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Fixed Points of Nešić Type Contraction Maps in Convex Metric Spaces

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Abstract

We define Nešić type contraction maps in convex metric spaces and prove the existence and uniqueness of fixed points of these maps in convex metric spaces. Our results extend the results of Nešić ([1], Results on fixed points of asymptotically regular mappings) from the metric space setting to convex metric spaces.

Keywords: Convex structure, Nešić contraction map, Nešić type contraction map.
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1. Introduction

In 1970, Takahashi [2] introduced the concept of convex structure in metric spaces and named the metric space together with convex structure as convex metric space and studied the existence of fixed points of nonexpansive maps in convex metric spaces.

Definition 1.1. (Takahashi [2]) Let (X, d) be a metric space. Let $W : X \times X \times [0, 1] \to X$. If for all $x, y \in X$ and for any $\lambda \in [0, 1]$,

$$d(u, W(x, y, \lambda)) \leq \lambda d(u, x) + (1 - \lambda)d(u, y) \tag{1.1}$$

for any $u \in X$, then we say that W is a convex structure on X.

A metric space (X, d) endowed with a convex structure W is called a convex metric space and we denote by (X, d, W). We observe that any normed linear space is a convex metric space, with convex structure

$$W(x, y, \lambda) = \lambda x + (1 - \lambda)y \tag{1.2}$$

$x, y \in X$ and $\lambda \in [0, 1]$.

Properties of the convex metric space are given in the following lemma.

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Lemma 1.2. (*Berinde and Păcurar* [4]) *Let* (X, d, W) *be a convex metric space. Then we have the following.*
i) $d(x, y) = d(x, W(x, y, \lambda)) + d(W(x, y, \lambda), y)$
ii) $d(x, W(x, y, \lambda)) = (1 - \lambda)d(x, y); d(y, W(x, y, \lambda)) = \lambda d(x, y)$
iii) $W(x, x, \lambda) = x;\quad W(x, y, 0) = y \quad$ *and* $\quad W(x, y, 1) = x$; *and*
iv) $|\lambda_1 - \lambda_2| d(x, y) \leq d(W(x, y, \lambda_1), W(x, y, \lambda_2))$
for all $x, y \in X$ *and* $\lambda, \lambda_1, \lambda_2 \in I = [0, 1]$.

Nešić [1] introduced the following contraction.

Definition 1.3. (Nešić [1]) Let (X, d) be a metric space and $T : X \to X$. If there exist nonnegative reals a, b, c such that

$$d(Tx, Ty) \leq ad(x, y) + b[d(x, Tx) + d(y, Ty)] + c[d(x, Ty) + d(y, Tx)] + F(d(x, Tx)d(y, Ty)) \tag{1.3}$$

for all $x, y \in X$, where $0 \leq a, c, a + 2c < 1, b + c < 1$ then we say that T is a Nešić contraction map.

Theorem 1.4. (*Nešić* [1]) *Let* (X, d) *be a complete metric space and* $T : X \to X$, *a Nešić contraction map. If* T *is asymptotically regular at some point of* X, *then* T *has a unique fixed point in* X.

Theorem 1.5. (*Nešić* [1]) *Let* (X, d) *be a metric space and* $T : X \to X$, *a Nešić contraction map. If* T *is asymptotically regular at a point* x *in* X *and the sequence of iterates* $\{T^n x\}$ *has a subsequence converging to a point* z *in* X, *then* z *is the unique fixed point of* T *and* $\{x_n\}$ *also converges to* z.

Motivated by the works of Nešić [1], in Section 2 of this paper, we define Nešić type contraction map in convex metric space and prove the existence and uniqueness of fixed points in complete convex metric space. Further, we extend Theorem 1.5 to the case of convex metric spaces. We use the following lemma in our discussion.

Lemma 1.6. (*Babu and Sailaja* [3]) *Suppose* (X, d) *is a metric space. Let* $\{x_n\}$ *be a sequence in* X *such that* $d(x_n, x_{n+1}) \to 0$ *as* $n \to \infty$. *If* $\{x_n\}$ *is not a Cauchy sequence then there exist* $\epsilon > 0$ *and sequences of positive integers* $\{m_k\}$ *and* $\{n_k\}$ *with* $m_k > n_k > k$ *such that* $d(x_{m_k}, x_{n_k}) \geq \epsilon$, $d(x_{m_k - 1}, x_{n_k}) < \epsilon$ *and*
i) $\lim_{k\to\infty} d(x_{m_k}, x_{n_k}) = \epsilon$ *ii)* $\lim_{k\to\infty} d(x_{m_k - 1}, x_{n_k}) = \epsilon$
iii) $\lim_{k\to\infty} d(x_{m_k - 1}, x_{n_k + 1}) = \epsilon$ *iv)* $\lim_{k\to\infty} d(x_{m_k - 1}, x_{n_k - 1}) = \epsilon$.

