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Measures of Noncompactness on Ω-distance Spaces.

Asif Hussain Jan*, Tanweer Jalal

Department of Mathematics, National Institute of Technology, Hazratbal, Srinagar -190006, Jammu and Kashmir, India

Abstract

The aim of this article is to present a new framework for studying measures of noncompactness in G-metric spaces. First, we introduce the concept of Ω-distance space as an Ω-measure of non-compactness on G-metric spaces. Finally, we use our main result to characterize G-metric completeness.

Keywords: G-metric space, Measures of noncompactness, Ω-distance.
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1. Introduction and Preliminaries

Measures of non-compactness is a function that determines how non-compact a set is. In 1930 Kuratowski [14] developed and explored the first measure of non-compactness, the function α. Darbo [6], an Italian mathematician, used the Kuratowski measure in 1955 to examine a class of operators (condensing operators) whose features can be described as being intermediate between those of contraction and compact mappings. Goldenstein et al [7] established the Hausdorff measure of non-compactness (χ) in 1957 (and later investigated by Goldenstein and Markus [8], and Measure of non-compactness β by Istratescu [9] in 1972). A relevant measure of noncompactness in a given space is one that satisfies some requirement for relative compactness and can be stated by a simple formula. In some spaces, the Hausdorff measure of noncompactness satisfies these requirements. However, developing an useful measure of noncompactness in desirable spaces is not an easy process. Several authors explored an axiomatic method to construct a broad concept of measure of noncompactness in order to overcome this hurdle. For further information on measures of noncompactness, see [3, 4, 10, 11, 15, 16, 17].

We take an axiomatic approach to this concept in this work, which includes the most significant definitions. Let $\mathbb{N}$ represent the set of natural numbers, $\mathbb{R}$ the set of real numbers, and $\mathbb{R}^+$ the set of nonnegative real numbers.

*Asif Hussain Jan
Email addresses: $asif_{06}phd20@nitsri.net$ (Asif Hussain Jan), $tjalal@nitsri.ac.in$ (Tanweer Jalal)

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If (X, d) is a metric space and T its subset. $P(X)$ denotes the set of all subsets of space X, whereas $\bar{T}$ and $\delta(T)$ signify the closure and diameter of T, respectively.

Definition 1.1. [5] Let (X, d) be a metric space. The function ϕ: $P(X) \longrightarrow [0,\infty)$ is said to be measure of non-compactness if it satisfies the following conditions:

1. $\phi(T)=\infty$ if and only if T is unbounded;
2. $\phi(T) = \phi(\bar{T})$;
3. $\phi(T) = 0$, then T is totally bounded;
4. if $T \subseteq R$, then $\phi(T) \leq \phi(R)$;
5. if X is complete, and $\{S_n\}$ is a sequence of nonempty closed subsets of X such that $S_{n+1} \subseteq S_n$ for all $n \in \mathbb{N}$ and $\lim\limits_{n\to\infty} \phi(S_n)=0$, then $\bigcap\limits_{n=1}^{\infty} (S_n)$ is a nonempty compact subset of X.

Definition 1.2. A set T in a metric space (X, d) is said to be totally bounded if for every $\epsilon>0$ it can be covered by a finite number of open balls of radius ϵ.

The Kuratowski measure of non-Compactness is defined as follows [5, 6, 18, 21]

Definition 1.3. Let (X, d) be a metric space. Kuratowski measure of non-compactness of a bounded set $T \subseteq X$, denoted by $\alpha(T)$, is the infimum of all $\epsilon>0$ such that T can be covered by a finite number of sets whose diameter is less than ϵ.

Theorem 1.4. *(Kuratowski[14]) Let (X, d) be a complete metric space, and let $\{S_n\}$ be a sequence of nonempty closed and bounded subsets of X such that $S_{n+1} \subseteq S_n$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} \alpha(S_n)=0$. Then, $\bigcap\limits_{n=1}^{\infty} S_n$ is a nonempty and compact subset of X.*
.

In 2005, Mustafa and Sims[20] introduced the notion of a G-metric space as generalization of the usual metric space. In this paper, we introduce the notion of Ω-measure of non-compactness on G-metric space with respect Ω-distance in the sense of saadati et al[22], which is a generalization of the concept of a ω-distance due to Kada et al. [12].

