniformly convex and uniformly smooth Banach space E. Let $A \subset E \times E^*$ be a monotone operator satisfying $D(A) \subset K$ and for all $r > 0$, let $J_r = (J + rA)^{-1} J$. Let T be a relatively nonexpansive mapping of K to itself such that $F(T) \cap A^{-1}0 \neq \emptyset$. For $x_1 = x \in K$ and $n \in \mathbb{N}$, the sequence $\{x_n\}$ is defined by

$$\begin{cases} u_n = J^{-1}\left(\lambda_n J x_n + (1-\lambda_n) J T J_{r_n} x_n\right), \\ K_n = \{u \in K_{n-1} \cap Q_{n-1} : \varphi(u, u_n) \le \varphi(u, x_n)\} \\ Q_n = \{u \in K_{n-1} \cap Q_{n-1} : \langle x_n - u, Jx - Jx_n \rangle \ge 0\} \\ x_{n+1} = \Pi_{K_n \cap Q_n} x, \end{cases}$$

where J is the duality mapping on E and $\{\lambda_n\} \subset [0,1]$ and $\{r_n\}$ is a sequence in $[a,\infty)$ for some $a>0$. Given that $\liminf_{n\to\infty}(1-\lambda_n)>0$, the strong convergence of the sequence of iteration to $\Pi_{F(T)\cap A^{-1}}x$ was established, where $\Pi_{F(T)\cap A^{-1}}$ is the generalized projection from K onto $F(T)\cap A^{-1}0$ (See e.g., [14]).

Finding a common element of the zero point set of a maximal monotone operator and the fixed point set of a hemi-relatively nonexpansive mapping in a Banach space by using the hybrid method has also been considered. It is an extention of the above result and a strong convergence result was obtained (See e.g., [14]).

Motivated by the previous results in this direction, this paper will consider the class of generalized nonexpansive mappings in Banach spaces (See, e.g., [4, 19, 25, 21, 5]). The goal is to find a common element of the zero point set of a maximal monotone operator and the fixed point set of a generalized nonexpansive mapping in a Banach space. A new monotone hybrid algorithm is presented and the conditions which guarantee the strong convergence of the generated sequence are established.

2. Preliminaries

Throughout this paper, the sets of all positive integers and real numbers will be denoted respectively by $\mathbb{N}$ and $\mathbb{R}$. Suppose E^* is the dual of a real Banach space E. The strong convergence of a given sequence $\{x_n\}$ of E to a given point $p\in E$ will be denoted by $x_n\to x$. Let $D(E)$ be the unit sphere centered at the origin of E. Then, E is said to be smooth if the limit

$$\lim_{\theta\to\infty}\frac{\|x+\theta y\|-\|x\|}{\theta}$$

exists for all $x,y\in D(E)$. The space E is said to be uniformly smooth if the limit exists uniformly in $x,y\in D(E)$. The space E is strictly convex if $\|\frac{x+y}{2}\|<1$ whenever $x,y\in D(E)$ and $x\neq y$. It is said to be uniformly convex if for each $\epsilon\in(0,2]$, there exists $\delta>0$ such that $\|\frac{x+y}{2}\|<1-\delta$ whenever $x,y\in D(E)$ and $\|x-y\|\geq\epsilon$.

Definition 2.1. The normalized duality mapping J from E to 2^{E^*} is defined by

$$Jx=\{x^*\in E^*:\langle x,x^*\rangle=\|x\|\|x^*\|,\|x^*\|=\|x\|\}\ \forall\ x\in E.$$

J is known to be uniformly norm-to-norm continuous on bounded sets of E if E is uniformly smooth. Let E be a smooth Banach space. The function $\varphi:E\times E\to\mathbb{R}$ is defined by

$$\varphi(x,y)=\|x\|^2-2\langle x,Jy\rangle+\|y\|^2,$$

for all $x,y\in E$. Observe that in a framework of Hilbert space, $\varphi(x,y)=\|x-y\|^2\geq0$. For all $x,y,z\in E$, the following are well known

(i) $(\|x\|-\|y\|)^2\leq\varphi(x,y)\leq(\|x\|+\|y\|)^2$,

(ii) $\varphi(x,y)=\varphi(x,z)+\varphi(z,y)+2\langle x-z,Jz-Jy\rangle$,

(iii) $\varphi(x,y)=\langle x,Jx-Jy\rangle+\langle x-y,Jy\rangle\leq\|x\|\|Jx-Jy\|+\|x-y\|\|y\|$.

