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Generalized Monotone Method for System of Riemann-Liouville Fractional Reaction Diffusion Equation

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Abstract

In this paper, our aim is to obtain the integral representation for the solution of linear Riemann-Liouville (R-L) reaction diffusion equation of order q, where $0 < q < 1$, in term of Green's function. We have developed a generalized monotone method for non-linear weakly coupled system of R-L reaction diffusion equation when the forcing term is the sum of increasing and decreasing functions. The generalized monotone method yields monotone sequences which converges uniformly and monotonically to coupled minimal and maximal solutions. Under uniqueness assumption, we prove the existence of a unique solution for the non-linear system of R-L reaction diffusion equation.

Keywords: Eigenfunction, Non-linear weakly coupled system, Coupled upper and lower solutions.
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1. Introduction

Computation of explicit solution of non-linear dynamic equation is rarely possible. It is more so with non-linear fractional dynamic equations with initial and boundary conditions. In general, the existence and uniqueness of solution of the fractional dynamic equation has been established mostly, using some kind of fixed point approach. See [2, 4, 5, 9, 11, 19, 20, 23] and the reference therein for the existence, uniqueness and applications of fractional dynamic equations. The method of upper and lower solutions combined with the monotone iterative technique not only guarantess the interval of existence but also the method is both theoretical and computational. See [6, 7, 8, 24] for the monotonic method and generalized monotone method for non-linear dynamic equations. In this case , we obtain a sequence of approximate solutions which

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are either monotonically increasing or monotonically decreasing if the approximation is the lower solution or upper solution respectively. However, from practical application problems, the non-linear forcing term will be a sum of increasing and decreasing function as in the population models and chemical combustion models, see [16]. In order to handle such problems, a generalized monotone method has been developed in [12, 13, 14, 15, 22].

In this work, we consider the non-linear system of R-L reaction diffusion equation where the forcing function is the sum of increasing and decreasing functions. We develop a generalized monotone method for the non-linear weakly coupled system of R-L reaction diffusion equation using coupled lower and upper solutions. Initially, we have obtained a representation form for the solution of linear weakly coupled system of R-L reaction diffusion equation using the eigen function expansion method and Green's identity. We have also developed the maximum principle and comparison results. These results are used to prove the sequences developed in the generalized monotone method converge to the coupled minimal and maximal solutions of the non-linear system of fractional diffusion equations. The convergence of the sequences is monotonic and uniform in the weighted norm.

The rest of paper is arranged in the following way.

In section 2, definitions and basic results are discussed that plays vital role in the main results. In section 3, comparison results. These results are useful in main results proving that the sequences developed in the generalized monotone method converge to coupled minimal and maximal solutions of the non-linear system of fractional reactions diffusion equation. Finally under uniqueness assumption, we prove that there exists a unique solution to the non-linear system of R-L reaction diffusion equation.

2. Preliminaries

In this section , we recall some known definitions and known results which are useful to develop our main result. Here and throughout, the notation $\Gamma(q)$ denotes the gamma function of order q.

Definition 2.1 The R-L fractional integral of $u(t)$ of order q is defined by

$$D_t^{-q}u(t) = \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}u(s)ds \tag{2.1}$$

where $0 < q \leq 1$

Definition 2.2 The R-L (left-sided) fractional derivative of $u(t)$ of order q when $0 < q \leq 1$, is defined as

$$D_t^q u(t) = \frac{1}{\Gamma(1-q)}\frac{d}{dt}\int_0^t (t-s)^{q-1}u(s)ds, t > 0 \tag{2.2}$$

Definition 2.3 The two parameter Mittag-liffler function is defined as

$$E_{q,r}(\lambda t^q) = \sum_{k=0}^{\infty}\frac{(\lambda t^q)^k}{\Gamma(qk+r)} \tag{2.3}$$

If $r = q$, (2.3) reduces to

$$E_{q,q}(\lambda t^q) = \sum_{k=0}^{\infty}\frac{(\lambda t^q)^k}{\Gamma q(k+1)} \tag{2.4}$$

If $r = 1$, the mittag-leffer function is defined as

$$E_{q,1}(\lambda t^q) = \sum_{k=0}^{\infty}\frac{(\lambda t^q)^k}{\Gamma(qk+r)}. \tag{2.5}$$

Further, if $q = r = 1$, $E_{1,1} = e^{\lambda t}$ is the exponential function.

For more details, see [1, 17, 21] In our next definition we assume $p = 1 - q$. When $0 < q < 1$, $J = (0, T]$ and $J_0 = [0, T]$.

Definition 2.4 A function $\phi(t) \in C(J, \mathbb{R})$ is a C_p continuous function, if $t^{1-q}\phi(t) \in C(J_0, \mathbb{R})$. The set of C_p continuous functions is denoted by $C_p(J, \mathbb{R})$ Further, given a function $\phi(t) \in C_p(J, \mathbb{R})$, we call the function $t^{1-q}\phi(t)$ the continuous extension of $\phi(t)$.
Note that any continuous function in J_0 is also a C_p continuous function.

Consider the initial value problem for the linear R-L fractional differential equation of order q as

$$D^q u = \lambda u + f(t), \Gamma(q)u(t)t^{1-q}|_{t=0} = u^0, \tag{2.6}$$

where λ is a real number and $f \in C[J_0, \mathbb{R}]$. The integral representation of the solution of equation (2.6) is:

$$u(t) = u^0 t^{q-1} E_{q,q}(\lambda t^q) + \int_0^t (t-s)^{q-1} E_{q,q}[\lambda(t-s)^q] f(s) ds. \tag{2.7}$$

For details, see [3, 5, 17, 21]. The next result is a basic comparison result involving the q^{th} order fractional R-L derivative with respect to time.

Lemma 2.1 [6, 7]. Let $m(x,t) \in C_p[J_0, \mathbb{R}]$ be such that for some $t_1 \in (0, T]$, $m(x, t_1) = 0$, and $t^{1-q}m(x,t) \leq 0$ on $[0, t_1]$, then $D^q m(x, t_1) \geq 0$.

3. Auxiliary Results

In this section, we obtain a representation form for the solution of the system of linear R-L fractional reaction diffusion equation with the fractional time derivative. We achieve this by using the eigenfunction expansion method. Then we develop comparison results for the system of non-linear R-L fractional reaction diffusion equation with initial and boundary conditions. The first comparison theorem is with respect to the natural lower and upper solutions when the non-linear term is of the form $F_i(x,t,u)$ where $F_i(x,t,u)$ satisfies the one sided Lipschitz condition. The second comparison theorem is relative to coupled lower and upper solutions. In this case, we assume the non-linear term as the sum of two functions $f_i(x,t,u)$ and $g_i(x,t,u)$, where $f_i(x,t,u)$ non-decreasing function in u, $g_i(x,t,u)$ is non-increasing function in u for (x,t)in $[0, L] \times [0, T]$. In order to present our result, consider the system of linear R-L fractional diffusion equation with initial and boundary conditions of the form

$$\frac{\partial^q u_i}{\partial t^q} - k\frac{\partial^2 u_i}{\partial x^2} = Q_i(x,t) \quad on \quad Q_T. \tag{3.1}$$

$$u_i(0,t) = A_i(t), u_i(L,t) = B_i(t) \quad in \quad \Gamma_T.$$

$$\Gamma(q)t^{1-q}u_i(x,t)|_{t=0} = f_i^0(x) \quad x \in \Omega$$

Where $i = 1, 2$, $\Omega = [0, L]$, $J = (0, T]$, $Q_T = J \times \Omega$, $k > 0$ and $\Gamma_T = (0, T) \times \partial\Omega$.

