Available online at https://rsgmodelling.co.uk/
Commun. Nonlinear Anal. 1 (2023), 1–18
Research Article

Communications in Nonlinear Analysis
Publisher
Research & Scinece Group Ltd.

Convergence Theorems for Monotone Generalized α-Nonexpansive Mappings in Ordered Banach Space by a New Four-Step Iteration Process with Application

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Abstract

We introduce a four-step iterative algorithm and show that the algorithm converges faster than a number of existing iterative algorithms for contraction mappings. We prove strong and weak convergence results for approximating fixed points of monotone generalized α-nonexpansive mappings. Further, we utilize our proposed algorithm to solve Split Feasibility Problem (SFP). Our result complements, extends and generalizes some existing results in literature.

Keywords: Monotone, generalized α-Nonexpansive mapping, Ordered Banach space, Fixed point, contraction mapping, Split Feasibility Problem.
2010 MSC: 34A12, 47H09, 47H30.

1. Introduction

Let Θ be a mapping with domain $D(\Theta)$ and range $R(\Theta)$ in an ordered Banach space ϖ endowed with the partial order $\leq$, and ϑ a nonempty closed convex subset of ϖ. Then, $\Theta : D(\Theta) \longrightarrow R(\Theta)$ is said to be:

(1) monotone [22] if

$$\Theta x \leq \Theta y \ \ \forall\, x, y \in D(\Theta) \ \ with \ \ x \leq y, \tag{1.1}$$

(2) monotone nonexpansive [22] if Θ is monotone and

$$\|\Theta x - \Theta y\| \leq \|x - y\| \ \ \forall\, x, y \in D(\Theta) \ \ with \ \ x \leq y. \tag{1.2}$$

Remark 1.1. If Θ does not satisfy the monotone condition, then Θ is said to be nonexpansive [31].

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Received 2022-10-11

(3) monotone quasi-nonexpansive [32] if there exists a fixed point set $F(\Theta) \neq \emptyset$ and

$$\|\Theta x - p\| \leq \|x - p\|, \tag{1.3}$$

$\forall p \in F(\Theta)$ and $x \in \vartheta$, with $x \leq p$ or $x \geq p$.

(4) monotone α-nonexpansive [32] if Θ is monotone and for some $\alpha < 1$

$$\|\Theta x - \Theta y\|^2 \leq \alpha\|\Theta x - y\|^2 + \alpha\|\Theta y - x\|^2 + (1 - 2\alpha)\|x - y\|^2, \tag{1.4}$$

$\forall x, y \in \vartheta,\ x \leq y$.

Remark 1.2. If Θ does not satisfy the monotone condition, then Θ is said to be α-nonexpansive [16].

(5) Suzukis generalized nonexpansive if Θ satisfy condition (C), that is

$$\begin{aligned} \frac{1}{2}\|x - \Theta x\| &\leq \|x - y\|, \quad \textit{implies} \\ \|\Theta x - \Theta y\| &\leq \|x - y\|, \quad \forall\ x, y \in \vartheta \end{aligned} \tag{1.5}$$

(6) monotone generalized α-nonexpansive [24] if Θ is monotone and there exists $\alpha \in [0, 1)$ such that

$$\begin{aligned} & \frac{1}{2}\|x - \Theta x\| \leq \|x - y\| \\ \Longrightarrow\ & \|\Theta x - \Theta y\| \leq \alpha\|\Theta x - y\| + \alpha\|\Theta y - x\| + (1 - 2\alpha)\|x - y\|, \end{aligned} \tag{1.6}$$

$\forall\ x, y \in \vartheta$. If $\alpha = 0$, then (1.6) reduces to (1.5).

Obviously, a monotone generalized α-nonexpansive mapping includes nonexpansive, firmly nonexpansive, Suzukis generalized nonexpansive mapping as special cases and partially extends monotone α-nonexpansive mapping.

In 1965, Browder [8] first initiated the study of existence of fixed points of nonexpansive mappings and obtained a fixed point theorem for nonexpansive mappings on a bounded closed and convex subset of a Hilbert space.

Consequently, Browder [9] and Gohde [6] generalized the result of Browder [8] from a Hilbert space to a uniformly convex Banach space. Also Kirk [30] used the normal structure property in a reflexive Banach space to generalize the same result.

Thereafter, quite a number of extensions and generalizations of nonexpansive mappings have been studied by many mathematicians.

In 2008, Suzuki [28] introduced an interesting generalization of nonexpansive mappings called Suzuki type generalized nonexpansive mapping (see Definition 5), and obtained some existence and convergence results in Banach spaces.

In 2011, Aoyama and Kohsaka [16] introduced a new class of nonexpansive mappings namely α-nonexpansive mappings (see Definition 4) and obtained a fixed point theorem for such mappings in a uniformly convex Banach space.

Remarkably, it has been established that nonexpansive mappings are continuous on their domains but Suzuki type generalized nonexpansive mappings and α-nonexpansive mappings need not be continuous, and are therefore more important in theoretical and application point of view.

In 2016, Song *et al.* [32] introduced the concept of monotone α-nonexpansive mappings in ordered Banach space and obtained some existence and convergence theorems for the Mann iteration under some suitable conditions.

Pant and Shukla [24] introduced the class of generalized α-nonexpansive mapping which contains both the Suzuki type generalized mappings and α-nonexpansive mappings. (If $\alpha = 0$, generalized α-nonexpansive mapping reduces to Suzuki type generalized mappings).

Again, Shukla *et al* [26] introduced the class of monotone generalized α-nonexpansive mapping and obtained some existence and convergence theorems for Mann iteration process.

The Mann iteration for approximating fixed points of nonexpansive mappings is defined by:

$$x_{k+1} = (1-b_k)x_k + b_k\Theta x_k, \ k \geq 1 \tag{1.7}$$

where $\{b_k\} \in (0,1)$.

Another widely used iteration method in approximating fixed point of nonexpansive mapping is the Ishikawa iteration scheme introduced by S. Ishikawa [27] in 1976 and defined by:

$$\begin{aligned} x_{k+1} &= (1-a_k)x_k + a_k\Theta y_k \\ y_n &= (1-b_k)x_k + b_k\Theta x_k, \end{aligned} \tag{1.8}$$

where $\{a_k\}, \{b_k\} \in (0,1)$, $k \in N$.

In the recent past, quite a number of iteration processes have been constructed to approximate the fixed points of various classes of mappings such as the Noor [21], Agarwal *et al.* [25] (S-iteration), Abbas and Nazir [20], Thakur *et al.* [2], Piri *et al.* and M-iterations [19] and many others.

Recently, Garodia and Uddin [3] introduced a new iteration process as follows:
For an arbitrary $x_1 \in \vartheta$, define a sequence $\{x_k\}$ by

$$\begin{aligned} x_{k+1} &= \Theta y_k \\ y_k &= (1-b_k)\Theta x_k + b_k\Theta z_k \\ z_k &= \Theta((1-a_k)x_k + a_k\Theta x_k) \end{aligned} \tag{1.9}$$

where $\{a_k\}, \{b_k\} \in (0,1)$, $k \in N$.