Definition 2.2. Nonexpansive mappings: Let K be a closed subset of a Banach space E. A self-mapping T of K is called nonexpansive if

$$\|Tx-Ty\|\leq\|x-y\|\ \forall\ x,y\in K.$$

A self-mapping $T:K\to K$ is hemi-relatively nonexpansive if $F(T)\neq\emptyset$ and

$$\varphi(p,Tx)\leq\varphi(p,x)\text{ for all }x\in K\text{ and }p\in F(T),$$

where $F(T) = \{x \in K : x = Tx\}$. A point $p \in K$ is said to be an asymptotic fixed point of T if K contains a sequence $\{x_n\}$ which converges weakly to p and $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$. Denote the set of all asymptotic fixed points of T by $\hat{F}(T)$. If $\hat{F}(T) = F(T) \neq \emptyset$, then a hemi-relative nonexpansive mapping $T : K \to K$ is said to be relatively nonexpansive. Interested readers on asymptotic behavior of a relatively nonexpansive mapping are referred to [7]. A mapping $T : K \to E$ is called generalized nonexpansive whenever $F(T) \neq \emptyset$ and

$$\varphi(p, Tx) \leq \varphi(p, x) \text{ for all } x \in K \text{ and } p \in F(T).$$

Definition 2.3. Monotone mappings: Let $A \subset E \times E^*$ with domain $D(A) = \{x \in E : Ax \neq \emptyset\}$ and range $R(A) = \cup \{Ax : x \in D(A)\}$ be a multi-valued mapping. $A^{-1}0$ will denote the set of all points $x \in E$ such that $0 \in Ax$. A is said to be monotone provided that $\langle x - y, x^* - y^* \rangle \geq 0$, $(x, x^*), (y, y^*) \in A$. A is said to be strictly monotone if $\langle x - y, x^* - y^* \rangle > 0, (x, x^*), (y, y^*) \in T(x = y)$. A is said to be maximal if its graph $G(T) = \{(x, x^*) : x^* \in Ax\}$ is not properly contained in the graph of any other monotone mapping. If A is maximal monotone, then the solution set $A^{-1}0$ is closed and convex. It is well known that if E is a strictly convex, smooth, and reflexive Banach space, then a multi-valued monotone mapping A is maximal if and only if $R(J + rA) = E^*$ for all $r > 0$, where $R(J + rA)$ is the range of $J + rA$ (For more details, see, e.g., [22]).

Definition 2.4. Resolvent: Let E be a strictly convex, smooth, and reflexive Banach space and $A \subset E \times E^*$ a maximal monotone mapping. Given $r > 0$ and $x \in E$, then there exists a unique $x_r \in D(A)$ such that $Jx \in Jx_r + rAx_r$. Thus one can define a single-valued mapping $J_r : E \to D(A)$, which is being called the resolvent of A by

$$J_r x = \{z \in D(A) : Jx \in Jz + rAz\}.$$

$J_r x$ consists of one point and for all $r > 0$, $A^{-1}0 = F(J_r)$, where $F(J_r)$ is the set of fixed points of J_r. Also, for all $r > 0$ and $x \in E$, the Yosida approximation $A_r : C \to E^*$ is defined by

$$A_r x = \frac{1}{r}(J - JJ_r)x.$$

For all $r > 0$ and $x \in E$, the following hold (See e.g, [16, 10])

(i) $\varphi(p, J_r x) + \varphi(J_r x, x) \leq \varphi(p, x)$ for all $p \in A^{-1}0$.

(ii) $(J_r x, A_r x) \in A$.

Definition 2.5. Metric projection: Let K be a nonempty closed convex subset of a Hilbert space H. A mapping $P_K : H \to K$ of H onto K satisfying

$$\|x - P_K x\| = \min_{y \in K} \|x - y\|,$$

is called the metric projection. This set is known to be singleton. The metric projection has the important property that; for $x \in H$ and $x_0 \in K, x_0 = P_K x$ if and only if

$$\langle x - x_0, x_0 - y \rangle \geq 0 \ \forall \ y \in K.$$

Definition 2.6. Generalized projection: Let K be nonempty subset of a Banach space E. A mapping $\Pi_K : E \to K$ of E onto K satisfying

$$\varphi(\Pi_K x, x) = \min_{y \in K} \varphi(y, x),$$

is called the generalized projection and it known to be singleton.