In order for the initial boundary value problem to be compatible, we assume that $f_i^0(0) = A_i(0) = f_i^0(L) = B_i(0) = 0$, $\Gamma(q)t^{1-q}u_i(x,t)|_{t=0} = f_i^0(x)$. Here and throughout this work,we assume the initial and boundary condition satisfy the compatibility conditions. Using the method of eigenfunction expansion on equation(3.1), we have the solution of the form:

$$u_i(x,t) = \sum_{k=0}^{\infty} b_n(t)\phi_n(x), \tag{3.2}$$

where the eigenfunctions of the related homogeneous problem are known to be $\phi_n(x) = sin\frac{n\pi x}{L}$ and its corresponding eigenvalues are $\lambda_n = (\frac{n\pi}{L})^2$. Using the same approach as in [10, 18]. We can compute $b_n(t)$, where $b_n(t)$ will be the solution of the ordinary linear R-L differential equation.

Using the standard arguments, one can compute $b_n(t)$ as follows.

$$b_n(t) = b_n^0 t^{q-1} E_{q,q}(-k\lambda_n t^q) + \int_0^t (t-s)^{q-1} E_{q,q}(-k\lambda_n t^q) q_n(s) + k\frac{2n\pi}{L^2}[A_i(s) - (-1)^n B_i(s)]ds \tag{3.3}$$

where

$$b_n^0 = \frac{2}{L}\int_0^L f_i^0(y)\phi_n(y)dy \tag{3.4}$$

$$q_n(t) = \frac{2}{L}\int_0^L Q_i(y,t)\phi_n(y)dy. \tag{3.5}$$

Therefore,

$$\begin{aligned} b_n(t) =& \frac{2}{L}\int_0^L f_i^0(y)\phi_n(y)dy t^{q-1} E_{q,q}(-k\lambda_n t^q) \\ &+ \int_0^t (t-s)^{q-1} E_{q,q}(-k\lambda_n t^q)\frac{2}{L}\int_0^L Q_i(y,s)\phi_n(y)dyds \\ &+ k\frac{2n\pi}{L^2}\int_0^t (t-s)^{q-1} E_{q,q}(-k\lambda_n t^q)[A_i(s) - (-1)^n B_i(s)]ds. \end{aligned}$$

So, using $b_n(t)$ in (3.2), we can get the solution $u_i(x,t)$ of the form

$$\begin{aligned} u_i(x,t) =& \int_0^L t^{q-1}[\sum_{k=1}^{\infty}\frac{2}{L}E_{q,q}(-k\lambda_n t^q)\phi_n(x)\phi_n(y)]f_i^0(y)dy \\ &+ \int_0^t\int_0^L[\sum_{k=1}^{\infty}\frac{2}{L}(t-s)^{q-1}E_{q,q}(-k\lambda_n(t-s)^q)\phi_n(x)\phi_n(y)]Q_i(y,s)dyds \\ &+ k\int_0^t[\frac{2n\pi}{L^2}(t-s)^{q-1}E_{q,q}(-k\lambda_n(t-s)^q)\phi_n(x)]A_i(s)ds \\ &- k\int_0^t[\frac{2n\pi}{L^2}(t-s)^{q-1}E_{q,q}(-k\lambda_n(t-s)^q)\phi_n(x)]B_i(s)ds. \end{aligned}$$

Finally, we can write

$$\begin{aligned} u_i(x,t) =& \int_0^L t^{q-1}G(x,y,t)f_i^0(y)dy + \int_0^t\int_0^L G(x,y,t-s)Q_i(y,s)dyds \\ &+ k\int_0^t G_y(x,0,t-s)A_i(s)ds - k\int_0^t G_y(x,L,t-s)B_i(s)ds, \end{aligned}$$

where

$$G(x,y,t) = \sum_{k=0}^{\infty}\frac{2}{L}E_{q,q}(-k\lambda_n t^q)\phi_n(x)\phi_n(y).$$

This result is useful in our main result for computing the linear approximations of the generalized monotone iterates. Here, we can find the steady state condition with homogeneous boundary conditions in which the source term $Q_i(x,t) = Q_i(x)$ is independent of time:

$$k\frac{\partial^2 u_i}{\partial x^2} + Q_i(x) = 0.$$

Now the form $\frac{\partial^2 u_i}{\partial x^2} = g(x)$, in which $g(x) = \frac{-Q_i(x)}{k}$.
Therefore,

$$u_i(x,t) = \int_0^L f_i^0(y) t^{q-1} G(x,t;y,0)dy + \int_0^L -kg(y)[\int_0^t G(x,t;y,s)ds]dy, \tag{3.6}$$

where

$$t^{q-1}G(x,t;y,s) = t^{q-1}\sum_{k=1}^{\infty} \frac{2}{L} E_{q,q}(-k\lambda_n t^q)\phi_n(x)\phi_n(y)$$

As $t \to \infty$, $G(x,t;y,0) \to 0$ such that the effect of the initial condition $t^{1-q}u_i(x,t)|_{t=0} = f_i^0(x)$ vanishes as $t \to \infty$. But, as $t^{q-1}G(x,t;y,s) \to 0$ as $t \to \infty$, the steady source is still important as $t \to \infty$ since $\int_0^t E_{q,q}(-k\lambda_n(t-s)^q)ds = \frac{1-E_{q,q}(-k\lambda_n t^q)}{k(\frac{n\pi}{L})^2}$. Thus, as $t \to \infty$,

$$u_i(x,t) \to u_i(x) = \int_0^L g(y)G(x,y)dy,$$

where

$$G(x,y) = -\sum_{k=1}^{\infty} \frac{2}{L}\phi_n(x)\phi_n(y).$$

Hence, we obtained the steady-state temperature distribution $u_i(x)$ by taking the limit as $t \to \infty$ of the time-dependent problem with a steady source $Q(x) = -kg(x)$.
We recall lemmas regarding the Mittage-Leffler function series from.

Lemma 3.1[4] Let $E_{q,1}(-\lambda t^q)$ be the Mittage-Leffler function of order q, where $0 < q \leq 1$. Then, $\frac{E_{q,1}(-\lambda_1 t^q)}{E_{q,1}(-\lambda_2 t^q)} < 1$, where $\lambda_1, \lambda_2 > 0$ such that $\lambda_1 = \lambda_2 + k$ for $k > 0$.