2. Main Results

Definition 2.1. Let (X, d) be a metric space, $T : X \to X$. Let $F : \mathbb{R}^+ \to \mathbb{R}^+$ be a map such that $F(t) < t$ for $t > 0$ and $F(0) = 0$. If there exist nonnegative reals a, b, c and $\lambda \in [0, 1)$ such that

$$d(Tx, Ty) \leq ad(x, y) + b[d(x, Tx) + d(y, Ty)] + c[d(x, Ty) + d(y, Tx)] + F(d(x, Tx)) \tag{2.1}$$

for all $x, y \in X$, where

$$b + |c| \leq a + 2b + c + |c| + (1 - a)\lambda < 0 \tag{2.2}$$

then we say that T is a Nešić type contraction map.

Theorem 2.2. *Let* (X, d, W) *be a complete convex metric space and* $T : X \to X$ *be a Nešić type contraction map. If* $a + 2c < 1$, *then* T *has a unique fixed point in* X.

Proof. Let $x_0 \in X$ be arbitrary. We define a sequence $\{x_n\}_{n=0}^{\infty}$ by

$$x_{n+1} = W(x_n, Tx_n, \lambda) \tag{2.3}$$

for $n = 0, 1, 2...$ where $\lambda \in [0, 1)$.
Without loss of generality, we assume that $x_{n+1} \neq x_n$ for all $n \geq 0$.

From (ii) of Lemma 1.2, we have
$d(x_n, x_{n+1}) = d(x_n, W(x_n, Tx_n, \lambda)) = (1-\lambda)d(x_n, Tx_n)$.
Therefore

$$d(x_n, Tx_n) = \frac{1}{1-\lambda} d(x_n, x_{n+1}) \text{ for } n = 0, 1, 2, \dots \,. \tag{2.4}$$

Now
$d(x_n, Tx_{n-1}) = d(Tx_{n-1}, W(x_{n-1}, Tx_{n-1}), \lambda) = \lambda d(x_{n-1}, Tx_{n-1})$
$= \frac{\lambda}{1-\lambda} d(x_{n-1}, x_n)$, by (2.4) so that

$$d(x_n, Tx_{n-1}) = \frac{\lambda}{1-\lambda} d(x_{n-1}, x_n). \tag{2.5}$$

By using the triangle inequality, we have
$d(x_{n-1}, Tx_n) \le d(x_{n-1}, x_n) + d(x_n, Tx_n) = d(x_{n-1}, x_n) + \frac{1}{1-\lambda} d(x_n, x_{n+1})$.
If $c \ge 0$, then

$$cd(x_{n-1}, Tx_n) \le cd(x_{n-1}, x_n) + \frac{c}{1-\lambda} d(x_n, x_{n+1}). \tag{2.6}$$

Again by applying the triangle inequality, it follows that
$d(x_{n-1}, Tx_n) \ge d(x_{n-1}, x_n) - d(x_n, Tx_n)$
$= d(x_{n-1}, x_n) - \frac{1}{1-\lambda} d(x_n, x_{n+1})$.
If $c < 0$, then

$$cd(x_{n-1}, Tx_n) \le cd(x_{n-1}, x_n) - \frac{c}{1-\lambda} d(x_n, x_{n+1}). \tag{2.7}$$

From (2.6) and (2.7), we get

$$cd(x_{n-1}, Tx_n) \le cd(x_{n-1}, x_n) + \frac{|c|}{1-\lambda} d(x_n, x_{n+1}) \tag{2.8}$$

for $n = 1, 2, 3 \dots$. Put $x = x_n$ and $y = x_{n-1}$ in (2.1), we get

$$d(Tx_n, Tx_{n-1}) \le ad(x_n, x_{n-1}) + b[d(x_n, Tx_n) + d(x_{n-1}, Tx_{n-1})] + c[d(x_n, Tx_{n-1}) + d(x_{n-1}, Tx_n)] + F(d(x_n, Tx_n)) \tag{2.9}$$