Definition 2.7. Retraction: Let K be nonempty subset of a Banach space E. A mapping $R : E \to K$ is called sunny if

$$R(Rx + \alpha(x - Rx)) = Rx,$$

for all $x \in E$ and all $\alpha \geq 0$. If $Rx = x$ for all $x \in K$, it is also called a retraction. A retraction which is also sunny and nonexpansive is called a sunny nonexpansive retraction. If E is a smooth Banach space, the sunny nonexpansive retraction of E onto K is denoted by R_K. K is said to be a sunny generalized nonexpansive retract of E provided that there exists a sunny generalized nonexpansive retraction R from E onto K.

Reference will be made to the following results on sunny generalized nonexpansive retraction (See e.g, [10, 15]).

Lemma 2.8. *Let K be a nonempty closed subset of a smooth and strictly convex Banach space E. Let R_K be a retraction of E onto K. Then R_K is sunny and generalized nonexpansive if and only if*

$$\langle x - R_K x, JR_K x - Jy\rangle \geq 0$$

for each $x \in E$ and $y \in K$.

Lemma 2.9. *Let K be a nonempty closed subset of a smooth and strictly convex Banach space E such that there exists a sunny generalized nonexpansive retraction R from E onto K and let $(x, z) \in E \times K$. Then the following hold:*

(i) $z = Rx$ if and only if $\langle x - z, Jy - Jz\rangle \leq 0$ for all $y \in K$;

(ii) $\varphi(x, R_K y) + \varphi(R_K y, y) \leq \varphi(x, y)$.

Lemma 2.10. *Let E be a smooth, strictly convex and reflexive Banach space and let K be a nonempty closed subset of E. Then the following are equivalent:*

(i) K is a sunny generalized nonexpansive retract of E;

(ii) K is a generalized nonexpansive retract of E;

(iii) JK is closed and convex.

The following well known results will also be needed.

Lemma 2.11. *Let E be a uniformly convex and smooth Banach space and let $\{u_n\}$ and $\{v_n\}$ be two sequences in E such that either $\{u_n\}$ or $\{v_n\}$ is bounded. If $\lim_{n\to\infty} \varphi(u_n, v_n) = 0$, then $\lim_{n\to\infty} \|u_n - v_n\| = 0$ (See [13]).*

Lemma 2.12. *Let E be a uniformly convex and smooth Banach space and let $r > 0$. Then there exists a strictly increasing, continuous and convex function $g : [0, 2r] \to [0, \infty)$ such that $g(0) = 0$ and*

$$g(\|x - y\|) \leq \varphi(x, y)$$

for all $x, y \in B_r(0)$, where $B_r(0) = \{z \in E : \|z\| \leq r\}$ (See e.g, [13]).

3. Main Results

Lemma 3.1. *Let E be a strictly convex, smooth, and reflexive Banach space and let $A \subset E \times E^*$ be a maximal monotone mapping with $A^{-1}0 \neq \emptyset$. For each $r > 0$, let $J_r : E \to E$ be the resolvent of A. Then J_r is a generalized nonexpansive mapping.*

Proof. Let $x \in E, y \in F(J_r)$ and $r > 0$. Since A is maximal monotone, recall that $A^{-1}0 = F(J_r)$. Apply Definition 2.4(i) to have

$$\varphi(y, J_r x) + \varphi(J_r x, x) \leq \varphi(y, x) \text{ for all } y \in A^{-1}0.$$

By Definition 2.1(i), $\varphi(J_r x, x) \geq 0$. Consequently

$$\varphi(y, J_r x) \leq \varphi(y, x).$$

□

Lemma 3.2. *Let K be a nonempty closed subset of a smooth and strictly convex Banach space E. Let R_K be a sunny generalized nonexpansive retraction from E onto K. Then for all $x, y \in E$,*

$$\varphi(x, R_K y) \leq \varphi(x, y).$$

Proof. By Lemma 2.9 (ii),

$$\varphi(x, R_K y) + \varphi(R_K y, y) \leq \varphi(x, y) \quad \forall\ x, y \in E.$$