Definition 1.5. [22] Let X be a non-empty set and let $G : X \times X \times X \to R^+$ be a function satisfying the following axioms:

(i) If $G(x, y, z) = G(y, z, x) = G(z, x, y) = 0$ if $x = y = z$
(ii) $G(x, x, y) > 0$ for all $x, y \in X$, where $x \neq y$,
(iii) $G(x, x, z) \leq G(x, y, z)$ for all $x, y, z \in X$, with $z \neq y$,
(iv) $G(x, y, z) = G(p\{x, y, z\})$, where p is permutation of x, y, z (symmetry),
(v) $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$ for all $x, y, z, a \in X$(rectangle inequality).

Then the function G is called a generalized metric, or more specifically G-metric on X, and the pair (X, G) is called G-metric space.

Definition 1.6. [20] Let (X, G) be a G-metric space and let (x_n) be a sequence of points of X. We say that (x_n) is G-convergent to x if for any $\epsilon > 0$, there exists $k \in \mathbb{N}$ such that $G(x, x_n, x_m) < \epsilon$ for all $n, m \geq k$.

Definition 1.7. [20] Let (X, G) be a G-metric space. A sequence (x_n) is called G-Cauchy sequence if, for each $\epsilon > 0$ there exists a positive integer $m^* \in \mathbb{N}$ such that $G(x_n, x_m, x_l) < \in$ for all $n, m, l \geq m^*$ i.e $G(x, x_n, x_l) \longrightarrow 0$ as $n, m, l \longrightarrow \infty$.

Definition 1.8. [20] A G-metric space (X, G) is said to be G-complete if every G-Cauchy sequence in (X, G) is G-convergent in X.

Definition 1.9. [20] Let (X, G) be a G-metric space and let $\{F_n\}$ be a descending sequence $(F_1 \supseteq F_2 \subseteq F_3 \supseteq ...)$ of nonempty G-closed subsets of X such that $\sup\{G(x, y, z) : x, y, z \in F_n\} \to 0$ as $n \to \infty$, then (X, G) is G-complete if and only if $\cap_{n=1}^{\infty} F_n$ consists of exactly one point.

Definition 1.10. [20] A G-metric space is said to be compact G-metric space if it is G-complete and G-totally bounded.

Definition 1.11. [22] Let (X, G) be a G-metric space. Then $K : X \times X \times X$ to $\mathbb{R}^+$ is said to be an Ω-distance on X if the following conditions are satisfied:

(i) $K(x, y, z) \leq K(x, a, a) + K(a, y, z)$ for all $x, y, z, a \in X$,
(ii) for any $x, y \in X, K(x, y, .), K(x, ., y) : X \to [0, \infty)$ are lower semi-continuous.
(iii) for each $\epsilon > 0$, there exists $\delta > 0$ such that $K(x, a, a) \leq \delta$ and $K(a, y, z) \leq \delta$ imply $G(x, y, z) \leq \epsilon$.

Example 1.12. [22] Let (X, d) be a metric space and $G : X \times X \times X$ to $\mathbb{R}^+$ defined by

$$G(x, y, z) = \max\{d(x, y), d(y, z), d(x, z)\}$$

for all $x, y, z \in X$. Then $\Omega = G$ is an Ω-distance on X.

2. Main Results

We will generalize Arandjelovic's [1, 19] axiomatic definition of noncompactness measures on G-metric spaces with a Ω-distance. The notions of diameter and Kuratowski measure of noncompactness are then generalised in this setting, demonstrating that they are indeed Ω-measures of noncompactness. In addition, we present a new characterization of G-metric completeness.

Definition 2.1. Let (X, G) be a G-metric space with an Ω-distance K, let T be its subset. Then K-diameter of set T, denoted by $\delta_K(T)$, is defined by

$$\delta_K(T) = \sup_{x,y,z \in T} K(x, y, z) \tag{i}$$

If the above supremum exists and is finite, we say that set T is K-bounded otherwise T is K-unbounded.

Definition 2.2. Let (X, G) be a G-metric space with an Ω-distance K. The function $\phi : P(x) \to [0, \infty)$ (where P(x) is the partitive set of X) is said to be Ω-measure of non-compactness if it satisfies the following conditions:

1. $\phi(T) = \infty$ if and only if T is K-unbounded;
2. $\phi(T) = \phi(\bar{T})$;
3. if $\phi(T) = 0$, then T is totally bounded;
4. if $T \subseteq R$, then $\phi(T) \leq \phi(R)$;
5. if X is complete and $\{S_n\}$ is a sequence of non-empty compact subset of X such that $S_{n+1} \subseteq S_n$ for all $n \in \mathbb{N}$ and $\lim\limits_{n\to\infty} \phi(S_n) = 0$, then $\bigcap\limits_{n=1}^{\infty} (S_n)$ is a non-empty compact subset of X.