Lemma 3.2[4] Let $E_{q,q}(-\lambda t^q)$ be the Mittage-Leffler function of order q, where $0 < q \leq 1$. Then, $\frac{E_{q,q}(-\lambda_1 t^q)}{E_{q,q}(-\lambda_2 t^q)} < 1$, where $\lambda_1, \lambda_2 > 0$ such that $\lambda_1 = \lambda_2 + k$ for $k > 0$.

Now, we show the convergence of the above solution using Lemma 3.1 and Lemma 3.2 above. We can split the solution of (3.1)as $u_i^1(x,t)$, $u_i^2(x,t)$ and $u_i^3(x,t)$ respectively as follows:
(a) $u_i^1(x,t)$ is the solution of (3.1), when $Q_i(x,t) = 0$, $A_i(t) = 0 = B_i(t)$,
(b) $u_i^2(x,t)$ is the solution of (3.1), when $A_i(t) = 0 = B_i(t)$, $f_i^0 = 0$,
(c) $u_i^3(x,t)$ is the solution of (3.1), when $Q_i(x,t) = 0$, $f_i^0 = 0$.

Theorem 3.1[4] $u_i^1(x,t)$, $u_i^2(x,t)$ and $u_i^3(x,t)$ converge when $|f_i^0 = 0| < N_1$, $|Q_i(x,t)| < N_2$, $N_1, N_2 > 0$; $|A_i(t)| < M_1$ and $|B_i(t)| < M_2$, $M_1, M_2 > 0$ respectively.

Now, we consider the weekly coupled system of non-linear R-L fractional reaction diffusion equations of the type:

$$\begin{gathered}
\frac{\partial^q u_i}{\partial t^q} - k_i \frac{\partial^2 u_i}{\partial x^2} = f_i(x,t,u_i) + g_i(x,t,u_i), \quad (x,t) \in Q_T, \\
\Gamma(q)t^{1-q}u_i(x,t)|_{t=0} = f_i^0(x), \quad x \in \Omega, \\
u_i(0,t) = A_i(t), u_i(L,t) = B_i(t) \quad on \quad \Gamma_T, \\
\Omega = [0,L], J = (0,T], Q_T = J \times \Omega, k > 0, \\
\Gamma_T = (0,T) \times \partial\Omega \quad i = 1,2 \\
f_i, g_i \in C^{2,q}[\Omega \times J \times \mathbb{R}, \mathbb{R}].
\end{gathered} \tag{3.7}$$

In this work, we study the classical solution of (3.7) $u_i(x,t) \in C_p^{2,q}$ on Q_T, and $u_i(x,t) \in C_p$ on Q_T. To develop the generalized monotone method for (3.7), non-linear coupled system of R-L we need to define.

Definition 3.1 If the functions $v_i(x,t), \quad w_i(x,t) \in C^{2,q}[Q_T, \mathbb{R}]$ are called the natural lower and upper solutions of (3.7) if

$$\frac{\partial^q v_i(x,t)}{\partial t^q} - k\frac{\partial^2 v_i(x,t)}{\partial x^2} \leq f_i(x,t,v_i(x,t)) + g_i(x,t,v_i(x,t)), \quad on \quad Q_T \tag{3.8}$$

$$\Gamma(q)(t-t_0)^{1-q} v_i(x,t)|_{t=0} \leq f_i^0(x), \quad x \in \Omega$$

$$v_i(x,0) \leq A_i(t), v_i(L,t) \leq B_i(t) \quad in \quad \Gamma_T$$

and

$$\frac{\partial^q w_i(x,t)}{\partial t^q} - k\frac{\partial^2 w_i(x,t)}{\partial x^2} \geq f_i(x,t,w_i(x,t)) + g_i(x,t,w_i(x,t)), \quad on \quad Q_T \tag{3.9}$$

$$\Gamma(q)(t-t_0)^{1-q} w_i(x,t)|_{t=0} \geq f_i^0(x), \quad x \in \Omega$$

$$w_i(x,0) \geq A_i(t), w_i(L,t) \geq B_i(t) \quad in \quad \Gamma_T.$$

Definition 3.2 If the functions $v_i(x,t), \quad w_i(x,t) \in C^{2,q}[Q_T, \mathbb{R}]$ are called coupled lower and upper solutions of type I if

$$\frac{\partial^q v_i(x,t)}{\partial t^q} - k\frac{\partial^2 v_i(x,t)}{\partial x^2} \leq f_i(x,t,v_i(x,t)) + g_i(x,t,w_i(x,t)), \quad on \quad Q_T \tag{3.10}$$

$$\Gamma(q)(t-t_0)^{1-q} v_i(x,t)|_{t=0} \leq f_i^0(x), \quad x \in \Omega$$

$$v_i(x,0) \leq A_i(t), v_i(L,t) \leq B_i(t) \quad in \quad \Gamma_T,$$

$$\frac{\partial^q w_i(x,t)}{\partial t^q} - k\frac{\partial^2 w_i(x,t)}{\partial x^2} \geq f_i(x,t,w_i(x,t)) + g_i(x,t,v_i(x,t)), \quad on \quad Q_T \tag{3.11}$$

$$\Gamma(q)(t-t_0)^{1-q} w_i(x,t)|_{t=0} \geq f_i^0(x), \quad x \in \Omega$$

$$w_i(x,0) \geq A_i(t), w_i(L,t) \geq B_i(t) \quad in \quad \Gamma_T.$$

The next result is a comparison result relative to lower and upper solutions of (3.7) of natural type. For that purpose, we write $F_i(x,t,u_i) = f_i(x,t,u_i) + g_i(x,t,u_i)$.

Theorem 3.2 Assume that
(i) $v_i(x,t), w_i(x,t) \in C^{2,q}[Q_T, \mathbb{R}]$ are natural lower and upper solutions of (3.7), respectively and $\Gamma(q)t^{1-q}v_i(x,t)|_{t=0} \leq$ $\Gamma(q)t^{1-q}w_i(x,t)|_{t=0}$, $v_i(0,t) \leq w_i(0,t)$, $v_i(L,t) \leq w_i(L,t)$.
(ii)$F_i(x,t,u_i)$ satisfies the one sided Lipschitz condition

$$F_i(x,t,u_i^1) - F_i(x,t,u_i^2) \leq L(u_i^1 - u_i^2),$$

whenever $u_i^1 \geq u_i^2$ and $L > 0$. Then $v_i(x,t) \leq w_i(x,t)$ on $J \times \Omega$.

Proof. Initially, we prove the theorem when one of the inequalities in (i) is strict. For that purpose, let $m_i(x,t) = v_i(x,t) - w_i(x,t)$. We claim that $m_i(x,t) < 0, \quad (x,t) \in \Omega \times J$. Suppose that the conclusion is not true, then there exists a $t_1 \in J$ and $x_1 \in \Omega$ such that $t^{q-1}m_i(x_1,t_1) < 0$ on $[0,t_1)$, $m_i(x_1,t_1) = 0$. It easy to check $\frac{\partial m_i(x_1,t_1)}{\partial x} = 0$ and $\frac{\partial^2 m_i(x_1,t_1)}{\partial x^2} \leq 0$.