They proved that the iteration process (1.9) converges faster than a number of existing iteration processes in literature for contractive-like mappings. Using the iteration process (1.9), they proved some weak and strong convergence theorems for generalized α-nonexpansive mappings in real Banach space.

Motivated and inspired by the ongoing research in this direction, we introduce a new four-step iteration process called UI iterative scheme to approximate the fixed points of monotone generalized α-nonexpansive mappings. We define the process as follows: For an arbitrary $x_1 \in \vartheta$, define a sequence $\{x_k\}$ by

$$\begin{aligned} x_{k+1} &= \Theta y_k, \\ y_k &= \Theta((1-b_k)\Theta w_k + b_k\Theta z_k), \\ z_k &= \Theta((1-a_k)\Theta x_k + a_k\Theta w_k), \\ w_k &= \Theta((1-c_k)x_k + c_k\Theta x_k) \end{aligned} \tag{1.10}$$

where $\{a_k\}, \{b_k\}, \{c_k\} \in (0,1)$, $k \in N$.

Our purpose in this paper first is, to prove analytically that the UI iteration scheme (1.10) converges faster than the iteration process (1.9) for contraction mappings. Also, we prove some existence and convergence theorems for the iteration process (1.10). Again, with a numerical example, we show that our iteration process (1.10) converges faster than a number of iteration processes in literature. Finally, we apply our iteration process (1.10) to the solution of Split Feasibility Problem (SFP).

2. Preliminaries

Throughout this paper, let ϖ be an ordered Banach space with the norm $\|\cdot\|$ and the partial order $\leq$. Let $F(\Theta) = \{x \in \varpi : \Theta x = x\}$ denote the set of all fixed points of a mapping $\Theta : \varpi \longrightarrow \varpi$.

Definition 2.1. A Banach space ϖ is said to be:

(i) Strictly convex if $\frac{1}{2}\|x+y\| < 1$
for all $x, y \in \varpi$ with $\|x\| = \|y\| = 1$ and $x \neq y$.

(ii) Uniformly convex if, for all $\epsilon \in (0, 2]$, there exists $\delta > 0$ such that
$\frac{1}{2}\|x+y\| \leq 1-\delta$, for all $x, y \in \varpi$ with $\|x\| \leq 1$, $\|y\| \leq 1$ and $\|x-y\| \geq \epsilon$.

Definition 2.2. [34]: A Banach space ϖ is said to satisfy the Opial's condition if for each weakly convergent sequence $\{x_k\}$ in ϖ, $\{x_k\}$ converges weakly to a point $x \in \varpi$, implies

$$\limsup_{k\to\infty} \|x_k - x\| < \limsup_{k\to\infty} \|x_k - y\|,$$

for all $y \in \varpi$ with $y \neq x$.

Definition 2.3. Let ϑ be a nonempty subset of a Banach space ϖ and $\{x_k\}$ be a bounded sequence in ϖ. For each $x \in \varpi$, we define the following:

(i) Asymptotic radius of $\{x_k\}$ at x by

$$r(x, \{x_k\}) := \limsup_{k\to\infty} \|x_k - x\|$$

(ii) Asymptotic radius of $\{x_k\}$ relative to ϑ by

$$r(\vartheta, \{x_k\}) := \inf\{r(x, x_k) : x \in \vartheta\}$$

(iii) Asymptotic center of $\{x_k\}$ relative to ϑ by

$$A(\vartheta, \{x_k\}) := \{x \in \vartheta : r(x, \{x_k\}) = r(\vartheta, \{x_k\})\}$$

It is known that in a uniformly convex Banach space, $A(\vartheta, \{x_k\})$ consists of exactly one point. Also, $A(\vartheta, \{x_k\})$ is nonempty and convex when ϑ is weakly compact and convex.

Definition 2.4. [29] Let $\{U_k\}$ and $\{V_k\}$ be two sequences of real numbers that converge to u and v respectively. Then, $\{U_k\}$ converges faster to u than $\{V_k\}$ does to v if

$$\lim_{k\to\infty} \frac{\|U_k - u\|}{\|V_k - v\|} = 0.$$

Lemma 2.5. *(see Shukla et al [26], Lemma 3.7)*
Let ϑ be a nonempty closed convex subset of an ordered Banach space $(\varpi, \leq)$ and $\Theta : \vartheta \longrightarrow \vartheta$ be a monotone generalized α-nonexpansive mapping. Then, for all $x, y \in \vartheta$ with $x \leq y$, the following inequalities hold.

(i) $\|\Theta x - \Theta^2 x\| \leq \|x - \Theta x\|$

(ii) Either $\frac{1}{2}\|x - \Theta x\| \leq \|x - y\|$ *or* $\frac{1}{2}\|\Theta x - \Theta^2 x\| \leq \|\Theta x - y\|$

(iii) (a) $\|\Theta x - \Theta y\| \leq \alpha\|\Theta x - y\| + \alpha\|\Theta y - x\| + (1-2\alpha)\|x - y\|$ *or*

(b) $\|\Theta^2 x - Ty\| \leq \alpha\|\Theta^2 x - y\| + \alpha\|\Theta x - \Theta y\| + (1-2\alpha)\|\Theta x - y\|$

Lemma 2.6. *(See Shukla et al. [26] , Lemma 3.6)*
Let ϑ be a nonempty subset of an ordered Banach space $(\varpi, \leq)$ and $\Theta : \vartheta \longrightarrow \vartheta$ be a generalized α-nonexpansive mapping. Then $F(\Theta)$ is closed. Moreover, if ϖ is strictly convex and ϑ is convex, then $F(\Theta)$ is also convex.

Lemma 2.7. *(Xu [11] , Theorem 2)*
For any real numbers $q > 1$ *and* $r > 0$*, a Banach space* E *is uniformly convex if and only if there exists a continuous strictly increasing convex function* $f : [0, +\infty) \longrightarrow [0, +\infty)$ *with* $f(0) = 0$ *such that*

$$\|tx + (1-t)y\|^q = t\|x\|^p + (1-t)\|y\|^q - \omega(q,t)f(\|x-y\|),$$

for all $x, y \in B_r(0) = \{x \in E : \|x\| \le r\}$ *and* $t \in [0,1]$,
where, $\omega(q,t) = t^q(1-t) + t(1-t)^q$.

In particular, taking $q = 2$ *and* $t = \frac{1}{2}$

$$\|\frac{x+y}{2}\|^2 \le \frac{1}{2}\|x\|^2 + \frac{1}{2}\|y\|^2 - \frac{1}{4}f(\|x-y\|),$$

Lemma 2.8. *(Schu [15])*
Let ϖ *be a uniformly convex Banach space and* $\{\lambda_k\}$ *be a sequence with* $0 < \liminf_{k\to\infty} \lambda_k \le \limsup_{k\to\infty} \lambda_k < 1$. *Suppose* $\{x_k\}$ *and* $\{y_k\}$ *are two sequences of* ϖ *such that* $\limsup_{k\to\infty} \|x_k\| \le r$, $\limsup_{k\to\infty} \|y_k\| \le r$ *and* $\lim_{k\to\infty} \|\lambda_k x_k + (1-\lambda_k)y_k\| = r$. *Then* $\lim_{k\to\infty} \|x_k - y_k\| = 0$

3. Main Result

3.1. Convergence Result

In this section, we show that the UI iteration scheme (1.10) converges faster than the iteration process (1.9) for contraction mappings.