By Definition 2.1(i), $\varphi(R_K y, y) \geq 0$. Wherefore,

$$\varphi(x, R_K y) \leq \varphi(x, y).$$

□

Theorem 3.3. *Let K be a nonempty closed convex subset of a uniformly convex and uniformly smooth Banach space E. Let $A \subset E \times E^*$ be a maximal monotone mapping and for all $r > 0$, the resolvent $J_r : E \to E$ is associated to A. Let $R_K : E \to K$ be a sunny and generalized nonexpansive retraction from E onto K and $T : K \to E$ be a closed generalized nonexpansive mapping such that $F(T) \cap A^{-1}0 \neq \emptyset$. For each $n \in N$, the sequence $\{x_n\}$ is generated as by*

$$\begin{cases} x_1 = x \in K, \ K_0 = Q_0 = K, \\ u_n = J^{-1}\left(\lambda_n J x_n + (1-\lambda_n) J T R_K\left(J_{r_n} x_n\right)\right), \\ v_n = J^{-1}\left(\beta_n J u_n + (1-\beta_n) J T R_K\left(J_{r_n} x_n\right)\right), \\ K_n = \{u \in K_{n-1} \cap Q_{n-1} : \varphi(u, v_n) \leq \varphi(u, x_n)\} \\ Q_n = \{u \in K_{n-1} \cap Q_{n-1} : \langle x_n - u, Jx - Jx_n \rangle \geq 0\} \\ x_{n+1} = R_{K_n \cap Q_n} x, \end{cases}$$

where J is the duality mapping on E, $\{\lambda_n\}$ and $\{\beta_n\}$ are sequences in $[0,1]$ such that $\liminf_{n\to\infty}(1-\lambda_n) > 0$ and $\lim_{n\to\infty} \beta_n = 1$, while $\{r_n\}$ is a sequence in $[a, \infty)$ for some $a > 0$. Then the sequence $\{x_n\}$ converges strongly to $R_{F(T)\cap A^{-1}0}x$, where $R_{F(T)\cap A^{-1}0}$ is the sunny nonexpansive retraction from K onto $F(T) \cap A^{-1}0$.

Proof. **Step 1**: We show that JK_n and JQ_n are closed and convex for all $n \in \mathbb{N}$. It is obvious from the definition of K_n and Q_n that JK_n is closed and JQ_n is closed and convex for each $n \in \mathbb{N}$. The task is to show that JK_n is convex.

Observe that

$$\varphi(u, v_n) \leq \varphi(u, x_n)$$

implies that for all $u \in JK_n$,

$$\|x_n\|^2 - \|v_n\|^2 - 2\langle u, Jx_n - Jv_n \rangle \geq 0,$$

which is affine in u, and thus JK_n is convex. Consequently by Lemma 2.10, $K_n \cap Q_n$ is a closed and convex subset of E for all $n \in \mathbb{N}$.

Step 2: We show that $F(T)\cap A^{-1}0\subset K_n\cap Q_n$. Let $p\in F(T)\cap A^{-1}0$ and put $z_n=R_K\left(J_{r_n}x_n\right)$. By the generalized nonexpansive property of T and J_{r_n}, we have

$$\begin{aligned}
\varphi\left(p,u_n\right) &= \varphi\left(p,J^{-1}\left(\lambda_nJx_n+(1-\lambda_n)JTz_n\right)\right)\\
&= \|p\|^2-2\left\langle p,\lambda_nJx_n+(1-\lambda_n)JTz_n\right\rangle+\|\lambda_nJx_n+(1-\lambda_n)JTz_n\|^2\\
&\le \|p\|^2-2\lambda_n\left\langle p,Jx_n\right\rangle-2(1-\lambda_n)\left\langle p,JTz_n\right\rangle+\lambda_n\|x_n\|^2+(1-\lambda_n)\|Tz_n\|^2\\
&= \lambda_n\varphi\left(p,x_n\right)+(1-\lambda_n)\varphi\left(p,Tz_n\right)\\
&\le \lambda_n\varphi\left(p,x_n\right)+(1-\lambda_n)\varphi\left(p,z_n\right)\\
&= \lambda_n\varphi\left(p,x_n\right)+(1-\lambda_n)\varphi\left(p,R_K\left(J_{r_n}x_n\right)\right)\\
&\le \lambda_n\varphi\left(p,x_n\right)+(1-\lambda_n)\varphi\left(p,J_{r_n}x_n\right)\\
&\le \lambda_n\varphi\left(p,x_n\right)+(1-\lambda_n)\varphi\left(p,x_n\right)\\
&= \varphi\left(p,x_n\right).
\end{aligned} \tag{3.1}$$