Now, we introduce the first Ω-measure of noncompactness.

Theorem 2.3. *If (X, G) is a G-metric space with an Ω-distance K, then $\delta_K(T)$ is an Ω-measure of non compactness on X.*

Proof. It suffices to show that the function δ_K satisfies all the conditions in Definition (2.2).

1. This is obvious from Definition (2.1).
2. Since $T \subseteq \bar{T}$, Definition (2.1) implies that $\delta_K(T) \leq \delta_K(\bar{T})$, the converse inequality must be proved. Assume that $\delta_K(T) < \delta_K(\bar{T})$. Then there exists $x, y, z \in T$ such that $\delta(T) < K(x, y, z)$. Since $x, y, z \in \bar{T}$, there are three sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ in T that converge to x, y and z respectively. We get by using $K's$ Lower-semicontinuity with respect to both variables:
$\delta_K(T) < K(x, y, z) \leq \lim\limits_{n\to\infty} \inf K(x_n, y, z) \leq \lim\limits_{n\to\infty} \inf \lim\limits_{n\to\infty} \inf K(x_n, y_m, z_l) \leq \delta_K(T)$
whhich is a contradiction.
3. Assume that T is a subset of X, with $\delta_K(T) = 0$. According to the Definition (2.1), $K(x, y, z) = 0$ for all $x, y, z \in T$. If T has four distinct points x, y, z and a, then we have $K(x, y, z) = 0$ and $K(a, y, z) = 0$. By Definition (1.11) $K(x, y, z) \leq K(x, a, a) + K(a, y, z)$ i.e $K(x, a, a) = 0$. So $K(x, y, z) = 0$ and $K(a, y, z) = 0$, by Defintion (1.5) $x = y = z$, a contradiction. Hence T is either an empty set or a singleton. In both cases, T is totally bounded.
4. Evident again by definition (2.1).
5. Let $\{S_n\}$ be a decreasing sequence of subsets and S be the intersection. We have $\delta_K(S) \leq \delta_K(S_n) \to 0$ by (4) because $S \subseteq S_n$ for all $n \in \mathbb{N}$. As a result, $\delta_K(S) = 0$, implying that S is either empty or a singleton set according to (3). Let us choose a point $x_n \in S$ for all $n \in \mathbb{N}$. We will show that the sequence $\{x_n\}$ converges to a point $x \in S$, implying that $S = \{x\}$. Indeed, for all $m, n, l \in \mathbb{N}$ with $m, n, l > N_0 (N_0 \in \mathbb{N}), K(x_n, x_m, x_l) \leq \delta_K(S_n) \to 0$ as $n \to \infty$ which by Definition (1.7) implies that $\{x_n\}$ is a cauchy sequence in X, so it converges to a point $x \in X$ because X is complete. It is easy to see that $x \in S_n$ for all $n \in \mathbb{N}$ and therefore $x \in S$. Hence $S = \{x\}$, which is non-empty and compact.

□

Now, we introduce the Kuratowski K-measure of non-compactness.

Definition 2.4. let (X, G) be a G-metric space with an Ω-distance K, and let T be its subset. Then we define Kuratowski K-measure of non-compactness of set T, denoted by $\alpha_K(\mathrm{T})$, as the infimum of all $\epsilon > 0$ such that T can be covered by a finite number of subsets in X whose K-diameter is less tha ϵ.

The following properties are the consiquences of Definition (2.4).

Lemma 2.5. *Let T, T_1 and T_2 be bounded subsets of a complete G-metric space (X, d) with an Ω-distance. Then*

1. $\alpha_K(T) = 0$ *if and only if* $\bar{T}$ *is compact, (regularity),*
2. $\alpha_K(T) = \alpha_K(\bar{T})$, *(invariance under the passage of closure),*
3. $T_1 \subset T_2$, *implies* $\alpha_K(T_1) \leq \alpha_K(T_2)$, *(monotonicity),*
4. $\alpha_K(T \cup T_1 \cup T_2) = \max\{\alpha_K(T), \alpha_K(T_1), \alpha_K(T_2)\}$,*(maximum property),*
5. $\alpha_K(T \cap T_1 \cap T_2) \leq \min\{\alpha_K(T), \alpha_K(T_1), \alpha_K(T_2)\}$.