Theorem 3.1. *Let* Θ *be a contraction mapping defined on a nonempty closed convex subset* ϑ *of a Banach space* ϖ *with a contraction factor* $\delta \in (0,1)$ *and* $F(\Theta) \ne \phi$. *If* $\{x_k\}$ *is a sequence defined by* (1.10), *then* $\{x_k\}$ *converges faster than the iteration process* (1.9).

Proof. Let $q \in F(\Theta)$. From (1.10), we have

$$\begin{aligned}
\|w_n - q\| &= \|\Theta((1-c_k)x_k + c_k\Theta x_k) - q\| \\
&\le \delta((1-c_k)\|x_k - q\| + c_k\|\Theta x_k - q\|) \\
&\le \delta((1-c_k) + c_k\delta)\|x_k - q\| \\
&\le \delta(1-(1-\delta)c_k)\|x_k - q\| \\
&\le \delta\|x_k - q\|
\end{aligned} \tag{3.1}$$

Using (1.10) and (3.1), we have

$$\begin{aligned}
\|z_n - q\| &= \|\Theta((1-a_k)\Theta x_k + a_k\Theta w_k) - q\| \\
&\le \delta((1-a_k)\delta\|x_k - q\| + a_k\delta\|w_k - q\|) \\
&\le \delta^2(1-(1-\delta)a_k)\|x_k - q\| \\
&\le \delta^2\|x_k - q\|
\end{aligned} \tag{3.2}$$

Using (1.10) , (3.1) and (3.2), we have

$$\begin{aligned}
\|y_n - q\| &= \|\Theta((1-b_k)\Theta w_k + b_k\Theta z_k) - q\| \\
&\le \delta((1-b_k)\delta\|w_k - q\| + b_k\delta\|z_k - q\|) \\
&\le \delta^2((1-b_k)\delta\|x_k - q\| + b_k\delta^2\|x_k - q\|) \\
&\le \delta^3(1-(1-\delta)b_k)\|x_k - q\| \\
&\le \delta^3\|x_k - q\|
\end{aligned} \tag{3.3}$$

Using (1.10) and (3.3), we have

$$\begin{aligned} \|x_{k+1} - q\| &= \|\Theta y_k - q\| \\ &\leq \delta\|y_k - q\| \\ &\leq \delta(\delta^3\|x_k - q\|) \\ &= \delta^4\|x_k - q\| \\ &\cdot \\ &\cdot \\ &\cdot \\ &\leq \delta^{4k}\|x_1 - q\| \end{aligned} \tag{3.4}$$

$$Let \quad p_k = \delta^{4k}\|x_1 - q\| \tag{3.5}$$

Also from (1.9), we have

$$\begin{aligned} \|z_k - q\| &= \|\Theta((1 - a_k)x_k + a_k\Theta x_k) - q\| \\ &\leq \delta((1 - a_k\|x_k - q\| + a_k\|\Theta x_k - q\|) \\ &\leq \delta((1 - a_k) + a_k\delta)\|x_k - q\| \\ &\leq \delta(1 - (1 - \delta)a_k)\|x_k - q\| \\ &\leq \delta\|x_k - q\| \end{aligned} \tag{3.6}$$

Using (1.9) and (3.6), we have

$$\begin{aligned} \|y_k - q\| &= \|\Theta((1 - b_k)\Theta x_k + b_k\Theta z_k) - q\| \\ &\leq \delta((1 - b_k)\delta\|x_k - q\| + b_k\delta\|z_k - q\|) \\ &\leq \delta^2(1 - (1 - \delta)b_k)\|x_k - q\| \\ &\leq \delta^2\|x_k - q\| \end{aligned} \tag{3.7}$$

Using (1.9) and (3.7), we have

$$\begin{aligned} \|x_{k+1} - q\| &= \|\Theta y_k - q\| \\ &\leq \delta\|y_k - q\| \\ &\leq \delta(\delta^2\|x_k - q\|) \\ &= \delta^3\|x_k - q\| \\ &\cdot \\ &\cdot \\ &\cdot \\ &\leq \delta^{3k}\|x_1 - q\| \end{aligned} \tag{3.8}$$

$$Let \quad r_k = \delta^{2k}\|x_1 - q\| \tag{3.9}$$

So from (3.5) and (3.9), we have that

$$\frac{p_k}{r_k} = \frac{\delta^{4k}\|x_1 - q\|}{\delta^{3k}\|x_1 - q\|} = \delta^k \longrightarrow 0, \qquad as \;\; k \to \infty$$

Hence (1.10) converges faster than (1.9). □

3.2. NUMERICAL EXAMPLE

We now show the comparison between the rate of convergence of the UI iteration process (1.10) and other well known iteration algorithms in literature.

Example 3.2. Let $\vartheta = [1, 15]$ and $\Theta : [1, 15] \longrightarrow [1, 15]$ defined by $\Theta(\upsilon) = \frac{1}{3}\upsilon + \frac{3}{4}$

For Table 1 , we use the following parameters:
Choose $\alpha_k = \frac{7k}{10}$, $\beta_k = \frac{13k}{20}$, $\gamma_k = \frac{4k}{5}$, and the initial value $t_1 = 5$.
Obviously, the fixed point of Θ is $p = 1.125$, with a contraction constant $\delta = \frac{1}{3}$.

Table 1 shows the behaviour of the UI iteration process (1.10) in comparison with the iteration processes of Noor [21], Agarwal *et al.* (S-iteration) [25], Abbas and Nazir [20], Thakur *et al.* [2], M-iterations [19], Piri *et al.* [14] and Garodia and Uddin [3] to the fixed point of Θ in 30-iterations with $\|t_n - p\| < 10^{-15}$ as the stop criterion.