Then, using Lemma 2.1 we get $\frac{\partial^q m_i(x,t)}{\partial t} \geq 0$
From the hypothesis, we also have

$$\begin{aligned}
&\frac{\partial^q m_i(x_1,t_1)}{\partial t_1^q}\\
&\qquad = \frac{\partial^q v_i(x_1,t_1)}{\partial t_1^q} - \frac{\partial^q w_i(x_1,t_1)}{\partial t_1^q}\\
&\qquad < k\frac{\partial^2 v_i(x_1,t_1)}{\partial x^2} + F_i(x_1,t_1,v_i(x_1,t_1)) - k\frac{\partial^2 w_i(x_1,t_1)}{\partial x^2} - F_i(x_1,t_1,w_i(x_1,t_1))\\
&\qquad < F_i(x_1,t_1,v_i(x_1,t_1)) - F_i(x_1,t_1,w_i(x_1,t_1)) = 0
\end{aligned}$$

which is a contradiction. Therefore, $v_i(x,t) < w_i(x,t)$ on Q_T.
In order to prove the theorem for the non strict inequalities, let

$$\begin{aligned}
\overline{w_i}(x,t) &= w_i(x,t) + \epsilon t^{q-1}E_{q,q}[2Lt^q],\\
\overline{v_i}(x,t) &= v_i(x,t) - \epsilon t^{q-1}E_{q,q}[2Lt^q].
\end{aligned}$$

From this it follows

$$\overline{w_i}(0,t) > \overline{v_i}(0,t),$$

$$\overline{w_i}(L,t) > \overline{v_i}(L,t),$$

$\Gamma(q)t^{1-q}\overline{w_i}(x,t)|_{t=0} > \Gamma(q)t^{1-q}w_i(x,t)|_{t=0} > \Gamma(q)t^{1-q}v_i(x,t)|_{t=0} > \Gamma(q)t^{1-q}\overline{v_i}(x,t)|_{t=0}$. Then,

$$\begin{aligned}
&\frac{\partial \overline{w_i}(x,t)}{\partial t^q} - k\frac{\partial^2 \overline{w_i}(x,t)}{\partial x^2}\\
&\qquad = \frac{\partial w_i(x,t)}{\partial t^q} - k\frac{\partial^2 w_i(x,t)}{\partial x^2} + \frac{\partial}{\partial t^q}\epsilon t^{q-1}E_{q,q}[2Lt^q]\\
&\qquad \geq F_i(x,t,w_i(x,t)) + \epsilon t^{q-1}2LE_{q,q}[2Lt^q]\\
&\qquad = F_i(x,t,w_i(x,t)) + 2L\epsilon t^{q-1}E_{q,q}[2Lt^q] - F_i(x,t,\overline{w_i}(x,t)) + F_i(x,t,\overline{w_i}(x,t))\\
&\qquad \geq -L(\overline{w_i} - w_i) + F_i(x,t,\overline{w_i}(x,t)) + \epsilon 2Lt^{q-1}E_{q,q}[2Lt^q]\\
&\qquad = -L\epsilon t^{q-1}E_{q,q}[2Lt^q] + F_i(x,t,\overline{w_i}(x,t)) + \epsilon 2Lt^{q-1}E_{q,q}[2Lt^q]\\
&\qquad = F_i(x,t,\overline{w_i}(x,t)) + \epsilon Lt^{q-1}E_{q,q}[2Lt^q]\\
&\qquad > F_i(x,t,\overline{w_i}(x,t)) \quad on \quad Q_T.
\end{aligned}$$

Similarly,

$$\frac{\partial \overline{v_i}(x,t)}{\partial t^q} - k\frac{\partial^2 \overline{v_i}(x,t)}{\partial x^2} > F_i(x,t,\overline{v_i}(x,t)) \quad on \quad Q_T$$

By the strict inequality result, $\overline{v_i} < \overline{w_i}$ on Q_T. Letting $\epsilon \to 0$ we have $v_i \leq w_i$ on Q_T.

The next result is related to coupled lower and upper solutions of type I related to (3.7).

Theorem 3.3 Assume that
(i) $v_i(x,t), w_i(x,t) \in C^{2,q}[Q_T, \mathbb{R}]$ are coupled lower and upper solutions of type I of (3.7)respectively.
(ii)Assume $F_i(x,t,u_i) = f_i(x,t,u_i) + g_i(x,t,u_i)$, where f_i is a nondecreasing function and g_i is a nonincreasing function respectively for $(x,t) \in Q_T$ in u
(iii)Let $f_i(x,t,u_i)$ and $g_i(x,t,u_i)$ satisfy the one sided Lipschitz condition

$$\begin{aligned} f_i(x,t,u_i^1) - f_i(x,t,u_i^2) &\leq L(u_i^1 - u_i^2), \\ g_i(x,t,u_i^1) - g_i(x,t,u_i^2) &\geq -M(u_i^1 - u_i^2), \end{aligned}$$

whenever $u_i^1 \geq u_i^2$ and $L, M > 0$. Then $v_i(x,t) \leq w_i(x,t)$ on $J \times \Omega$.

Proof. Initially, we prove the theorem when one of the inequalities in (i) is strict. For that purpose,let $m_i(x,t) = v_i(x,t) - w_i(x,t)$. It is easy to see that $m_i(x,0) < 0$ on Ω. Also, $m_i(0,t) < 0$ and $m_i(L,t) < 0$, $t \in J$. Suppose that the conclusion is not true, then there exists a $t_1 \in J$ and $x_1 \in \Omega$ such that $t^{q-1}m_i(x_1,t_1) < 0$ on $[0,t_1)$, $m_i(x_1,t_1) = 0$. This implies $v_i(x_1,t_1) = w_i(x_1,t_1)$ and $\frac{\partial^2 m_i(x_1,t_1)}{\partial x^2} \leq 0$. Where $t_1 > 0$ and $x_1 \in (0,L)$. Using lemma 2.1 we get $\frac{\partial^q m_i(x_1,t_1)}{\partial t_1^q} \geq 0$.
From the hypothesis, we also have

$$\begin{aligned} &\frac{\partial^q m_i(x_1,t_1)}{\partial t_1^q} \\ &\qquad = \frac{\partial^q v_i(x_1,t_1)}{\partial t_1^q} - \frac{\partial^q w_i(x_1,t_1)}{\partial t_1^q} \\ &\qquad < k\frac{\partial^2 v_i(x_1,t_1)}{\partial x^2} + f_i(x_1,t_1,v_i(x_1,t_1)) + g_i(x_1,t_1,w_i(x_1,t_1)) \\ &\qquad - k\frac{\partial^2 w_i(x_1,t_1)}{\partial x^2} - f_i(x_1,t_1,w_i(x_1,t_1)) - g_i(x_1,t_1,v_i(x_1,t_1)) \\ &\qquad \leq 0 \end{aligned}$$

which leads to a contradiction. Therefore,$v_i(x,t) < w_i(x,t)$ on Q_T.
In order to prove the theorem for the non strict inequalities, let

$$\begin{aligned} \overline{w_i}(x,t) &= w_i(x,t) + \epsilon(t-t_0)^{q-1}E_{q,q}[2(L+M)(t-t_0)^q], \\ \overline{v_i}(x,t) &= v_i(x,t) - \epsilon(t-t_0)^{q-1}E_{q,q}[2(L+M)(t-t_0)^q]. \end{aligned}$$

One can show $\overline{v_i}(x,t)$ and $\overline{w_i}(x,t)$ satisfy the hypothesis with strict inequalities. Using the strict inequality result, $\overline{v_i} < \overline{w_i}$ on Q_T. Letting $\epsilon \to 0$ we have $v_i \leq w_i$ on Q_T.