for $n = 1, 2, \dots$. Now
$d(Tx_n, x_n) - d(x_n, Tx_{n-1}) \le d(Tx_n, Tx_{n-1})$
$\le ad(x_n, x_{n-1}) + b[d(x_n, Tx_n) + d(x_{n-1}, Tx_{n-1})] +$
$c[d(x_n, Tx_{n-1}) + d(x_{n-1}, Tx_n)] + F(d(x_n, Tx_n))$. (By (2.9))
From (2.4), (2.5) and (2.8) we have
$\frac{1}{1-\lambda} d(x_n, x_{n+1}) - \frac{\lambda}{1-\lambda} d(x_n, x_{n-1}) = d(Tx_n, x_n) - d(x_n, Tx_{n-1})$
$\le d(Tx_n, Tx_{n-1})$
$\le ad(x_n, x_{n-1}) + \frac{b}{1-\lambda} d(x_{n+1}, x_n) + \frac{b}{1-\lambda} d(x_n, x_{n-1}) + \frac{c\lambda}{1-\lambda} d(x_n, x_{n-1})$
$+ cd(x_n, x_{n-1}) + \frac{|c|}{1-\lambda} d(x_n, x_{n+1}) + F(\frac{1}{1-\lambda} d(x_n, x_{n+1}))$,
and hence
$\frac{1-b-|c|}{1-\lambda} d(x_n, x_{n+1}) \le \frac{a+b+c+(1-a)\lambda}{1-\lambda} d(x_n, x_{n-1}) + F(\frac{1}{1-\lambda} d(x_n, x_{n+1}))$. Therefore
$d(x_n, x_{n+1}) \le \frac{a+b+c+(1-a)\lambda}{1-b-|c|} d(x_n, x_{n-1}) + \frac{1-\lambda}{1-b-|c|} \frac{1}{1-\lambda} d(x_n, x_{n+1})$
$= \frac{a+b+c+(1-a)\lambda}{1-b-|c|} d(x_n, x_{n-1}) + \frac{1}{1-b-|c|} d(x_n, x_{n+1})$.
Suppose $\limsup d(x_n, x_{n+1}) = r(\text{say}),\ r > 0$.
Now on taking limit superior, we have
$\limsup d(x_n, x_{n+1}) \le \frac{a+b+c+(1-a)\lambda}{1-b-|c|} \limsup d(x_n, x_{n-1}) + \frac{1}{1-b-|c|} \limsup d(x_n, x_{n+1})$, and hence
$r \le \frac{a+b+c+(1-a)\lambda}{1-b-|c|} r + \frac{r}{1-b-|c|}$

$$r \le \frac{a+b+c+(1-a)\lambda+1}{1-b-|c|} r. \tag{2.10}$$

Since $a+2b+c+|c|+(1-a)\lambda<0$, we have $a+b+c+(1-a)\lambda<-b-|c|$, and hence $1+a+b+c+(1-a)\lambda<1-b-|c|$ so that $\frac{1+a+b+c+(1-a)\lambda}{1-b-|c|}<1$.
Therefore, from (2.10), we have, $r\leq\theta r<r$, where $\theta=\frac{1+a+b+c+(1-a)\lambda}{1-b-|c|}$, a contradiction.
Hence $r=0$. Therefore $\limsup d(x_n,x_{n+1})=0$.
Therefore

$$\lim_{n\to\infty} d(x_n,x_{n+1})=0. \tag{2.11}$$

Now we prove that $\{x_n\}$ is Cauchy.
If $\{x_n\}$ is not Cauchy then there exist $\epsilon>0$ and sequence of positive integers $\{m_k\}$ and $\{n_k\}$ with $n_k>m_k>k$ such that

$$d(x_{n_k},x_{m_k})\geq\epsilon \text{ and } d(x_{n_k-1},d_{m_k})<\epsilon \tag{2.12}$$

and (i) to (iv) of Lemma 1.6 hold.
Then from (2.12), we have

$$\begin{aligned}
\epsilon &\leq d(x_{n_k},x_{m_k})\leq d(x_{n_k},x_{n_k-1})+d(x_{n_k-1},x_{m_k})\\
&= d(x_{n_k},x_{n_k-1})+d(x_{n_k-1},W(x_{m_k-1},Tx_{m_k-1},\lambda))\\
&\leq d(x_{n_k},x_{n_k-1})+\lambda d(x_{n_k-1},x_{m_k-1})+(1-\lambda)d(x_{n_k-1},Tx_{m_k-1})\\
&\leq d(x_{n_k},x_{n_k-1})+\lambda d(x_{n_k-1},x_{m_k-1})+(1-\lambda)[d(x_{n_k-1},Tx_{n_k-1})+d(Tx_{n_k-1},Tx_{m_k-1})]\\
&\leq d(x_{n_k},x_{n_k-1})+\lambda d(x_{n_k-1},x_{m_k-1})+(1-\lambda)[d(x_{n_k-1},Tx_{n_k-1})+ad(x_{n_k-1},x_{m_k-1})+b(d(x_{n_k-1},Tx_{n_k-1})\\
&\qquad +d(x_{m_k-1},Tx_{m_k-1}))+c(d(x_{n_k-1},Tx_{m_k-1})+d(x_{m_k-1},Tx_{n_k-1}))+F(d(x_{n_k-1},Tx_{n_k-1}))].
\end{aligned}$$