TABLE 1

n	NOOR	AGARWAL *et al.*	ABBAS-NAZIR	THAKUR *et al.*	UI ITERATION
1	5.0000000000	5.0000000000	5.0000000000	5.0000000000	5.0000000000
2	2.6953796296	2.0248611111	1.5027407407	1.4249537037	1.1370520119
3	1.7614108855	1.3339677469	1.1618227270	1.1482186385	1.1250374841
4	1.3829114040	1.1735269546	1.1285895340	1.1267972946	1.1250001166
5	1.2295209846	1.1362690372	1.1253499131	1.1251391239	1.1250000004
6	1.1673580968	1.1276169209	1.1250341100	1.1250107692	1.1250000000
7	1.1421660109	1.1256077072	1.1250033251	1.1250008336	1.1250000000
8	1.1319566849	1.1251411231	1.1250003241	1.1250000645	1.1250000000
9	1.1278192610	1.1250327719	1.1250000316	1.1250000050	1.1250000000
10	1.1261425316	1.1250076104	1.1250000031	1.1250000004	1.1250000000
11	1.1254630215	1.1250017673	1.1250000003	1.1250000000	1.1250000000
12	1.1251876438	1.1250004104	1.1250000000	1.1250000000	1.1250000000
13	1.1250760444	1.1250000953	1.1250000000	1.1250000000	1.1250000000
14	1.1250308177	1.1250000221	1.1250000000	1.1250000000	1.1250000000
15	1.1250124892	1.1250000051	1.1250000000	1.1250000000	1.1250000000
16	1.1250050613	1.1250000012	1.1250000000	1.1250000000	1.1250000000
17	1.1250020512	1.1250000003	1.1250000000	1.1250000000	1.1250000000
18	1.1250008313	1.1250000001	1.1250000000	1.1250000000	1.1250000000
19	1.1250003369	1.1250000000	1.1250000000	1.1250000000	1.1250000000
20	1.1250001365	1.1250000000	1.1250000000	1.1250000000	1.1250000000
21	1.1250000553	1.1250000000	1.1250000000	1.1250000000	1.1250000000
..	....	....	....	....	....
28	1.1250000001	1.1250000000	1.1250000000	1.1250000000	1.1250000000
29	1.1250000000	1.1250000000	1.1250000000	1.1250000000	1.1250000000
30	1.1250000000	1.1250000000	1.1250000000	1.1250000000	1.1250000000

TABLE 1 CONTD.

n	M-ITERATION	PIRI *et al.*	GARODIA-UDDIN	UI ITERATION
1	5.0000000000	5.0000000000	5.0000000000	5.0000000000
2	1.3546296296	1.2551234568	1.1918158436	1.1370520119
3	1.1386076818	1.1293695778	1.1261520921	1.1250374841
4	1.1258063811	1.1251467315	1.1250198653	1.1250001166
5	1.1250477855	1.1250049273	1.1250003425	1.1250000004
6	1.1250028317	1.1250001655	1.1250000059	1.1250000000
7	1.1250001678	1.1250000056	1.1250000001	1.1250000000
8	1.1250000099	1.1250000002	1.1250000000	1.1250000000
9	1.1250000006	1.1250000000	1.1250000000	1.1250000000
10	1.1250000000	1.1250000000	1.1250000000	1.1250000000
..	....	....	....	....
30	1.1250000000	1.1250000000	1.1250000000	1.1250000000

For Table 2 we use the following parameters:
Choose $\alpha_k = \frac{3k}{8k+4}$, $\beta_k = \frac{1}{k+4}$, $\gamma_k = \frac{k}{(2k+6)^2}$, and the initial value $t_1 = 10$.

TABLE 2

n	M-ITERATION	PIRI *et al.*	GARODIA-UDDIN	UI ITERATION
1	10.000000000	10.000000000	10.000000000	10.000000000
2	1.8481481481	1.7799793827	1.4185864609	1.1912742958
3	1.1839231824	1.1733378019	1.1347118885	1.1254949050
4	1.1298011482	1.1285673537	1.1253212709	1.1250036957
5	1.1253912047	1.1252632725	1.1250106277	1.1250000276
6	1.1250318759	1.1250194296	1.1250003516	1.1250000002
7	1.1250025973	1.1250014339	1.1250000116	1.1250000000
8	1.1250002116	1.1250001058	1.1250000004	1.1250000000
9	1.1250000172	1.1250000078	1.1250000000	1.1250000000
10	1.1250000014	1.1250000006	1.1250000000	1.1250000000
11	1.1250000001	1.1250000000	1.1250000000	1.1250000000
12	1.1250000000	1.1250000000	1.1250000000	1.1250000000
..	....	....	....	....
28	1.1250000000	1.1250000000	1.1250000000	1.1250000000
29	1.1250000000	1.1250000000	1.1250000000	1.1250000000
30	1.1250000000	1.1250000000	1.1250000000	1.1250000000

TABLE 2 CONTD.

n	NOOR	AGARWAL *et al.*	ABBAS-NAZIR	THAKUR *et al.*	UI ITERATION
1	10.000000000	10.000000000	10.000000000	10.000000000	10.000000000
2	7.5031929630	3.9717844444	2.6841862963	2.6841862963	1.1912742958
3	5.7088135744	2.0381472308	1.3989224683	1.3989224683	1.1254949050
4	4.4192476038	1.4179051642	1.1731235109	1.1731235109	1.1250036957
5	3.4924757053	1.2189535623	1.1334544810	1.1334544810	1.1250000276
6	2.8264328883	1.1551369622	1.1264853083	1.1264853083	1.1250000002
7	2.3477681437	1.1346668659	1.1252609434	1.1252609434	1.1250000000
8	2.0037663289	1.1281007869	1.1250458433	1.1250458433	1.1250000000
9	1.7565426722	1.1259946222	1.1250080539	1.1250080539	1.1250000000
10	1.5788705383	1.1253190394	1.1250014149	1.1250014149	1.1250000000
11	1.4511829717	1.1251023365	1.1250002486	1.1250002486	1.1250000000
12	1.3594177954	1.1250328259	1.1250000437	1.1250000437	1.1250000000
13	1.2934689502	1.1250105294	1.1250000077	1.1250000077	1.1250000000
14	1.2460735181	1.1250033775	1.1250000013	1.1250000013	1.1250000000
15	1.2120118604	1.1250010834	1.1250000002	1.1250000002	1.1250000000
16	1.1875327815	1.1250003475	1.1250000000	1.1250000000	1.1250000000
17	1.1699404109	1.1250001115	1.1250000000	1.1250000000	1.1250000000
18	1.1572973085	1.1250000358	1.1250000000	1.1250000000	1.1250000000
19	1.1482110947	1.1250000115	1.1250000000	1.1250000000	1.1250000000
20	1.1416811088	1.1250000037	1.1250000000	1.1250000000	1.1250000000
21	1.1369882063	1.1250000012	1.1250000000	1.1250000000	1.1250000000
22	1.1336155598	1.1250000004	1.1250000000	1.1250000000	1.1250000000
23	1.1311917412	1.1250000001	1.1250000000	1.1250000000	1.1250000000
24	1.1294498163	1.1250000000	1.1250000000	1.1250000000	1.1250000000
..	....	....	....	....	....
28	1.1261870233	1.1250000000	1.1250000000	1.1250000000	1.1250000000
29	1.1258530776	1.1250000000	1.1250000000	1.1250000000	1.1250000000
30	1.1256130810	1.1250000000	1.1250000000	1.1250000000	1.1250000000

4. Convergence of The Iteration Process

In this section, we consider the convergence of the four-step UI iteration process defined in (1.10) for a monotone generalized α-nonexpansive mapping Θ in an ordered Banach space $(\varpi, \leq)$.