The next result is the maximum principle for the R-L parabolic equation in one dimensional space which will be useful in proving the uniqueness of the solution.

Corollary 3.1 Let

$$\begin{aligned} \frac{\partial^q m_i(x,t)}{\partial t^q} - k\frac{\partial^2 v_i(x,t)}{\partial x^2} &\leq 0 \quad on \quad Q_T, \\ m_i(0,t) \leq 0, m_i(L,t) &\leq 0 \quad on \quad \Gamma_T, \\ \Gamma(q)t^{1-q}m_i(x,t)|_{t=0} &\leq 0 \quad on \quad \Omega. \end{aligned}$$

Then $m_i(x,t) \leq 0$ on Q_T.

Proof. Suppose $m_i(x,t)$ has positive maximum at (x_1,t_1). Let $m_i(x_1,t_1) = K$. Let $\overline{m_i}(x,t) = m_i(x,t) - K$. Then $t^{q-1}\overline{m_i}(x,t) \leq 0$ on $(0,t_1]$ and $\overline{m_i}(x_1,t_1) = 0$. Using lemma (2.1) we get $\frac{\partial^q m_i(x_1,t_1)}{\partial t_1^q} \geq 0$. Also $\frac{\partial^2 m_i(x_1,t_1)}{\partial x^2} \leq 0$. Combining these two, we get $\frac{\partial^q m_i(x_1,t_1)}{\partial t_1^q} - k\frac{\partial^2 v_i(x_1,t_1)}{\partial x^2} \geq 0$.
Also, we have

$$\frac{\partial^q \overline{m_i}(x,t)}{\partial t^q} - K\frac{\partial^2 \overline{m_i}(x,t)}{\partial x^2} = \frac{\partial^q m_i(x,t)}{\partial t^q} - K\frac{\partial^2 m_i(x,t)}{\partial x^2} - K\frac{t^{q-1}}{\Gamma q} < \frac{\partial^q m_i(x,t)}{\partial t} - k\frac{\partial^2 m_i(x,t)}{\partial x^2} < 0. \quad (3.12)$$

which gives a contradiction. Hence, $m(x,t) \leq 0$.

The solution of the linear problem is unique which follows from this maximum principle. This maximum principle is used to show the uniqueness of iterates and monotonicity of this iterates.

4. Main Result

In this section, we develop a generalized monotone method for the nonlinear system of R-L fractional reaction diffusion equation (3.7) using coupled lower and upper solutions of type I. The generalized monotone method yields monotone sequences which converge uniformly and monotonically to coupled minimal and maximal solutions of (3.7). Further using uniqueness condition, we prove the uniqueness of the solution of (3.7).

Theorem 4.1 (i) Let (v_i^0, w_i^0) be the coupled lower and upper solutions of (3.7) such that $t^{1-q}v_i^0 \leq t^{1-q}w_i^0$ on Q_T.
(ii) Suppose that $f_i(x,t,u_i)$ is nondecreasing and $g_i(x,t,u_i)$ is nonincreasing in (u_i), $i = 1,2$ on Q_T, respectively. Then there exist monotone sequences $\left\{t^{1-q}v_i^n(x,t)\right\}$ and $\left\{t^{1-q}w_i^n(x,t)\right\}$ such that $t^{1-q}v_i^n(x,t) \to t^{1-q}\rho_i(x,t) = (v_1,v_2)$ and $t^{1-q}w_i^n(x,t) \to t^{1-q}\gamma_i(x,t) = (w_1,w_2)$ uniformly and monotonically on Q_T, where $\rho_i(x,t)$ and $\gamma_i(x,t)$ are coupled minimal and maximal solutions of (3.7) respectively.

Proof. We construct the sequences $\{v_i^n(x,t)\}$ and $\{w_i^n(x,t)\}$ as follows:

$$\frac{\partial^q v_i^n(x,t)}{\partial t^q} - k\frac{\partial^2 v_i^n(x,t)}{\partial x^2} = f_i(x,t,v_i^{n-1}(x,t)) + g_i(x,t,w_i^n(x,t)), \quad on \quad Q_T \quad (4.1)$$

$$\Gamma(q)(t)^{1-q}v_i^n(x,t)|_{t=0} = f^0(x), \quad x \in \Omega$$

$$v_i^n(x,0) = A(t), v_i^n(L,t) = B(t) \quad in \quad \Gamma_T,$$

and

$$\frac{\partial^q w_i^n(x,t)}{\partial t^q} - k\frac{\partial^2 w_i^n(x,t)}{\partial x^2} = f_i(x,t,w_i^{n-1}(x,t)) + g_i(x,t,v_i^n(x,t)), \quad on \quad Q_T \quad (4.2)$$

$$\Gamma(q)(t)^{1-q}w_i^n(x,t)|_{t=0} = f^0(x), \quad x \in \Omega$$

$$w_i^n(x,0) = A(t), w_i^n(L,t) = B(t) \quad in \quad \Gamma_T,$$

It is easy to observe that $v_i^1(x,t)$ and $w_i^1(x,t)$ exist and unique by the representation form of linear equation and Corollary 3.1. By induction and the assumptions on f_i and g_i, we can prove that the solution $v_i^n(x,t)$ and $w_i^n(x,t)$ exist and unique by Corollary 3.1, for any n. Let us prove first $v_i^0(x,t) \leq v_i^1(x,t)$ and $w_i^1(x,t) \leq w_i^0(x,t)$ on Q_T. Let $\rho_i(x,t) = v_i^0(x,t) - v_i^1(x,t)$. Then

$$\begin{aligned}\frac{\partial^q \rho_i(x,t)}{\partial t^q} - k\frac{\partial^2 \rho_i(x,t)}{\partial x^2}&\\ &= \frac{\partial^q v_i^0(x,t)}{\partial t^q} - k\frac{\partial^2 v_i^0(x,t)}{\partial x^2} - [\frac{\partial^q v_i^1(x,t)}{\partial t^q} - k\frac{\partial^2 v_i^1(x,t)}{\partial x^2}]\\ &\leq f_i(x,t,v_i^0) + g_i(x,t,w_i^0) - [f_i(x,t,v_i^0) + g_i(x,t,w_i^0)] = 0\end{aligned}$$