On letting $k\to\infty$ and by (iv) of Lemma 1.6 we get
$\epsilon\leq\limsup_{n\to\infty} d(x_{n_k},x_{m_k})\leq\lambda\epsilon+(1-\lambda)(a+2c)\epsilon<\epsilon$,
a contradiction.
Hence $\{x_n\}$ is a Cauchy sequence.
Since X is complete, there exists x^* in X such that $\lim\limits_{n\to\infty} x_n=x^*$.
On letting $n\to\infty$ in (2.4) and (2.5), we get $\lim\limits_{n\to\infty} Tx_n=x^*$.
Put $x=x^*$ and $y=x_n$ in (2.1), we get
$d(Tx^*,Tx_n)\leq ad(x^*,x_n)+b[d(x^*,Tx^*)+d(x_n,Tx_n)]+c[d(x^*,Tx_n)+d(x_n,Tx^*)]+F(d(x^*,Tx^*))$.
On letting $n\to\infty$, we get

$$\begin{aligned}
d(Tx^*,x^*) &\leq ad(x^*,x^*)+b[d(x^*,Tx^*)+d(x^*,x^*)]+c[d(x^*,x^*)+d(x^*,Tx^*)]+F(d(x^*,Tx^*))\\
d(Tx^*,x^*) &\leq bd(x^*,Tx^*)+cd(x^*,Tx^*)+d(x^*,Tx^*)\\
&=(b+c+1)d(x^*,Tx^*)\\
&<d(x^*,Tx^*)
\end{aligned}$$

a contracdiction. Therefore $Tx^*=x^*$.
Hence T has a fixed point in X.
Now we prove that T has a unique fixed point in X.
Suppose that there exists $x_1,x_2\in X$ and $Tx_1=x_1$ and $Tx_2=x_2$.
Put $x=x_1$ and $y=x_2$ in (2.1), we get
$d(Tx_1,Tx_2)\leq ad(x_1,x_2)+b[d(x_1,Tx_1)+d(x_2,Tx_2)]+c[d(x_1,Tx_2)+d(x_2,Tx_1)]+F(d(x_1,Tx_1))$.
i.e., $d(x_1,x_2)\leq ad(x_1,x_2)+b[d(x_1,x_1)+d(x_2,x_2)]+c[d(x_1,x_2)+d(x_2,x_1)]+F(d(x_1,x_1))$,
so that $d(x_1,x_2)\leq(a+2c)d(x_1,x_2)<d(x_1,x_2)$
a contradiction, since $a+2c<1$.
Therefore $d(x_1,x_2)=0$ so that $x_1=x_2$, and T has a unique fixed point in X. □

Theorem 2.3. *Let (X,d,W) be a convex metric space and $T:X\to X$ be a Nešić type contraction map. Let $x_0\in X$. We define the sequence $\{x_n\}$ in X by $x_{n+1}=W(x_n,Tx_n,\lambda)$ for $n=0,1,2...$. If $\{x_{n_k}\}$ is a subsequence of the sequence $\{x_n\}$ such that $\{x_{n_k}\}$ converges to u then u is a fixed point of T. Further, if $a+2c<1$ then this fixed point is unique and the sequence $\{x_n\}$ also converges to u.*

Proof. Let $x_0\in X$ be arbitrary. We consider the sequence $\{x_n\}_{n=0}^{\infty}$ defined by

$$x_{n+1}=W(x_n,Tx_n,\lambda) \tag{2.13}$$

for $n = 0, 1, 2...$, where $\lambda \in [0, 1)$.
Let $\{x_{n_k}\}$ be a convergent subsequence of $\{x_n\}$ which converges to u in X.
We now show that u is a fixed point of T.
Now consider