Lemma 4.1. *Let ϑ be a nonempty closed convex subset of an ordered Banach space $(\varpi, \leq)$ and $\Theta : \vartheta \longrightarrow \vartheta$ be a monotone generalized α-nonexpansive mapping. Then for all $x, y \in \vartheta$, with $x \leq y$*

(i) $\|\Theta x - \Theta y\| \leq \frac{4}{1-\alpha}\|x - \Theta x\| + \|x - y\|$,

(ii) *Θ is monotone quasi-nonexpansive if $F(\Theta) \neq 0$ and $p \in F(\Theta)$ with $x \leq p$ or $p \leq x$.*

Proof. (i) From Lemma 2.5(iii a), we have for all $x, y \in \vartheta$ either

$$(a)\ \ \|\Theta x - \Theta y\| \leq \alpha\|\Theta x - y\| + \alpha\|\Theta y - x\| + (1-2\alpha)\|x - y\| \quad or$$

$$(b)\ \ \|\Theta^2 x - \Theta y\| \leq \alpha\|\Theta^2 x - y\| + \alpha\|\Theta x - \Theta y\| + (1-2\alpha)\|\Theta x - y\|$$

In the first case, since $\|\Theta x - \Theta^2 x\| \leq \|x - \Theta x\|$, we have

$$\begin{aligned}
\|\Theta x - \Theta y\| &\leq \alpha\|\Theta x - y\| + \alpha\|\Theta y - x\| + (1-2\alpha)\|x - y\| \\
&\leq \alpha(\|\Theta x - \Theta^2 x\| + \|\Theta^2 x - y\|) + \alpha(\|\Theta x - \Theta y\| + \|x - \Theta x\|) \\
&\quad +(1-2\alpha)\|x - y\| \\
&\leq 2\alpha\|x - \Theta x\| + \alpha\|\Theta^2 x - y\| + \alpha\|\Theta x - \Theta y\| + (1-2\alpha)\|x - y\| \\
&\leq 2\alpha\|x - \Theta x\| + \alpha(\|\Theta^2 x - x\| + \|x - y\|) + \alpha\|\Theta x - \Theta y\| + (1-2\alpha)\|x - y\| \\
&\leq 2\alpha\|x - \Theta x\| + \alpha(\|\Theta^2 x - \Theta x\| + \|\Theta x - x\|) + \alpha\|\Theta x - \Theta y\| \\
&\quad +\alpha\|x - y\| + (1-2\alpha)\|x - y\| \\
&\leq 2\alpha\|x - \Theta x\| + 2\alpha\|\Theta x - x\| + \alpha\|\Theta x - \Theta y\| + (1-\alpha)\|x - y\| \\
&= 4\alpha\|x - \Theta x\| + \alpha\|\Theta x - \Theta y\| + (1-\alpha)\|x - y\| \qquad (4.1)
\end{aligned}$$

which implies that

$$\|\Theta x - \Theta y\| \leq \frac{4\alpha}{1-\alpha}\|x - \Theta x\| + \|x - y\| \qquad (4.2)$$

In the other case of Lemma 2.5 (iii b), we further have

$$\begin{aligned}
\|\Theta x - \Theta y\| &\leq \|x - \Theta x\| + \|x - \Theta y\| \\
&\leq \|x - \Theta x\| + (\|x - \Theta^2 x\| + \|\Theta^2 x - \Theta y\|) \\
&\leq \|x - \Theta x\| + (\|x - \Theta x\| + \|\Theta x - \Theta^2 x\|) + (\alpha\|\Theta^2 x - y\| + \alpha\|\Theta x - \Theta y\| \\
&\quad +(1-2\alpha)\|\Theta x - y\|) \\
&\leq 3\|x - \Theta x\| + \alpha(\|\Theta^2 x - \Theta x\| + \|\Theta x - y\|) + \alpha\|\Theta x - \Theta y\| \\
&\quad +(1-2\alpha)\|\Theta x - y\| \\
&= 3\|x - \Theta x\| + \alpha\|\Theta^2 x - \Theta x\| + \alpha\|\Theta x - \Theta y\| + \alpha\|\Theta x - y\| \\
&\quad +(1-2\alpha)\|\Theta x - y\| \\
&\leq 3\|x - \Theta x\| + \alpha\|x - \Theta x\| + \alpha\|\Theta x - \Theta y\| + (1-\alpha)\|\Theta x - y\| \\
&\leq 3\|x - \Theta x\| + \alpha\|x - \Theta x\| + \alpha\|\Theta x - \Theta y\| \\
&\quad +(1-\alpha)(\|\Theta x - x\| + \|x - y\|) \\
&\leq 4\|x - \Theta x\| + \alpha\|\Theta x - Ty\| + (1-\alpha)\|x - y\| \qquad (4.3)
\end{aligned}$$

which implies

$$\|\Theta x - \Theta y\| \leq \frac{4}{1-\alpha}\|x - \Theta x\| + \|x - y\|. \tag{4.4}$$

The desired conclusion follows from (4.2) and (4.4) for all $x, y \in K$ and $\alpha \in [0,1)$

(ii) By the definition of monotone generalized α-nonexpansive mapping, we have

$$\begin{aligned} \|\Theta x - p\| &= \|\Theta x - \Theta p\| \\ &\leq \alpha\|\Theta x - p\| + \alpha\|\Theta p - x\| + (1-2\alpha)\|x - p\| \\ &= \alpha\|\Theta x - p\| + (1-\alpha)\|x - p\| \end{aligned} \tag{4.5}$$

where $p \in F(\Theta)$, and so $\|\Theta x - p\| \leq \|x - p\|$, that is Θ is monotone quasi nonexpansive.

□

Theorem 4.2. *Let ϑ be a nonempty closed convex subset of a uniformly convex ordered Banach space $(\varpi, \leq)$ and $\Theta : \vartheta \to \vartheta$ be a monotone generalized α-nonexpansive mapping. Suppose that the sequence $\{x_k\}$ defined by (1.10) is bounded and $\liminf_{k\to\infty} \|x_k - \Theta x_k\| = 0$. Then $F_{\geq}(\Theta) \neq \phi$.*

Proof. Since $\{x_k\}$ is a bounded sequence and $\liminf_{k\to\infty} \|x_k - \Theta x_k\| = 0$, then there exists a subsequence $\{x_{k_i}\}$ of $\{x_k\}$ such that

$$\lim_{k\to\infty} \|x_{k_i} - \Theta x_{k_i}\| = 0 \tag{4.6}$$

The asymptotic center of $\{x_{k_i}\}$ with respect to ϑ is denoted by $A(\vartheta, \{x_{k_i}\}) = \{x^*\}$ such that $x_{k_i} \leq x^*$ for all $k \in \aleph$ such that x^* is unique. From the definition of asymptotic radius, we have

$$r(\Theta x^*, \{x_{k_i}\}) = \limsup_{k\to\infty} \|x_{k_i} - \Theta x^*\| \tag{4.7}$$

Using Lemma 4.1(i) and (4.6), we further obtain

$$\begin{aligned} r(\Theta x^*, \{x_{k_i}\}) &\leq \limsup_{k\to\infty} [\|x_{k_i} - Tx_{k_i}\| + \|\Theta x_{k_i} - \Theta x^*\|] \\ &= \limsup_{k\to\infty} \|\Theta x_{k_i} - \Theta x^*\| \\ &\leq \limsup_{k\to\infty} [\frac{4}{1-\alpha}\|x_{k_i} - \Theta x_{k_i}\| + \|x_{k_i} - x^*\|] \\ &= r(x^*, \{x_{k_i}\}) \end{aligned} \tag{4.8}$$