$\rho_i(0,t) = 0, \rho_i(L,t) = 0$ on Q_T and $\Gamma(q)t^{q-1}\rho_i(x,t)|_{t=0} = 0$ on Γ_T. Therefore, by Corollary 3.1, it follows that $\rho_i(x,t) \leq 0$ on Q_T and $t^{1-q}v_i^0(x,t) \leq t^{1-q}v_i^1(x,t)$ on Q_T.
Assume that $v_i^{k-1}(x,t) \leq v_i^k(x,t)$. Now we show $v_i^k(x,t) \leq v_i^{k+1}(x,t)$. Let $\rho_i(x,t) = v_i^k(x,t) - v_i^{k+1}(x,t)$. Then

$$\begin{aligned}\frac{\partial^q \rho_i(x,t)}{\partial t^q} - k\frac{\partial^2 \rho_i(x,t)}{\partial x^2}&\\ &= \frac{\partial^q v_i^k(x,t)}{\partial t^q} - k\frac{\partial^2 v_i^k(x,t)}{\partial x^2} - [\frac{\partial^q v_i^{k+1}(x,t)}{\partial t^q} - k\frac{\partial^2 v_i^{k+1}(x,t)}{\partial x^2}]\\ &\leq f_i(x,t,v_i^k) + g_i(x,t,w_i^k) - [f_i(x,t,v_i^{k+1}) + g_i(x,t,w_i^{k+1})] = 0\end{aligned}$$

$\rho_i(0,t) = 0, \rho_i(L,t) = 0$ on Q_T and $\Gamma(q)t^{q-1}\rho_i(x,t)|_{t=0} = 0$ on Γ_T. Therefore, by Corollary 3.1, it follows that $\rho_i(x,t) \leq 0$ on Q_T and $t^{1-q}v_i^k(x,t) \leq t^{1-q}v_i^{k+1}(x,t)$ on Q_T.
Hence by mathematical induction, we have

$$t^{1-q}v_i^0(x,t) \leq t^{1-q}v_i^1(x,t)...t^{1-q}v_i^k(x,t) \leq t^{1-q}v_i^{k+1}(x,t)...t^{1-q}v_i^{n-1}(x,t) \leq t^{1-q}v_i^n(x,t) \tag{4.3}$$

We show that $w_i^1(x,t) \leq w_i^0(x,t)$ on Q_T.
Let $\rho_i(x,t) = w_i^1(x,t) - w_i^0(x,t)$. Then

$$\begin{aligned}\frac{\partial^q \rho_i(x,t)}{\partial t^q} - k\frac{\partial^2 \rho_i(x,t)}{\partial x^2}&\\ &= \frac{\partial^q w_i^1(x,t)}{\partial t^q} - k\frac{\partial^2 w_i^1(x,t)}{\partial x^2} - [\frac{\partial^q w_i^0(x,t)}{\partial t^q} - k\frac{\partial^2 w_i^0(x,t)}{\partial x^2}]\\ &\leq f_i(x,t,w_i^0) + g_i(x,t,v_i^0) - [f_i(x,t,w_i^0) + g_i(x,t,v_i^0] = 0\end{aligned}$$

$\rho_i(0,t) = 0, \rho_i(L,t) = 0$ on Q_T and $\Gamma(q)t^{q-1}\rho_i(x,t)|_{t=0} = 0$ on Γ_T. Therefore, by Corollary 3.1, it follows that $\rho_i(x,t) \leq 0$ on Q_T and $t^{1-q}w_i^0(x,t) \leq t^{1-q}w_i^1(x,t)$ on Q_T.
Assume that $w_i^k(x,t) \leq w_i^{k-1}(x,t)$. To show that $w_i^{k+1}(x,t) \leq w_i^k(x,t)$.
Let $\rho_i(x,t) = w_i^{k+1}(x,t) - w_i^k(x,t)$. Then

$$\begin{aligned}\frac{\partial^q \rho_i(x,t)}{\partial t^q} - k\frac{\partial^2 \rho_i(x,t)}{\partial x^2}&\\ &= \frac{\partial^q w_i^{k+1}(x,t)}{\partial t^q} - k\frac{\partial^2 w_i^{k+1}(x,t)}{\partial x^2} - [\frac{\partial^q w_i^k(x,t)}{\partial t^q} - k\frac{\partial^2 w_i^k(x,t)}{\partial x^2}]\\ &\leq f_i(x,t,w_i^{k+1}) + g_i(x,t,v_i^{k+1}) - [f_i(x,t,w_i^k) + g_i(x,t,v_i^k)] = 0\end{aligned}$$

$\rho_i(0,t)=0, \rho_i(L,t)=0$ on Q_T and $\Gamma(q)t^{q-1}\rho_i(x,t)|_{t=0}=0$ on Γ_T. Therefore, by Corollary 3.1, it follows that $\rho_i(x,t)\leq 0$ on Q_T and $t^{1-q}w_i^{k+1}(x,t)\leq t^{1-q}v_i^k(x,t)$ on Q_T.
Hence by mathematical induction, we have

$$t^{1-q}w_i^n(x,t)\leq t^{1-q}w_i^{n-1}(x,t)...t^{1-q}w_i^{k+1}(x,t)\leq t^{1-q}w_i^k(x,t)...t^{1-q}w_i^1(x,t)\leq t^{1-q}w_i^0(x,t) \tag{4.4}$$

Then, we prove that $v_i^1(x,t)\leq w_i^1(x,t)$. Let $\rho_i(x,t)=v_i^1(x,t)-w_i^1(x,t)$. Then from hypothesis, we get

$$\begin{aligned}\frac{\partial^q\rho_i(x,t)}{\partial t^q}-k\frac{\partial^2\rho_i(x,t)}{\partial x^2}&\\ &=\frac{\partial^q v_i^1(x,t)}{\partial t^q}-k\frac{\partial^2 v_i^1(x,t)}{\partial x^2}-[\frac{\partial^q w_i^1(x,t)}{\partial t^q}-k\frac{\partial^2 w_i^1(x,t)}{\partial x^2}]\\ &\leq f_i(x,t,v_i^0(x,t))+g_i(x,t,w_i^0(x,t))-[f_i(x,t,v_i^0(x,t))+g_i(x,t,w_i^0(x,t))]=0\end{aligned}$$

$\rho_i(x,t)=0, \rho_i(L,t)=0$ on Q_T and $\Gamma(q)t^{q-1}\rho_i(x,t)|_{t=0}=0$ on Γ_T. Therefore, by Corollary 3.1, it follows that $\rho_i(x,t)\leq 0$ on Q_T and $t^{1-q}v_i^1(x,t)\leq t^{1-q}w_i^1(x,t)$ on Q_T. Hence,
$t^{1-q}v_i^0(x,t)\leq t^{1-q}v_i^1(x,t)\leq t^{1-q}w_i^1(x,t)\leq t^{1-q}w_i^0(x,t)$ on Q_T.
By mathematical induction and equations (4.3), (4.4) we have
$t^{1-q}v_i^0(x,t)\leq ...\leq t^{1-q}v_i^n(x,t)\leq t^{1-q}w_i^n(x,t)\leq ...\leq t^{1-q}w_i^0(x,t)$ on Q_T for all n.
Furthermore, if $t^{1-q}v_i^0(x,t)\leq t^{1-q}u_i(x,t)\leq t^{1-q}w_i^0(x,t)$ on Q_T, then for any $u_i(x,t)$ of (3.7), we establish the following inequality by the method of induction.