Therefore,

$$\begin{aligned}
\varphi\left(p,v_n\right) &= \varphi\left(p,J^{-1}\left(\beta_nJv_n+(1-\beta_n)JTz_n\right)\right)\\
&= \|p\|^2-2\left\langle p,\beta_nJv_n+(1-\beta_n)JTz_n\right\rangle+\|\beta_nJv_n+(1-\beta_n)JTz_n\|^2\\
&\le \|p\|^2-2\beta_n\left\langle p,Jv_n\right\rangle-2(1-\beta_n)\left\langle p,JTz_n\right\rangle+\beta_n\|v_n\|^2+(1-\beta_n)\|Tz_n\|^2\\
&= \beta_n\varphi\left(p,v_n\right)+(1-\beta_n)\varphi\left(p,Tz_n\right)\\
&\le \beta_n\varphi\left(p,v_n\right)+(1-\beta_n)\varphi\left(p,z_n\right)\\
&= \beta_n\varphi\left(p,v_n\right)+(1-\beta_n)\varphi\left(p,R_K\left(J_{r_n}x_n\right)\right)\\
&\le \beta_n\varphi\left(p,v_n\right)+(1-\beta_n)\varphi\left(p,J_{r_n}x_n\right)\\
&\le \beta_n\varphi\left(p,v_n\right)+(1-\beta_n)\varphi\left(p,x_n\right)\\
&\le \beta_n\varphi\left(p,x_n\right)+(1-\beta_n)\varphi\left(p,x_n\right)\\
&= \varphi\left(p,x_n\right).
\end{aligned}$$

So $p\in K_n$ for all $n\in\mathbb{N}$, which indicates that $F(T)\cap A^{-1}0\subset K_n$. Next is to show that $F(T)\cap A^{-1}0\subset Q_n$ for all $n\in\mathbb{N}$. Recall that by the strictly convexity property of E since E is uniformly convex, J is one-to-one. Therefore, for each $n\in\mathbb{N}$, $J\left(K_n\cap Q_n\right)=JK_n\cap JQ_n$ is closed convex. By Lemma 2.10, $K_n\cap Q_n$ is a sunny generalized nonexpansive retract of E. To apply induction to show that $F(T)\cap A^{-1}0\subset Q_n$ for all $n\in\mathbb{N}$, observe that for $n=1$, by definition, $F(T)\cap A^{-1}0\subset K=K_0\cap Q_0$. Assume that $F(T)\cap A^{-1}0\subset K_{k-1}\cap Q_{k-1}$ for some $k\in\mathbb{N}$. Since $x_k=R_{K_{k-1}\cap Q_{k-1}}x$, applying Lemma 2.8 gives

$$\langle x-x_k,Jx_k-Ju\rangle\ge 0,$$

for all $u\in K_{n-1}\cap Q_{n-1}$. Given that $F(T)\cap A^{-1}0\subset K_{k-1}\cap Q_{k-1}$, then

$$\langle x-x_k,Jx_k-Ju\rangle\ge 0,\quad \forall\ u\in F(T)\cap A^{-1}0. \tag{3.2}$$

The definition of Q_n and the inequality (3.2) implies that $F(T)\cap A^{-1}0\subset Q_k$ and consequently $F(T)\cap A^{-1}0\subset Q_n$ for all $n\in\mathbb{N}$. Therefore, for all $n\in\mathbb{N}$, $F(T)\cap A^{-1}0\subset K_n\cap Q_n$ and this justifies that $\{x_n\}$ is well defined.

Step 3: Next is to show that $x_n\to R_{F(T)\cap A^{-1}0}x$ as $n\to\infty$. The definition of Q_n implies that $x_n=R_{Q_n}x$. Apply Lemma 2.9(ii) to obtain

$$\varphi(x,x_n)=\varphi(x,R_{Q_n}x)\le\varphi(x,u)-\varphi(R_{Q_n}x,u)\le\varphi(x,u),$$

for all $F(T)\cap A^{-1}0\subset Q_n$, that is, $\{\varphi(x,x_n)\}$ is bounded. Furthermore, by definition of φ, it is known that $\{x_n\},\{u_n\}$ and $\{z_n\}$ are bounded. Therefore, $\lim\limits_{n\to\infty}\varphi(x,x_n)$ exists. For any positive integer k and for each

$n \in N$, it can be obtained from $x_n = R_{Q_n}x$ that

$$\begin{aligned}\varphi(x_n, x_{n+k}) &= \varphi(R_{Q_n}x, x_{n+k}) \\ &\leq \varphi(x, x_{n+k}) - \varphi(x, R_{Q_n}x) \\ &\leq \varphi(x, x_{n+k}) - \varphi(x, x_n),\end{aligned}$$

consequently,

$$\lim_{n\to\infty} \varphi(x_n, x_{n+k}) = 0. \tag{3.3}$$

On the grounds that $\{x_n\}$ is bounded, there exists $r > 0$ such that $\{x_n\}\, Br(0)$. By Lemma 2.12, there exists a strictly increasing, convex and continuous function $g : [0, 2r] \to [0, \infty)$ such that for $m, n \in \mathbb{N}$ with $m > n$,

$$g(\|x_m - x_n\|) \leq \varphi(x_m, x_n) \leq \varphi(x_m, x_0) - \varphi(x_n, x_0).$$