It follows from the uniqueness of x^* that $\Theta x^* = x^*$ which shows that $F(\Theta) \neq 0$. □

Theorem 4.3. *Let ϑ be a nonempty closed convex subset of uniformly convex ordered Banach space $(\varpi, \leq)$ and $\Theta : \vartheta \to \vartheta$ be a monotone generalized α-nonexpansive mapping. Assume $\{x_n\}$ defined by (1.10) is bounded and there exists $x_1 \in \vartheta$ such that $x_1 \leq \Theta x_1$ (or $\Theta x_1 \leq x_1$) . Let $F_{\geq}(\Theta) \neq \phi$ (or $F_{\leq}(\Theta) \neq \phi$) and $x_1 \leq p$, for every $p \in F(\Theta)$. Then the following assertions hold:*

(1) $\|x_{k+1} - p\| \leq \|x_k - p\|$ and the limit $\limsup_{k\to\infty} \|x_k - p\|$ exists for all $p \in F_{\geq}(\Theta)$;

(2) $\liminf_{k\to\infty} \|x_k - \Theta x_k\| = 0$, provided $\limsup_{k\to\infty} b_k(1-b_k) > 0$.

(3) $\lim_{k\to\infty} \|x_k - \Theta x_k\| = 0$, provided $\liminf_{k\to\infty} b_k(1-b_k) > 0$.

Proof. (1) By Lemma 4.1(ii) and (1.10), we obtain that

$$\begin{aligned} \|w_k - p\| &= \|\Theta((1-c_k)x_k + c_k\Theta x_k) - p\| \\ &\leq (1-c_k)\|x_k - p\| + c_k\|\Theta x_k - p\| \\ &\leq \|x_k - p\| \end{aligned} \tag{4.9}$$

Also, from (1.10) and (4.9)

$$\begin{aligned} \|z_k - p\| &= \|\Theta((1-a_k)\Theta x_k + a_k\Theta w_k) - p\| \\ &\leq (1-a_k)\|x_k - p\| + a_k\|w_k - p\| \\ &\leq \|x_k - p\| \end{aligned} \tag{4.10}$$

Again, from (1.10) , (4.9) and (4.10) we have

$$\begin{aligned} \|y_k - p\| &= \|\Theta((1-b_k)\Theta w_k + b_k\Theta z_k) - p\| \\ &\leq (1-b_k)\|w_k - p\| + b_k\|z_k - p\| \\ &\leq \|x_k - p\| \end{aligned} \tag{4.11}$$

Further, from (1.10) and (4.11) we have that

$$\begin{aligned} \|x_{k+1} - p\| &= \|\Theta y_k - p\| \\ &\leq \|y_k - p\| \\ &\leq \|x_k - p\| \end{aligned} \tag{4.12}$$

Thus, the sequence $\{\|x_k - p\|\}$ is nonincreasing and bounded, hence $\lim_{k\to\infty} \|x_k - p\|$ exists.

Now,

$$\begin{aligned} \|x_{k+1} - p\|^2 &= \|\Theta y_k - p\|^2 \\ &\leq \|y_k - p\|^2 \\ &= \|\Theta[(1-b_k)\Theta w_k + b_k\Theta z_k] - p\|^2 \\ &\leq (1-b_k)\|\Theta w_k - p\|^2 + b_k\|\Theta z_k - p\|^2 - b_k(1-b_k)f(\|x_k - \Theta x_k\|) \\ &\leq (1-b_k)\|x_k - p\|^2 + b_k\|x_k - p\|^2 - b_k(1-b_k)f(\|x_k - \Theta x_k\|) \\ &= \|x_k - p\|^2 - b_k(1-b_k)f(\|x_k - \Theta x_k\|) \end{aligned} \tag{4.13}$$

which implies that

$$b_k(1-b_k)f(\|x_k - \Theta x_k\|) \leq \|x_k - p\|^2 - \|x_{k+1} - p\|^2. \tag{4.14}$$

Letting $k \to \infty$, it follows from (1) that

$$\limsup_{k\to\infty} b_k(1-b_k)f(\|x_k - \Theta x_k\|) = 0 \tag{4.15}$$

(2) By condition (2) $\limsup_{k\to\infty} b_k(1-b_k) > 0$, and since $(\limsup_{k\to\infty} b_k(1-b_k))(\liminf_{k\to\infty} f(\|x_k - \Theta x_k\|)) \leq \limsup_{k\to\infty} b_k(1-b_k)f(\|x_k - \Theta x_k\|)$, then by (4.15), we have $\liminf_{k\to\infty} f(\|x_k - \Theta x_k\|) = 0$,
and by the property of f $\liminf_{k\to\infty} f(\|x_k - \Theta x_k\|) = 0$.

(3) Again by the assumption of (3), $\liminf_{k\to\infty} b_k(1-b_k) > 0$, and since $(\liminf_{k\to\infty} b_k(1-b_k))(\limsup_{k\to\infty} f(\|x_k-\Theta x_k\|)) \leq \limsup_{k\to\infty} b_k(1-b_k)f(\|x_k-\Theta x_k\|)$, then by (4.15), we have $\lim_{k\to\infty} f(\|x_k-\Theta x_k\|) = \limsup_{k\to\infty} f(\|x_k-\Theta x_k\|) = 0$, and by property of f $\lim_{k\to\infty}\|x_k-\Theta x_k\| = 0$. This completes the proof.

□

Theorem 4.4. *Let ϑ be a nonempty closed convex subset of a uniformly convex ordered Banach space $(\varpi,\leq)$ and $\Theta:\vartheta\to\vartheta$ be a monotone generalized α-nonexpansive mapping. Assume that ϖ satisfies Opials condition and the sequence $\{x_k\}$ defined by the iteration process (1.10) is bounded and there exists $x_1\in\vartheta$ such that $x_1\leq\Theta x_1$. Let $F_{\geq}(\Theta)\neq\phi$ and $x_1\leq q$, for every $q\in F(\Theta)$ and $\liminf_{k\to\infty} b_k(1-b_k)>0$, then the sequence $\{x_k\}$ converges weakly to a fixed point q of Θ.*

Proof. By the boundedness of $\{x_k\}$, there exists a subsequence $\{x_{k_i}\}\subset\{x_k\}$ weakly converging to a point $q\in\vartheta$ and $x_1\leq x_{k_i}\leq q$.

