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On Hardy-Rogers Type Contraction Mappings in Cone A_b-metric spaces

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Abstract

In this manuscript, a generalized fixed point theorem of Hardy-Rogers type contraction is proved in cone A_b-metric spaces, which relaxes the contraction condition. Also, some fixed point results for different contraction mappings are given in such spaces.

Keywords: Cone A_b-metric spaces, Hardy-Rogers Type Contractions, Fixed point
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1. Introduction and Preliminaries

Ever since S. Banach proved the Banach fixed point theorem in 1922, many authors have tried to generalize this conclusion. Usually these studies have been obtained by generalizing the concept of metric space or by generalizing the contraction mappings. There are different generalizations of metric space in the literature. Some of them are b-metric space [7], cone metric space [9], cone b-metric space [10], A-metric space [1], A_b-metric space [14] and cone A_b-metric space [13]. Many authors were proved fixed point theorems for analogue of Banach, Kannan, Chatterjea, Reich and Cirić contraction principles and for various generalized contractions in these generalized spaces (see, e.g., ([1], [2], [4], [10], [13], [14], [15], [17]).

In present manuscript, firstly, we will give an analogue of Hardy-Rogers type contraction in cone A_b-metric spaces. Secondly, we will prove a fixed point theorem for class of this mappings under appropriate conditions in cone A_b-metric spaces. Finally, we will give some results on fixed point using our main theorem.

We repeat some definitions and results, which will be needed in the sequel.

Definition 1.1. [9] Let V be a Banach space. A subset W of V is called a cone if and only if

1. W is non-empty closed and $W \neq \{0\}$;
2. $\alpha u + \beta v \in W$ for all $u, v \in W$ and non-negative real numbers α, β;
3. $W \cap (-W) = \{0\}$.

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Let V be a Banach space and $W \subset V$ be a cone. The partial ordering of the elements in V is defined as $u \leq v$ if and only if $v - u \in W$. Moreover, we will indicate that

$$\begin{aligned} u &< v \text{ iff } u \leq v \text{ and } u \neq v, \\ u &\ll v \text{ iff } v - u \in intW, \end{aligned}$$

where $intW$ denotes the interior of W. A cone W is called normal if there is a positive real number K such that

$$0 \leq u \leq v \text{ implies } \parallel u \parallel \leq K \parallel v \parallel$$

for all $u, v \in V$.

Definition 1.2. [13] Suppose that V is a real Banach space, W is a cone in V with $intW \neq \emptyset$ and "$\leq$" is partial ordering in V with respect to W. Let U be nonempty set. Suppose a mapping $A : U^t \to V$ satisfies the following conditions:

1) $0 \leq A(u_1, u_2, ..., u_{t-1}, u_t)$,
2) $A(u_1, u_2, ..., u_{t-1}, u_t) = 0$ if and only if $u_1 = u_2 = ... = u_{t-1} = u_t$,
3) $A(u_1, u_2, ..., u_{t-1}, u_t) \leq b \left[\begin{array}{c} A(u_1, u_1, ..., (u_1)_{t-1}, v) + A(u_2, u_2, ..., (u_2)_{t-1}, v) + ... \\ +A(u_{t-1}, u_{t-1}, ..., (u_{t-1})_{t-1}, v) + A(u_t, u_t, ..., (u_t)_{t-1}, v) \end{array} \right]$

for any $u_i, v \in U$, $(i = 1, 2, ..., t)$ and $b \geq 1$. Then, (U, A) is said to be a cone A_b-metric space.

It is clear that cone A_b- metric spaces is generalization ordinary metric spaces, b-metric space, cone metric space, cone b-metric space, A-metric spaces and cone A-metric spaces.

Example 1.3. [13] Let $V = \mathbb{R}^2$ and $W = \{(u, v) \in V : u, v \geq 0\}$ a normal cone in V. Let $U = \mathbb{R}$ and $A : U^t \to V$ be such that

$$A(u_1, u_2, ..., u_{t-1}, u_t) = A_*(u_1, u_2, ..., u_{t-1}, u_t)(\alpha, \beta)$$

where $\alpha, \beta > 0$ are constants and A_* is an A_b-metric on U. Then (U, A) is a cone A_b- metric space.