It can be deduced from the property of g that $\{x_n\}$ is a Cauchy sequence. Thus there exists $\nu \in K$ so that $x_n \to \nu$. By considering $x_{n+1} = R_{K_n \cap Q_n}x \in K_n$ and by the definition of K_n,

$$\varphi(x_{n+1}, x_n) - \varphi(x_{n+1}, v_n) \geq 0, \ \forall\, n \in \mathbb{N}. \tag{3.4}$$

By (3.3) and (3.4), it can be concluded that $\lim_{n\to\infty} \varphi(x_{n+1}, x_n) = \lim_{n\to\infty} \varphi(x_{n+1}, v_n) = 0$. In light of uniform convexity and smoothness of E, apply Lemma 2.11 to get

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = \lim_{n\to\infty} \|x_{n+1} - v_n\| = 0,$$

thus

$$\lim_{n\to\infty} \|x_n - v_n\| = 0.$$

The norm-to-norm uniform continuity of J on bounded sets leads to

$$\lim_{n\to\infty} \|Jx_{n+1} - Jx_n\| = \lim_{n\to\infty} \|Jx_{n+1} - Jv_n\| = \|Jx_n - Jv_n\| = 0. \tag{3.5}$$

On account of the fact that $\{u_n\}$ and $\{Tz_n\}$ are bounded and $\beta_n \to 1$ as $n \to \infty$, it can be deduced that

$$\|Ju_n - Jv_n\| = (1 - \beta_n)\|Ju_n - JTz_n\| \to 0,$$

thus

$$\|Jx_{n+1} - Ju_n\| \leq \|Jx_{n+1} - Jv_n\| + \|Jv_n - Ju_n\| \to 0 \text{ as } n \to \infty, \tag{3.6}$$

and by the norm-to-norm uniform continuity of J^{-1} on bounded sets,

$$\lim_{n\to\infty} \|x_{n+1} - u_n\| = 0.$$

Moreover,

$$\begin{aligned}\|Jx_{n+1} - Ju_n\| &= \|Jx_{n+1} - \lambda_n Jx_n - (1 - \lambda_n)JTz_n\| \\ &= \|(1 - \lambda_n)(Jx_{n+1} - JTz_n) - \lambda_n (Jx_n - Jx_{n+1})\| \\ &\geq (1 - \lambda_n)\|Jx_{n+1} - JTz_n\| - \lambda_n\|Jx_n - Jx_{n+1}\|.\end{aligned}$$

It follows that

$$\|Jx_{n+1} - JTz_n\| \leq \frac{1}{(1 - \lambda_n)} \left(\|Jx_{n+1} - Ju_n\| + \lambda_n\|Jx_n - Jx_{n+1}\|\right).$$

By (3.5) and (3.6) with $\liminf_{n\to\infty}(1-\lambda_n) > 0$, it is obtained that

$$\lim_{n\to\infty} \|Jx_{n+1} - JTz_n\| = 0.$$

Recall that J^{-1} is norm-to-norm uniformly continuous on bounded sets. Therefore

$$\lim_{n\to\infty} \|x_{n+1} - Tz_n\| = 0.$$

Note that

$$\|x_n - Tz_n\| \leq \|x_n - x_{n+1}\| + \|x_{n+1} - Tz_n\|,$$

leads to

$$\lim_{n\to\infty} \|x_n - Tz_n\| = 0.$$

Observe that by (3.1),

$$\varphi(p, z_n) \geq \frac{1}{(1-\lambda_n)} (\varphi(p, u_n) - \lambda_n \varphi(p, x_n)).$$