Proof. (1) and (3) are follows from Definition (2.4). Since $T \subseteq \bar{T}$, it is evidently $\alpha_K(T) \leq \alpha_K(\bar{T})$. Let $\epsilon > 0$, Z_i be a bounded subset of X with $\delta_K(Z_i) < \epsilon$ for $i = 1, 2, 3, ..., n$, and $T \subset \cup_{i=1}^{n} Z_i$. Then $\bar{T} \subset \overline{\cup_{i=1}^{n} Z_i} = \cup_{i=1}^{n} \bar{Z}_i$. Since $\delta_K(Z_i) = \delta_K(\bar{Z}_i)$, we conclude that $\alpha_K(T) \leq \alpha_K(\bar{T})$. This proves (2). From (3), we have $\alpha_K(T) \leq \alpha_K(T \cup T_1 \cup T_2)$ $\alpha_K(T_1) \leq \alpha_K(T \cup T_1 \cup T_2)$ and $\alpha_K(T_2) \leq \alpha_K(T \cup T_1 \cup T_2)$, and so

$$\max\{\alpha_K(T), \alpha_K(T_1), \alpha_K(T_2)\} \leq \alpha_K(T \cup T_1 \cup T_2). \tag{ii}$$

Let max $\{\alpha_K(T), \alpha_K(T_1), \alpha_K(T_2)\} = z$ and $\epsilon > 0$. By Definition (2.4) we know that T, T_1 and T_2 can be covered by finite number of subsets of diameter smaller than $z + \epsilon$. Obviously, the union of these covers is a finite cover of $T \cup T_1 \cup T_2$. Hence, we have $\alpha_K(T_1 \cup T_2) < z + \epsilon$, and now we obtain (4) from (ii). From $T \cap T_1 \cap T_2 \subset T_1$, $T \cap T_1 \cap T_2 \subset T_1$ and $T \cap T_1 \cap T_2 \subset T_2$ we obtain $\alpha_K(T \cap T_1 \cap T_2) \leq \alpha_K(T)$, $\alpha_K(T \cap T_1 \cap T_2) \leq \alpha_K(T_1)$ and $\alpha_K(T \cap T_1 \cap T_2) \leq \alpha_K(T_2)$. Hence $\alpha_K(T \cap T_1 \cap T_2) \leq \min\{\alpha_k(T), \alpha_K(T_1), \alpha_K(T_2)\}$. This proves inequality (5) and hence the proof is completed. □

Next theorem is the generalization of Theorem (1.4) on G-metric spaces.

Theorem 2.6. *Let (X, d) be a complete G-metric space, and let $\{S_n\}$ be a sequence of nonempty closed and bounded subsets of X such that $S_{n+1} \subseteq S_n$ for all $n \in N$ and $\lim_{n\to\infty} \alpha_K(S_n) = 0$. Then, $\bigcap_{n=1}^{\infty} S_n$ is a nonempty and compact subset of X.*

Proof. Let $S_\infty = \cap_{n=1}^{\infty} S_n$ be subset of X. Clearly S_∞ is a closed subset of X. Since $S_\infty \subset S_n$ for all $n = 1, 2, ...$, we get from (1) and (3) of Lemma(2.5) that S_∞ is a compact set. Now we show S_∞ is nonempty. Let $x_n \in S_n (n = 1, 2, 3, ...)$ and $X_n = \{x_i : i \geq n\}$ for $n = 1, 2, 3, ...$ Since $X_n \subset S_n$, we obtain from (1), (3), and (4) of above Lemma(2.5) that

$$\alpha_K(X_1) = \alpha_K(X_n) \leq \alpha_K(S_n) \tag{iii}$$

for each n. The assumption of our theorem and (iii) together $\alpha_K(X_1) = 0$, hence X_1 is a relatively compact set. Thus the sequence (x_n) has a convergent subsequence (x_{kn}) with $x = \lim x_{kn} \in X$, say. Since (S_n) is closed in X, we get $x \in (S_n)$ for all $n = 1, 2, ...$, that is, $x \in S_\infty$. This completes the proof. □

Theorem 2.7. *If (X, G) is a G-metric space with an Ω-distance K, then $\alpha_K(T)$ is a Ω-measure of non-compactness.*

Proof. Again we shall show that α_K satisfies all conditions of definition 2.2.