$$t^{1-q}v_i^0(x,t)\leq ...\leq t^{1-q}v_i^n(x,t)\leq t^{1-q}u_i(x,t)\leq t^{1-q}w_i^n(x,t)\leq ...\leq t^{1-q}w_i^0(x,t) \tag{4.5}$$

on Q_T for all n.
It is certainly true for $n=0$, by hypothesis. Assume the inequality (4.3) to be true for $n=k$, that is

$$t^{1-q}v_i^0(x,t)\leq ...\leq t^{1-q}v_i^k(x,t)\leq t^{1-q}u_i(x,t)\leq t^{1-q}w_i^k(x,t)\leq ...\leq t^{1-q}w_i^0(x,t) \tag{4.6}$$

on Q_T for all n.
Let$\rho_i(x,t)=v_i^{k+1}(x,t)-u_i^1(x,t)$. Then from hypothesis, we get

$$\begin{aligned}\frac{\partial^q\rho_i(x,t)}{\partial t^q}-k\frac{\partial^2\rho_i(x,t)}{\partial x^2}&\\ &=\frac{\partial^q v_i^{k+1}(x,t)}{\partial t^q}-k\frac{\partial^2 v_i^{k+1}(x,t)}{\partial x^2}-[\frac{\partial^q u_i(x,t)}{\partial t^q}-k\frac{\partial^2 u_i(x,t)}{\partial x^2}]\\ &\leq f_i(x,t,v_i^k(x,t))+g_i(x,t,w_i^k(x,t))-[f_i(x,t,u_i(x,t))+g_i(x,t,u_i(x,t))]\leq 0,\end{aligned}$$

$\rho_i(x,t)=0, \rho_i(L,t)=0$ on Ω and $\Gamma(q)t^{q-1}\rho_i(x,t)|_{t=0}=0$ on Γ_T. Therefore, by Corollary 3.1, it follows that $\rho_i(x,t)\leq 0$ on Q_T. Therefore $t^{1-q}v_i^{k+1}(x,t)\leq t^{1-q}u_i(x,t)$ on Q_T. Similarly, we can show that $t^{1-q}u_i(x,t)\leq t^{1-q}w_i^{k+1}(x,t)$ on Q_T.
Let $\rho_i(x,t)=u_i^1(x,t)-w_i^{k+1}(x,t)$. Then from hypothesis, we get

$$\begin{aligned}\frac{\partial^q\rho_i(x,t)}{\partial t^q}-k\frac{\partial^2\rho_i(x,t)}{\partial x^2}&\\ &=\frac{\partial^q u_i(x,t)}{\partial t^q}-k\frac{\partial^2 u_i(x,t)}{\partial x^2}-[\frac{\partial^q w_i^{k+1}(x,t)}{\partial t^q}-k\frac{\partial^2 w_i^{k+1}(x,t)}{\partial x^2}]\\ &\leq f_i(x,t,u_i(x,t))+g_i(x,t,u_i(x,t))-[f_i(x,t,w_i^k(x,t))+g_i(x,t,v_i^k(x,t))]\leq 0,\end{aligned}$$

$\rho_i(x,t) = 0, \rho_i(L,t) = 0$ on Ω and $\Gamma(q)t^{q-1}\rho_i(x,t)|_{t=0} = 0$ on Γ_T. Therefore, by Corollary 3.1, it follows that $\rho_i(x,t) \le 0$ on Q_T. Therefore $t^{1-q}u_i(x,t) \le t^{1-q}w_i^{k+1}(x,t)$ on Q_T.

Hence we constructed the monotone sequence $\{v_i^n(x,t)\}$, $\{w_i^n(x,t)\}$ of lower and upper solutions of integral representation of linear problem and an appropriate computation process, we show that the sequences $\left\{t^{q-1}v_i^n(x,t)\right\}$ and $\left\{t^{q-1}w_i^n(x,t)\right\}$ are uniformly bounded and equicontinuous. Using the Ascoli-Arzela theorem, we obtain subsequences of $\left\{t^{q-1}v_i^n(x,t)\right\}$ and $\left\{t^{q-1}w_i^n(x,t)\right\}$ which converge uniformly and monotonically on Q_T. Since the sequences $\left\{t^{q-1}v_i^n(x,t)\right\}$ and $\left\{t^{q-1}w_i^n(x,t)\right\}$ are monotone, the entire sequence $\left\{t^{q-1}v_i^n(x,t)\right\}$ and $\left\{t^{q-1}w_i^n(x,t)\right\}$ converges to $t^{1-q}\rho_i(x,t)$ and $t^{1-q}\rho_i(x,t)$ respectively. From this it follows that

$$\begin{aligned} t^{1-q}v_i^0(x,t) \le t^{1-q}v_i^1(x,t) \le ... \le t^{1-q}v_i^n(x,t) \le ... \le t^{1-q}\rho_i(x,t) \le t^{1-q}u_i(x,t) \\ \le t^{1-q}\gamma_i(x,t) \le ... \le t^{1-q}w_i^n(x,t) \le ... \le t^{1-q}w_i^0(x,t) \quad on \quad Q_T. \end{aligned}$$

Consequently, $\rho_i(x,t)$ and $\gamma_i(x,t)$ are coupled minimal and maximal solutions of (3.7) since

$$t^{1-q}v_i^0(x,t) \le t^{1-q}\rho_i(x,t) \le t^{1-q}u_i(x,t) \le t^{1-q}\gamma_i(x,t) \le t^{1-q}w_i^0(x,t) \quad on \quad Q_T.$$

Since $f_i(x,t,u_i)$ and $g_i(x,t,u_i)$ satisfy the one sided Lipschitz condition, we prove the uniqueness of the solution of (3.7). The next result is precisely this.

Theorem 4.2 Let all the assumptions of theorem 4.1 hold. Further, let $f(x,t,u_i)$ and $g(x,t,u_i)$satisfy the one sided Lipschitz condition of the form

$$\begin{aligned} f(x,t,u_i^1) - f(x,t,u_i^2) \le L_1(u_i^1 - u_i^2), \\ g(x,t,u_i^1) - g(x,t,u_i^2) \ge -L_2(u_i^1 - u_i^1), \end{aligned}$$

whenever $u_i^1 \ge u_i^2$ and $L_1, L_2 > 0$. Then the solution $u_i(x,t)$ of (3.7) exists and is unique.