Recall that $z_n =: R_K(J_{r_n} x_n)$, by Lemma 2.9 (ii),

$$\begin{aligned}
\varphi(z_n, x_n) &= \varphi(R_K(J_{r_n}x_n), x_n) \leq \varphi(p, x_n) - \varphi(p, R_K(J_{r_n}x_n)) \\
&= \varphi(p, x_n) - \varphi(p, z_n) \\
&\leq \varphi(p, x_n) - \frac{1}{(1-\lambda_n)}(\varphi(p, u_n) - \lambda_n\varphi(p, x_n)) \\
&= \frac{1}{(1-\lambda_n)}(\varphi(p, x_n) - \varphi(p, u_n)) \\
&= \frac{1}{(1-\lambda_n)}\left(\|x_n\|^2 - \|u_n\|^2 - 2\langle p, Jx_n - Ju_n\rangle\right) \\
&\leq \frac{1}{(1-\lambda_n)}\left(|\|x_n\|^2 - \|u_n\|^2| + 2|\langle p, Jx_n - Ju_n\rangle|\right) \\
&\leq \frac{1}{(1-\lambda_n)}(|\|x_n\| - \|u_n\||(\|x_n\| + \|u_n\|) + 2\|p\|\|Jx_n - Ju_n\|) \\
&\leq \frac{1}{(1-\lambda_n)}(\|x_n - u_n\|(\|x_n\| + \|u_n\|) + 2\|p\|\|Jx_n - Ju_n\|).
\end{aligned}$$

Since $\lim_{n\to\infty} \|x_n - u_n\| = 0$ and by (3.6), observe that $\lim_{n\to\infty} \varphi(z_n, x_n) = 0$. Wherefore by Lemma 2.11,

$$\lim_{n\to\infty} \|z_n - x_n\| = 0. \tag{3.7}$$

Bearing in mind that $x_n \to \nu$ and $\lim_{n\to\infty} \|z_n - x_n\| = 0$, thence $z_n \to \nu$. Given that T is closed and $z_n \to \nu$, then ν is a fixed point of T. Next is to show that $\nu \in A^{-1}0$. Using (3.7) and by the norm-to-norm uniform continuity of J on bounded sets, it is obtained that

$$\lim_{n\to\infty} \|Jx_n - Jz_n\| = 0.$$

For $r_n \geq a$, obtain that

$$\lim_{n\to\infty} \frac{1}{r_n}\|Jx_n - Jz_n\| = 0,$$

wherefore

$$\lim_{n\to\infty} \|A_{r_n}x_n\| = \lim_{n\to\infty} \frac{1}{r_n}\|Jx_n - Jz_n\| = 0.$$

For $(s, s^*) \in A$, the monotonicity of A gives that

$$\langle s - z_n, s^* - A_{r_n}x_n\rangle \geq 0 \text{ for all } n \in \mathbb{N}.$$

Letting $n \to \infty$ leads to

$$\langle s - \nu, s^* \rangle \geq 0.$$

The maximality of A ascertains that $\nu \in A^{-1}0$. Next is to show that $\nu = R_{F(T)\cap A^{-1}0}x$. Applying Lemma 2.9 gives

$$\varphi\left(\nu, R_{F(T)\cap A^{-1}0}x\right) + \varphi\left(R_{F(T)\cap A^{-1}0}x, x\right) \leq \varphi\left(\nu, x\right).$$

On account of $x_{n+1} = R_{K_n \cap Q_n}x$ and $\nu \in F(T) \cap A^{-1}0 \subset K_n \cap Q_n$, apply Lemma 2.9 to obtain

$$\varphi\left(R_{F(T)\cap A^{-1}0}x, x_{n+1}\right) + \varphi\left(x_{n+1}, x\right) \leq \varphi\left(R_{F(T)\cap A^{-1}0}x, x\right).$$

Then it can be deduced by the definition of φ that $\varphi\left(\nu, x\right) \leq \varphi\left(R_{F(T)\cap A^{-1}0}x, x\right)$ and $\varphi\left(\nu, x\right) \geq \varphi\left(R_{F(T)\cap A^{-1}0}x, x\right)$, thereby $\varphi\left(\nu, x\right) = \varphi\left(R_{F(T)\cap A^{-1}0}x, x\right)$. Thus, considering that $R_{F(T)\cap A^{-1}0}x$ is unique, the conclusion is that $\nu = R_{F(T)\cap A^{-1}0}x$. □

The following result can be deduced from Theorem 3.3, which is the main result of this paper.