1. This trivially follows from Definition (2.4).
2. It follows from (2) of Lemma (2.5).
3. Let Q be a subset of X such that $\delta_K(T) = 0$. Fix an arbitrary $\epsilon > 0$ and choose $\delta = \delta(\frac{\epsilon}{2})$ in definition. Let T_j be any element of the covering of Q corresponding to δ in definition, so $\delta_K(T_j) < \delta$. It follows then for every $x, y, z, a \in T_j$ we have $K(x, a, a) < \delta, K(a, y, z) < \delta$. This inturn implies $G(x, y, z) \leq \frac{\epsilon}{2} < \epsilon$ by Definition (1.11), which means that $\delta(T_j) < \epsilon$ i.e, diameter of every component in the covering of T is less than ϵ. Since $\epsilon > 0$ was arbitrary, we conclude that set T is totally bounded (see[20] Definition 11).

4. Obvious by definition (2.4).
5. Let $\{S_n\}$ be the sequence as defined in (5) of Definition (2.2) and let S be its intersection. Then S is closed because every S_n is closed. Since $S \subseteq S_n$ for every $n \in \mathbb{N}$, we have $\alpha_K(S) \leq \alpha_K(S_n) \to 0$ when $n \to \infty$. So, $\alpha_K(S) = 0$, which implies that S is totally bounded set. But S is complete, so S is actually relatively compact. It is also closed, which means that compact. That S is nonempty can be shown same way as in the proof Theorem (2.6).

□

3. Characterizations of G-metric completeness

Cantor's intersection theorem is well recognised for defining metric completeness. Similar results in the setting of so-called partial metric space have recently been reported (see [23]).

The topic of describing metric completeness using non-compactness measures has long been open (see[2]). Suzuki and Takahashi [24] are also notable for defining metric completeness using a generalised Banach's fixed point theorem on metric spaces with a ω-distance. The metric completeness was defined by Aleksadar Kostic[13] using the ω measure of non-compactness. As a result of these findings, we wonder if G-metric completeness can be described using the Ω-measure of non-compactness proposed in this study. In the next theorem, we will give a positive answer.

Theorem 3.1. *Let (X, G) be a G-metric space with an Ω-distance K. Then following conditions are equivalent:*

1. *X is complete.*
2. *Every sequence (S_n) of non-empty closed subsets in X such that $S_{n+1} \subseteq S_n$ for all $n \in \mathbb{N}$ and $\lim \delta_K(S_n) = 0$ has a singleton intersection.*
3. *Every sequence $\{S_n\}$ of non-empty closed subsets in X such that $S_{n+1} \subseteq S_n$ for all $n \in \mathbb{N}$ and $lim\delta_K(S_n) = 0$ has a compact intersection.*

Proof. The following chain of implications will be demonstrated.

(1) $\Longrightarrow$ (3):
Part (5) of the Theorem (2.7) demonstrate this.

(3) $\Longrightarrow$ (2):
Assume that the sequence $\{S_n\}$ fulfills the requirements in (2). Since by Definition (2.1), $\alpha_K(S_n) \leq \delta_K(S_n)$ for all $n \in \mathbb{N}$, it is clear that the sequence S_n also satisfies the conditions under (3), making its intersection S is a non-empty and compact subset of X. Also $\delta_K(S) = 0$, which can only happen if S is singleton.

(2) $\Longrightarrow$ (1):
Assume that (2) is true and X is not complete. Then in X , there is a cauchy sequence $\{x_n\}$ that is not convergent. The set S=$\{x_n : n \in \mathbb{N}\}$ is simply demonstrated to be bounded and closed. Suppose X is a G-metric space.

Define the mapping $K : X \times X \times X \to \mathbb{R}^+$ by

$$K(x, y, z) = \max\{d(x, y), d(x, z), d(y, z)\} \tag{iv}$$

Then, by above example, K ia an Ω-distance on X. Let $S_n = \{x_k : k \geq n\}$ for all $n \in \mathbb{N}$. Then we have that every S_n is a non-empty closed subset of X such that $S_{n+1} \subseteq S_n$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} \delta_K(S_n) = \lim_{n\to\infty} \sup_{n,m\geq l} K(x_n, x_m, x_l)$

$$=\lim_{n\to\infty}\sup_{n,m\geq l}\max\{d(x_n,x_m),\, d(x_m,x_l),d(x_n,x_l)\}=0,$$

since $\{x_n\}$ is Cauchy. But then by (2), we obtain that $\bigcap_{n=1}^{\infty} S_n = \{x\}$, which is impossible because $\{x_n\}$ would then converge to x.

□

Conflict of Interest

The authors have no conflict of interest regarding the publication of this article.

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