Proof. We have already proved (ρ_i, γ_i) are coupled minimal and maximal solutions of (3.7) on Q_T. Hence it is enough to show that $\gamma_i(x,t) \le \rho_i(x,t) \quad on \quad Q_T$.
It is known from theorem(4.1) that $\gamma_i(x,t) \le \rho_i(x,t) \quad on \quad Q_T$.
Let $p(x,t) = \gamma_i(x,t) - \rho_i(x,t)$. By the hypothesis, we get

$$\begin{aligned} &\frac{\partial^q p(x,t)}{\partial t^q} - k\frac{\partial^2 p(x,t)}{\partial x^2} \\ &\qquad = \frac{\partial^q \gamma_i(x,t)}{\partial t^q} - k\frac{\partial^2 \gamma_i(x,t)}{\partial x^2} - [\frac{\partial^q \rho_i(x,t)}{\partial t^q} - k\frac{\partial^2 \rho_i(x,t)}{\partial x^2}] \\ &\qquad \le f(x,t,\gamma_i(x,t)) + g(x,t,\rho_i(x,t)) - [f(x,t,\rho_i(x,t)) + g(x,t,\gamma_i(x,t))] \\ &\qquad \le t^{1-q}L_1|\gamma_i - \rho_i| + t^{1-q}L_2|\gamma_i - \rho_i| \\ &\qquad \le (L_1 + L_2)|p|, \end{aligned}$$

$p(x,t) = 0, p(L,t) = 0$ on Ω and $\Gamma(q)t^{q-1}p(x,t)|_{t=0} = 0$ on Γ_T. Therefore, by Corollary 3.1, it follows that $p(x,t) \le 0$. This proves that $\gamma_i(x,t) = \rho_i(x,t) = u(x,t)$ on Q_T and proof is complete.

5. Conclusion

In this work, initially we have obtained the maximal principle and comparison theorem relative to the non-linear weakly coupled system of R-L fractional reaction diffusion equation of (3.7) on Q_T. Using the comparison result as a tool, we have to developed a generalized monotone method for the R-L fractional reaction diffusion equation of (3.7). The generalized monotone method yields monotone sequence which converge uniformly and monotonically to coupled minimal and maximal solutions of (3.7). Under the uniqueness assumption, we have proved that the unique solution of $u_i(x,t)$ of (3.7) exist and unique.

References

[1] A.A.Kilbas, F.Mainardi and S.V.Rogosin, *Mittaf-Leffer Functions, Related Topic and Applications*, Springer Monographs in Mathematics, (2014).

[2] A.A.Kilbas and S. A. Marzan, *Cauchy problem for differential equation with Caputo derivative*, Fract. Calc. Appl. Anal., **7**, (3)(2004)297-321. 2

[3] A.A.Kilbas, H.M.Srivastava and J.J.Trujillo, *Theory and Applications of Fractional Differential Equations*, Amsterdam, Elsevier, (2006). 1
2

[4] Chhetri P.G. and Vatsala A.S., *The Convergence of the Solutions of Caputo fractional Reaction Diffusion Equations with Numerical Examples*, Neural, Parallel and scientific computations, **25** (2017), 295-306.

[5] Chhetri P.G. and Vatsala A.S., *Generalized Monotone Method for Riemann- Liouville Fractional Reaction Diffusion Equation with Applications*, Nonlinear Dynamics and System theory, **18**, (3)(2018), 259-272. 1, 3

[6] Denton Z. and Vatsala A.S., *Monotone Iterative Technique For Finite System of Non- linear Riemann- Liouville Fractional Differential Equations*, Opuscula Mathematica, **31**,(3)(2011) 327-339. 1, 2

[7] Denton Z. Ng P.W.and Vatsala A.S., *Quasilinearization Method Via Lower and Upper Solutions for Riemann-Liouville Fractional Differential Equations*, Nonlinear Dynamics and System Theory,**11**(3)(2011), 239-251.

[8] Donna Stutson and Vatsala,A.S., *A representations solution is obtained for the one dimentional Caputo fractional reaction diffusion equation*, Dynamic system and applictions, **6** (2012). 1, 2

[9] D.B. Dhaigude, J.A.Nanware and V.R.Nikam., *Monotone Technique for Weakly Coupled System of Caputo Fractional Differential Equations with Periodic Boundary Conditions*, Dyn.Contin.Discrete Impuls. Syst.Ser.A Math.Anal.,**19**,(2012) 575 - 584. 1, 2
1

[10] J.A.Nanware and D.B. Dhaigude., *Monotone Technique for Finite Weakly Coupled System of Caputo Fractional Differential Equations with Periodic Boundary Conditions*, Dyn.Contin.Discrete Impuls.Syst. Ser.A Math.Anal., **22**(1)(2015), 13-23.

[11] J.A.Nanware and D.B. Dhaigude., *Existence of Uniqueness of Solutions of Riemann- Liouville Fractional Differential Equations with Integral Bounadry Conditions*,International Journal of Nonlinear Science., **14**(4)(2012), 410 - 415. 1

[12] J.A.Nanware and D.B. Dhaigude, *System of initial value problems involving Riemann-Liouville sequential fractional derivative*, Communications in Applied Analysis, **22**(3)(2018), 353-368. 3

[13] J.A.Nanware,N.B.Jadhav and D.B. Dhaigude, *Monotone iterative technique for finite system of Riemann-Liouville fractional differential equations with integral boundary conditions*, Proc.Internal.Conf.on Mathematical Sciences, 2014, 235- 238. 1

[14] J.A.Nanware,N.B.Jadhav and D.B. Dhaigude, *Initial value problems for fractional differential equations involving Riemann-Liouville derivative*, Malaya J. Mat.,**5**(2)(2017) 337-345. 1
1

[15] J.A.Nanware,N.B. Jadhav and D.B.Dhaigude, *Existence and uniqueness of solutions of nonlinear implicit fractional differential equations*, DDyn,Contin.Discrete Impuls.Syst.Ser.A: Math.Anal., **27**(2020), 275-282. 1
1

[16] Pao, C.V., *Non-linear Parabolic and Elliptic Equations*, Springer Science + Business Media, LLC 1992.

[17] Podlubny,I., *Fractional Differential Equations.Academics Press*,San Diego, **198**, 1999. 1
2, 2

[18] Ramirez J.D. and Vatsala A.S.,*Generalized Monotone Iterative Technique for Caputo Fractional Differential Equations with Periodic Boundary Conditions via Initial Value Problem*,International Journal of Differential Equations, 17(2012), 842-843.

[19] R.Hilfer.,*Applications of Fractional Calculus in Physics, World Scientific*, Singapore 2000. 3
1

[20] Sowmya M. and Vatsala A.S.,*Generalized Monotone Method for Caputo Fractional Differential Equations via Coupled Lower and Upper Solutions with Superlinear Convergence*, Nonlinear Dynamics and System theory, **15**,(2)(2015), 198-208.

[21] V.Lakshmikantham S.Leela and D.J.Vasundhara Devi,*Theory of Fractional Dynamic Systems*,Cambridge Scientific Publishers, 2009. 1

[22] V.Lakshmikantham and A.S.Vatsala,*General Monotone Method For fractional Reaction Diffusion Equations*, Communications in Applied Analysis,**69** 16(2012), 165-174. 2, 2

[23] V.Lakshmikantham,*Theory of fractional functional differential equations*, Nonlinear Anal. (2008). 1

[24] Vatsala A.S. and Donna Stutson.,*Generalized Monotone Method For Fractional Reaction Diffusion Equation*, Communications in Applied Analysis, **16**, (2)(2012), 165-174 1