Lemma 1.4. *[13] Let (U, A) be a cone A_b-metric space. Then for all $u, v, z \in U$,*

1) $d(u, u, ..., u, v) \leq bd(v, v, ..., v, u)$,
2) $d(u, u, ..., u, z) \leq b(t - 1)d(u, u, ..., u, v) + bd(z, z, ..., z, v)$.

Lemma 1.5. *[13] Let (U, A) be a cone A_b-metric space and, let W be a normal cone with normal constant K.*

i) The sequence $\{u_n\}$ in U converges to u if and only if $A(u_n, u_n, ..., u_n, u) \to 0$ as $n \to \infty$.

ii) Let $\{u_n\}$ be a sequence in U. If $\{u_n\}$ converges to u and $\{u_n\}$ converges to v, then $u = v$. That is, the limit of a convergent sequence is unique.

iii) The sequence $\{u_n\}$ in U is a Cauchy sequence if and only if $A(u_n, u_n, ..., u_n, u_m) \to 0$ as $n, m \to \infty$.

2. Main Results

In this section, we first introduce the notion of Hardy-Rogers type contraction mappings in cone A_b-metric space as follows:

Definition 2.1. Let (U, A) be a cone A_b-metric space and $T : U \to U$ be a mapping. T is called a Hardy-Rogers type contraction mapping, if and only if, there exist $\alpha, \beta, \gamma \in \mathbb{R}^+$ with $\alpha b^2 + \beta \left(b^2 + 1\right) + \gamma b^2 \left[1 + b\left(t - 1\right)\right] < 1$ and $b < t$ such that for all $u, v \in U$,

$$\begin{aligned} A(Tu, Tu, \ldots, Tu, Tv) \leq\ & \alpha A(u, u, \ldots, u, v) + \\ & \beta \left[A(u, u, \ldots, u, Tu) + A(v, v, \ldots, v, Tv)\right] \\ & +\gamma \left[A(u, u, \ldots, u, Tv) + A(v, v, \ldots, v, Tu)\right]. \end{aligned} \tag{2.1}$$

It is clear that if we take $t = 2$ and $b = 1$ in the Definition 2.1, we obtain the contractive definition of Hardy-Rogers (2.2) in ordinary metric space (U, d) as follows:

Let (U, d) be a metric space and let $T : U \to U$ be a mapping. Then there exists constants $\alpha, \beta, \gamma \geq 0$ such that

$$d(Tu, Tv) \leq \alpha d(u, v) + \beta [d(u, Tu) + d(v, Tv)] + \gamma [d(u, Tv) + d(v, Tu)] \tag{2.2}$$

for all $u, v \in U$, where $\alpha + 2\beta + 2\gamma < 1$.
Our main result is as follows.

Theorem 2.2. *Let (U, A) be a complete cone A_b-metric space and let W be a normal cone with normal constant K. Suppose the mapping $T : U \to U$ is a Hardy-Rogers type contraction mapping as Definition 2.1. Then T has a unique fixed point in T and Picard iteration $\{u_n\}$ defined by $u_{n+1} = Tu_n$ converges to a fixed point of T.*

Proof. Let $u_0 \in X$ and a sequence $\{u_n\}$ be defined by $u_{n+1} = Tu_n$. Suppose $u_n \neq u_{n+1}$ for all n.

Taking $u = u_n$ and $v = u_{n+1}$ at the inequality (2.1), we get

$$\begin{aligned} A(u_n, u_n, \ldots, u_n, u_{n+1}) &= A(Tu_{n-1}, Tu_{n-1}, \ldots, Tu_{n-1}, Tu_n) \\ &\leq \alpha A(u_{n-1}, u_{n-1}, \ldots, u_{n-1}, u_n) + \\ &\quad \beta \left[\begin{array}{c} A(u_{n-1}, u_{n-1}, \ldots, u_{n-1}, Tu_{n-1}) \\ +A(u_n, u_n, \ldots, u_n, Tu_n) \end{array} \right] \\ &\quad +\gamma \left[\begin{array}{c} A(u_{n-1}, u_{n-1}, \ldots, u_{n-1}, Tu_n) \\ +A(u_n, u_n, \ldots, u_n, Tu_{n-1}) \end{array} \right] \\ &= \alpha A(u_{n-1}, u_{n-1}, \ldots, u_{n-1}, u_n) \\ &\quad +\beta \left[\begin{array}{c} A(u_{n-1}, u_{n-1}, \ldots, u_{n-1}, u_n) \\ +A(u_n, u_n, \ldots, u_n, u_{n+1}) \end{array} \right] \\ &\quad +\gamma [A(u_{n-1}, u_{n-1}, \ldots, u_{n-1}, u_{n+1})]. \end{aligned} \tag{2.3}$$