From Lemma (4.1) (i) and Theorem (4.3) (3) we can obtain

$$\begin{aligned}\limsup_{i\to\infty}\|\Theta x_{k_i}-\Theta q\| &\leq \limsup_{i\to\infty}[\frac{4}{1-\alpha}\|x_{k_i}-\Theta x_{k_i}\|+\|x_{k_i}-q\|]\\ &= \limsup_{i\to\infty}\|x_{k_i}-q\| \end{aligned}\tag{4.16}$$

Arguing by contradiction, we suppose that $q\neq\Theta q$. It follows from the Opial property of ϖ that

$$\begin{aligned}\limsup_{i\to\infty}\|x_{k_i}-q\| &< \limsup_{k\to\infty}\|x_{k_i}-\Theta q\|\\ &\leq \limsup_{i\to\infty}\|x_{k_i}-\Theta x_{k_i}\|+\limsup_{i\to\infty}\|\Theta x_{k_i}-\Theta q\|\\ &\leq \limsup_{i\to\infty}\|x_{k_i}-q\| \end{aligned}\tag{4.17}$$

This is a contradiction.Therefore, we conclude $q=\Theta q$; that is, $q\in F(\Theta)$.

Next, we show the uniqueness of the fixed point:
Now, suppose there exists another subsequence $\{x_{k_j}\}\subset\{x_n\}$ which converges weakly to $w\neq q$, then we have that $w\in F(\Theta)$. Note that $\lim_{k\to\infty}\|x_k-w\|$ exists and

$$\begin{aligned}\lim_{k\to\infty}\|x_k-q\| &= \limsup_{i\to\infty}\|x_{k_i}-q\|\\ &< \limsup_{i\to\infty}\|x_{k_i}-w\| = \lim_{i\to\infty}\|x_k-w\|\\ &= \limsup_{j\to\infty}\|x_{k_j}-w\|\\ &< \limsup_{j\to\infty}\|x_{k_j}-q\| = \lim_{k\to\infty}\|x_k-q\| \end{aligned}\tag{4.18}$$

This is a contradiction again. Consequently, $w=q$ and $\{x_k\}$ converges weakly to $q\in F_{\geq}(\Theta)$. □

Theorem 4.5. *Let ϑ be a nonempty closed convex subset of a uniformly convex ordered Banach space $(\varpi,\leq)$ and $\Theta:\vartheta\to\vartheta$ be a monotone generalized α-nonexpansive mapping. Assume the sequence $\{x_k\}$ defined by the iteration process (1.10) is bounded and there exists $x_1\in\vartheta$ such that $x_1\leq\Theta x_1$. Let $F_{\geq}(\Theta)\neq\phi$ and $x_1\leq p$, for every $p\in F(\Theta)$ and If $\limsup_{k\to\infty} b_k(1-b_k)>0$, then the sequence $\{x_k\}$ converges strongly to a fixed point $p\in F_{\geq}(\Theta)$.*

Proof. If $\{x_k\}$ converges strongly to a point $p \in F_{\geq}(\Theta)$, then $\lim_{k\to\infty} \|x_k - p\| = 0$.
Since $0 \leq d(x_k, F_{\geq}(\Theta)) \leq \|x_k - p\|$, then, $\liminf_{k\to\infty} d(x_k, F_{\geq}(\Theta)) = 0$.

Conversely, suppose that $\liminf_{k\to\infty} d(x_k, F_{\geq}(\Theta)) = 0$.
From (4.12), $\lim_{k\to\infty} d(x_k, F_{\geq}(\Theta))$ exists. Thus,

$$\lim_{k\to\infty} d(x_k, F_{\geq}(\Theta)) = 0. \tag{4.19}$$

By Theorem (4.3), we have that $\{x_k\}$ is bounded with $x_k \leq p$.
WLOG, let $\{x_{k_j}\}$ be a subsequence of $\{x_k\}$ such that $\|x_{k_j} - p_j\| \leq 1/2^j$ for all $j \geq 1$, where $\{p_j\}$ is a sequence in $F_{\geq}(\Theta)$. Combining with (4.12), we have

$$\|x_{k_{j+1}} - p_j\| \leq \|x_{k_j} - p_j\| \leq 1/2^j \tag{4.20}$$

It follows from (4.20) that

$$\begin{aligned} \|p_{j+1} - p_j\| &\leq \|p_{j+1} - x_{k_{j+1}}\| + \|x_{k_{j+1}} - p_j\| \\ &\leq \frac{1}{2^{j+1}} + \frac{1}{2^j} \leq \frac{1}{2^{j-1}} \quad \to 0 \quad as \quad j \to \infty \end{aligned} \tag{4.21}$$

This shows that $\{p_j\}$ is a Cauchy sequence in $F_{\geq}(\Theta)$.
By Lemma (2.6), $F_{\geq}(\Theta)$ is closed, so $\{p_j\}$ converges to some $q \in F_{\geq}(\Theta)$.
Moreover, by the triangle inequality, we have

$$\|x_{k_{j+1}} - q\| \leq \|x_{k_j} - p_j\| + \|p_j - q\| \tag{4.22}$$

Taking $j \longrightarrow \infty$ implies that x_{k_j} converges strongly to q.
From (4.12) again, $\lim_{k\to\infty} \|x_k - q\|$ exists, and the sequence $\{x_k\}$ converges strongly to $q \in F_{\geq}(\Theta)$.

□

Theorem 4.6. *Let ϑ be a nonempty compact, closed, convex subset of a uniformly convex ordered Banach space $(\varpi, \leq)$ and $\Theta : \vartheta \to \vartheta$ be a monotone generalized α-nonexpansive mapping. Assume the sequence $\{x_k\}$ defined by the iteration process (1.10) is bounded and there exists $x_1 \in \vartheta$ such that $x_1 \leq \Theta x_1$ (or $\Theta x_1 \leq x_1$) . Let $F_{\geq}(\Theta) \neq \phi$ (or $F_{\leq}(\Theta) \neq \phi$) and $x_1 \leq p$, for every $p \in F(\Theta)$. Then, the sequence $\{x_k\}$ generated by (1.10) converges strongly to a fixed point $p \in F_{\geq}(\Theta)$ if and only if $\liminf_{k\to\infty} d(x_k, F_{\geq}(\Theta)) = 0$, where $d(x_k, F_{\geq}(\Theta))$ denotes the distance from x to $F_{\geq}(\Theta)$.*

Proof. Following the compactness of ϑ, there exists a subsequence $\{x_{k_i}\} \subset \{x_k\}$ such that $\{x_{k_i}\}$ converges strongly to a point $p \in \vartheta$. Since $\{x_k\}$ is bounded, it follows that $x_1 \leq x_{k_i} \leq p$ for all $i \geq 1$.
By Theorem (4.2), we have that $F_{\geq}(\Theta) \neq \phi$. It follows from Theorem (4.3) that $\{x_k\}$ is bounded and $\liminf_{k\to\infty} \|x_k - \Theta x_k\| = 0$.
WLOG, we can assume that $\lim_{k\to\infty} \|x_{k_i} - \Theta x_{k_i}\| = 0$.

On the other hand, Lemma (4.1)(i) guarantees that

$$\|\Theta x_{k_i} - \Theta p\| \leq \frac{4}{1-\alpha}\|x_{k_i} - \Theta x_{k_i}\| + \|x_{k_i} - p\| \tag{4.23}$$

By the boundedness of the sequence $\{x_{k_i}\}$,
$\lim_{i\to\infty} \|x_{k_i} - p\| = 0$ and $\lim_{i\to\infty} \|x_{k_i} - \Theta x_{k_i}\| = 0$,
and we have that

$$\limsup_{i\to\infty} \|\Theta x_{k_i} - \Theta p\| \leq 0 \tag{4.24}$$

which implies that $\lim_{i\to\infty} \|\Theta x_{k_i} - \Theta p\| = 0$.