Corollary 3.4. *Let K be a nonempty closed convex subset of a uniformly convex and uniformly smooth Banach space E. Let $A \subset E \times E^*$ be a maximal monotone mapping and for all $r > 0$, the resolvent $J_r : E \to E$ is associated to A. Let $R_K : E \to K$ be a sunny and generalized nonexpansive retraction from E onto K and $T : K \to E$ be a closed generalized nonexpansive mapping such that $F(T) \cap A^{-1}0 \neq \emptyset$. For each $n \in N$, the sequence $\{x_n\}$ is generated as by*

$$\begin{cases} x_1 = x \in K, \;\; K_0 = Q_0 = K, \\ u_n = J^{-1}\left(\lambda_n J x_n + (1-\lambda_n) J T R_K\left(J_{r_n} x_n\right)\right), \\ K_n = \{u \in K_{n-1} \cap Q_{n-1} : \varphi(u, v_n) \leq \varphi(u, x_n)\} \\ Q_n = \{u \in K_{n-1} \cap Q_{n-1} : \langle x_n - u, Jx - Jx_n \rangle \geq 0\} \\ x_{n+1} = R_{K_n \cap Q_n}x, \end{cases}$$

where J is the duality mapping on E, $\{\lambda_n\}$ is a sequence in $[0,1]$ such that $\liminf\limits_{n\to\infty}(1-\lambda_n) > 0$ and $\{r_n\}$ is a sequence in $[a, \infty)$ for some $a > 0$. Then the sequence $\{x_n\}$ converges strongly to $R_{F(T)\cap A^{-1}0}x$, where $R_{F(T)\cap A^{-1}0}$ is the sunny nonexpansive retraction from K onto $F(T) \cap A^{-1}0$.

Proof. By letting $\beta_n = 1$ for all $n \in \mathbb{N}$ in Theorem 3.3, the desired result is obtained. □

In the framework of Hilbert spaces, the main result of this paper is given as below.

Corollary 3.5. *Let K be a nonempty closed convex subset of a Hilbert space H. Let $A \subset H \times H$ be a maximal monotone mapping and for all $r > 0$, the resolvent $J_r : H \to H$ is associated to A. Let $P_K : H \to K$ be a metric projection from H onto K and $T : K \to H$ be a closed generalized nonexpansive mapping such that $F(T) \cap A^{-1}0 \neq \emptyset$. For each $n \in N$, the sequence $\{x_n\}$ is generated as by*

$$\begin{cases} x_1 = x \in K, \;\; K_0 = Q_0 = K, \\ u_n = \lambda_n x_n + (1-\lambda_n) T R_K\left(J_{r_n} x_n\right), \\ v_n = \beta_n u_n + (1-\beta_n) T R_K\left(J_{r_n} x_n\right), \\ K_n = \{u \in K_{n-1} \cap Q_{n-1} : \|u - v_n\| \leq \|u - x_n\|\} \\ Q_n = \{u \in K_{n-1} \cap Q_{n-1} : \langle x_n - u, x - x_n \rangle \geq 0\} \\ x_{n+1} = P_{K_n \cap Q_n}x, \end{cases}$$

where $\{\lambda_n\}$ and $\{\beta_n\}$ are sequences in $[0,1]$ such that $\liminf\limits_{n\to\infty}(1-\lambda_n) > 0$ and $\lim\limits_{n\to\infty} \beta_n = 1$, while $\{r_n\}$ is a sequence in $[a, \infty)$ for some $a > 0$. Then the sequence $\{x_n\}$ converges strongly to $R_{F(T)\cap A^{-1}0}x$, where $R_{F(T)\cap A^{-1}0}$ is the metric projection from K onto $F(T) \cap A^{-1}0$.

Proof. Recall that in a Hilbert space, $\varphi(x, y) = \|x - y\|^2$ for all $x, y \in H$ and J is the identity mapping. Therefore, the desired result readily follows from Theorem 3.3. □

4. Conclusion

PPA is a prominent method for solving nonlinear analysis and optimization problems. Its various modifications and expansions have been considered in the literature, both in Hilbert space and Banach spaces. A new monotone hybrid method which is amalgamated with PPA has been presented in this paper for finding a common element of the zero point set of a maximal monotone operator and the fixed point set of a generalized nonexpansive mapping. The required conditions on the associated parameters to obtain a strong convergence result are established. The subdifferential of a proper, convex, and lsc function has been already been cited as an example of a maximal monotone mapping. Real sequences which satisfy the conditions stated in the main theorem are $\{\lambda_n\} = \left\{\frac{1}{8} + \frac{1}{5n}\right\}$ and $\{\beta_n\} = \left\{1 - \frac{1}{2n}\right\}$.

Abbreviation

lsc: lower semicontinuous. conditions stated in the main theorem are $\{\lambda_n\} = \left\{\frac{1}{8} + \frac{1}{5n}\right\}$ and $\{\beta_n\} = \left\{1 - \frac{1}{2n}\right\}$.

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