$$\begin{aligned}
d(u,Tu) &\leq d(u,x_{n_k}) + d(x_{n_k},x_{n_k+1}) + d(x_{n_k+1},Tu)\\
&= d(u,x_{n_k}) + d(x_{n_k},x_{n_k+1}) + d(W(x_{n_k},Tx_{n_k},\lambda),Tu)\\
&\leq d(u,x_{n_k}) + d(x_{n_k},x_{n_k+1}) + \lambda d(x_{n_k},Tu) + (1-\lambda)d(Tx_{n_k},Tu)\\
&\leq d(u,x_{n_k}) + d(x_{n_k},x_{n_k+1}) + \lambda d(x_{n_k},Tu) + (1-\lambda)[ad(x_{n_k},u) + b(d(x_{n_k},Tx_{n_k})+\\
&\qquad d(u,Tu)) + c(d(x_{n_k},Tu) + d(u,Tx_{n_k})) + F(d(x_{n_k},Tx_{n_k}))]\\
&\leq d(u,x_{n_k}) + d(x_{n_k},x_{n_k+1}) + \lambda d(x_{n_k},Tu) + (1-\lambda)[ad(x_{n_k},u) + \tfrac{b}{1-\lambda}d(x_{n_k},x_{n_k+1})+\\
&\qquad bd(u,Tu) + cd(x_{n_k},Tu) + c(d(u,x_{n_k}) + d(x_{n_k},Tx_{n_k})) + F(\tfrac{1}{1-\lambda}d(x_{n_k},x_{n_k+1}))]\\
&< d(u,x_{n_k}) + d(x_{n_k},x_{n_k+1}) + \lambda d(x_{n_k},Tu) + (1-\lambda)[ad(x_{n_k},u) + \tfrac{b}{1-\lambda}d(x_{n_k},x_{n_k+1})+\\
&\qquad bd(u,Tu) + cd(x_{n_k},Tu) + cd(u,x_{n_k}) + \tfrac{c}{1-\lambda}d(x_{n_k},x_{n_k+1}) + \tfrac{1}{1-\lambda}d(x_{n_k},x_{n_k+1})].
\end{aligned}$$

On letting $k \to \infty$, we get

$$\begin{aligned}
d(u,Tu) &\leq d(u,u) + \lambda d(u,Tu) + (1-\lambda)[ad(u,u) + bd(u,Tu) + cd(u,Tu) + cd(u,u)]\\
&= \lambda d(u,Tu) + (1-\lambda)[(b+c)d(u,Tu)]\\
d(u,Tu) &\leq [\lambda + (1-\lambda)(b+c)]d(u,Tu) \qquad (\text{since } \lambda + (1-\lambda)(b+c) < 1)
\end{aligned}$$

a contradiction.
Therefore $Tu = u$.
Now, if $a + 2c < 1$ then uniqueness of u follows as in the proof of Theorem 2.2.
We now prove that $\lim_{n\to\infty} x_n = u$.
Since T is Nešić type contraction map, from (2.11) we have $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$.
We now consider

$$\begin{aligned}
d(u,x_n) &= d(Tu,x_n)\\
&\leq d(Tu,Tx_n) + d(Tx_n,x_n)\\
&\leq ad(u,x_n) + b[d(u,Tu) + d(x_n,Tx_n)] + c[d(x_n,Tu) + d(u,Tx_n)] + F(d(u,Tu)) + d(Tx_n,x_n)\\
&\leq ad(u,x_n) + b[d(u,Tu) + d(x_n,Tx_n)] + c[d(x_n,u) + d(u,Tu) + d(u,x_n) + d(x_n,Tx_n)]+\\
&\qquad F(d(u,Tu)) + d(Tx_n,x_n)\\
&= ad(u,x_n) + b[d(u,u) + d(x_n,Tx_n)] + c[d(x_n,u) + d(u,u) + d(u,x_n) + d(x_n,Tx_n)]+\\
&\qquad F(d(u,u)) + d(Tx_n,x_n)\\
&= (a+2c)d(u,x_n) + (b+c+1)d(x_n,Tx_n)\\
(1-a-2c)d(u,x_n) &\leq (b+c+1)\tfrac{1}{1-\lambda}d(x_n,x_{n+1})\\
d(u,x_n) &= \tfrac{b+c+1}{1-a-2c}\tfrac{1}{1-\lambda}d(x_n,x_{n+1}) \longrightarrow 0 \text{ as } n \to \infty.
\end{aligned}$$

Therefore the sequence $\{x_n\}$ converges to the fixed point u in X. □

Remark 2.4. In the metric space setting, Nešić assumed that T is asymptotically regular in Theorem 1.4 and Theorem 1.5, where as in the convex metric space setting, in the corresponding theorems (Theorem 2.2 and Theorem 2.3) we did not assume the asymptotically regularity of T, infact it follows from the Nešić type contraction map.

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