Therefore, we have

$$\limsup_{i\to\infty} \|x_{k_i} - \Theta p\| \le \limsup_{i\to\infty} (\|x_{k_i} - \Theta x_{k_i}\| + \|\Theta x_{k_i} - \Theta p\|) = 0, \tag{4.25}$$

which implies that $p \in F_{\geq}(\Theta)$.
By Theorem 4.3(1), $\lim_{k\to\infty} \|x_k - p\|$ exists and so $\lim_{k\to\infty} \|x_k - p\| = 0$. □

Theorem 4.7. *Let ϑ be a nonempty closed convex subset of a uniformly convex ordered Banach space $(\varpi, \leq)$ and $\Theta : \vartheta \to \vartheta$ be a monotone generalized α-nonexpansive mapping. Assume that ϖ satisfies Opials condition and the sequence $\{x_k\}$ defined by the iteration process (1.10) is bounded and there exists $x_1 \in \vartheta$ such that $\Theta x_1 \leq x_1$. Let $F_{\leq}(\Theta) \neq \phi$ and $x_1 \leq p$, for every $p \in F(\Theta)$ and $\liminf_{k\to\infty} b_k(1 - b_k) > 0$, then the sequence $\{x_k\}$ converges weakly to a fixed point p of Θ.*

Theorem 4.8. *Let ϑ be a nonempty closed convex subset of a uniformly convex ordered Banach space $(\varpi, \leq)$ and $\Theta : \vartheta \to \vartheta$ be a monotone generalized α-nonexpansive mapping. Assume the sequence $\{x_k\}$ defined by the iteration process (1.10) is bounded and there exists $x_1 \in \vartheta$ such that $\Theta x_1 \leq x_1$. Let $F_{\leq}(\Theta) \neq \phi$ and $x_1 \leq p$, for every $p \in F(\Theta)$ and If $\limsup_{k\to\infty} b_k(1 - b_k) > 0$, then the sequence $\{x_k\}$ converges strongly to a fixed point $p \in F_{\leq}(\Theta)$.*

Corollary 4.9. *Let ϑ be a nonempty closed convex subset of a uniformly convex Banach space ϖ and $\Theta : \vartheta \to \vartheta$ be a generalized α-nonexpansive mapping. Assume that ϖ satisfies Opials condition and the sequence $\{x_k\}$ defined by the iteration process (1.10) is bounded. Let $F(\Theta) \neq \phi$ and for every $p \in F(\Theta)$ and $\liminf_{k\to\infty} b_k(1 - b_k) > 0$, then the sequence $\{x_k\}$ converges weakly to a fixed point p of Θ.*

Corollary 4.10. *Let ϑ be a nonempty closed convex subset of a uniformly convex Banach space ϖ and $\Theta : \vartheta \to \vartheta$ be a generalized α-nonexpansive mapping. Assume that the sequence $\{x_k\}$ defined by the iteration process (1.10) is bounded. Let $F(\Theta) \neq \phi$ and for every $p \in F(\Theta)$ and $\limsup_{k\to\infty} b_k(1 - b_k) > 0$, then the sequence $\{x_k\}$ converges strongly to a fixed point p of Θ.*

Corollary 4.11. *Let ϑ be a nonempty compact, closed, convex subset of a uniformly convex Banach space ϖ and $\Theta : \vartheta \to \vartheta$ be a generalized α-nonexpansive mapping. Assume the sequence $\{x_k\}$ defined by the iteration process (1.10) is bounded. Let $F(\Theta) \neq \phi$ and for every $p \in F(\Theta)$. Then, the sequence $\{x_k\}$ generated by (1.10) converges strongly to a fixed point $p \in F(\Theta)$ if and only if $\liminf_{k\to\infty} d(x_k, F(\Theta)) = 0$, where $d(x_k, F(\Theta))$ denotes the distance from x to $F(\Theta)$.*

5. APPLICATION

In this section, we will use our UI iteration process (1.10) to find the solution of split feasibility problem. Let H_1 and H_2 be two real Hilbert spaces, C and Q be a nonempty, closed and convex subsets of H_1 and H_2, respectively and let $A : H_1 \to H_2$ be a bounded linear operator. Then, the split feasibility problem (SFP) can be mathematically described as finding a point $x \in C$ such that

$$x \in C, \quad Ax \in Q. \tag{5.1}$$

We assume that the solution set Ω of the SFP (5.1) is nonempty. Let

$$\Omega = \{x \in C : Ax \in Q\} = C \cap A^{-1}Q.$$

Then, Ω is a nonempty, closed and convex set. Censor and Elfving [33] solved the class of inverse problems with the help of SFP. In 2002, Byrne [4] introduced the famous CQ-algorithm for solving the SFP. In this, the iterative step x_k is calculated as follows:

$$x_{k+1} = P_C[I - \gamma A^*(I - P_Q)A]x_k, \quad k \geq 0 \tag{5.2}$$

where $0 < \gamma < \frac{2}{\|A^2\|}$, P_C and P_Q denote the projections onto sets C and Q respectively and $A^* : H_2^* \to H_1^*$ is the adjoint of A.
We have the following important lemma due to Feng et al. [23]

Lemma 5.1. *Let operator* $\Theta = P_C[I - \gamma A^*(I - P_Q)A]$, *where* $0 < \gamma < \frac{2}{\|A^2\|}$. *Then* Θ *is a nonexpansive map.*

Also, since we have assumed that solution set Ω of SFP is nonempty, it is easy to see that any $x^* \in C$ is the solution of SFP if and only if it solves the following fixed point equation:

$$P_C[I - \gamma A^*(I - P_Q)A]x = x, \quad x \in C.$$

So, the solution set Ω is equal to the fixed point set of Θ , i.e, $F(\Theta) = \Omega = C \cap A^{-1}Q \neq \phi$. For details, one can refer to ([12] , [13]).

Now, we present our main results.

Theorem 5.2. *If* $\{x_k\}$ *is the sequence generated by the iterative algorithm* (1.10) *with* $\Theta = P_C[I - \gamma A^*(I - P_Q)A]$ *then,* $\{x_k\}$ *converges weakly to the solution of SFP* (5.1)

Proof. By Lemma (5.1), Θ is a nonexpansive map and every nonexpansive mapping is a generalized 0-nonexpansive mapping, so the result follows from Theorem (4.4) . □

Theorem 5.3. *If* $\{x_k\}$ *is the sequence generated by the iterative algorithm* (1.10) *with* $\Theta = P_C[I - \gamma A^*(I - P_Q)A]$ *then,* $\{x_k\}$ *converges strongly to the solution of SFP* (5.1) *if and only if* $\liminf_{k\to\infty} d(x_k, \Omega) = 0$.

Proof. Proof follows from Theorem (4.6) . □

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