Applying Lemma 1.4 and the inequality (2.3), we get

$$\begin{aligned} A(u_n, u_n, \ldots, u_n, u_{n+1}) &\leq \alpha A(u_{n-1}, u_{n-1}, \ldots, u_{n-1}, u_n) \\ &\quad +\beta \left[\begin{array}{c} A(u_{n-1}, u_{n-1}, \ldots, u_{n-1}, u_n) \\ +A(u_n, u_n, \ldots, u_n, u_{n+1}) \end{array} \right] \\ &\quad +\gamma b [(t-1) A(u_{n-1}, u_{n-1}, \ldots, u_{n-1}, u_n) \\ &\quad +A(u_{n+1}, u_{n+1}, \ldots, u_{n+1}, u_n)] \\ &\leq \alpha A(u_{n-1}, u_{n-1}, \ldots, u_{n-1}, u_n) \\ &\quad +\beta \left[\begin{array}{c} A(u_{n-1}, u_{n-1}, \ldots, u_{n-1}, u_n) \\ +A(u_n, u_n, \ldots, u_n, u_{n+1}) \end{array} \right] \\ &\quad +\gamma b(t-1) A(u_{n-1}, u_{n-1}, \ldots, u_{n-1}, u_n) \\ &\quad +\gamma b^2 A(u_n, u_n, \ldots, u_n, u_{n+1}) \\ &= [\alpha + \beta + \gamma b(t-1)] A(u_{n-1}, u_{n-1}, \ldots, u_{n-1}, u_n) \\ &\quad + \left(\beta + \gamma b^2\right) A(u_n, u_n, \ldots, u_n, u_{n+1}). \end{aligned} \tag{2.4}$$

From (2.4), we have

$$\left[1 - \beta - \gamma b^2\right] A(u_n, u_n, \ldots, u_n, u_{n+1}) \leq [\alpha + \beta + \gamma b(t-1)] A(u_{n-1}, u_{n-1}, \ldots, u_{n-1}, u_n)$$

This implies

$$A(u_n, u_n, \ldots, u_n, u_{n+1}) \leq \left(\frac{\alpha + \beta + \gamma b(t-1)}{1 - \beta - \gamma b^2}\right) A(u_{n-1}, u_{n-1}, \ldots, u_{n-1}, u_n).$$

Using above inequality, we have

$$A\left(u_n, u_n, \ldots, u_n, u_{n+1}\right) \leq \delta A\left(u_{n-1}, u_{n-1}, \ldots, u_{n-1}, u_n\right)$$

where

$$\delta=\frac{\alpha+\beta+\gamma b(t-1)}{1-\beta-\gamma b^2}<1, \tag{2.5}$$

as $\alpha b^2+\beta\left(b^2+1\right)+\gamma b^2[1+b(t-1)]<1$.
Repeating iteratively, we obtain

$$\begin{aligned} A\left(u_n, u_n, \ldots, u_n, u_{n+1}\right) & \leq \delta A\left(u_{n-1}, u_{n-1}, \ldots, u_{n-1}, u_n\right) \\ & \leq \delta^2 A\left(u_{n-2}, u_{n-2}, \ldots, u_{n-2}, u_{n-1}\right) \\ & \vdots \\ & \leq \delta^n A\left(u_0, u_0, \ldots, u_0, u_1\right). \end{aligned} \tag{2.6}$$

Assume that $m>n$. Using Lemma 1.4 and the above inequality, we get

$$\begin{aligned} A\left(u_n, u_n, \ldots, u_n, u_m\right) \leq\; & b(t-1) A\left(u_n, u_n, \ldots, u_n, u_{n+1}\right)+ \\ & b^2 A\left(u_{n+1}, u_{n+1}, \ldots, u_{n+1}, u_m\right) \\ \leq\; & b(t-1) A\left(u_n, u_n, \ldots, u_n, u_{n+1}\right)+ \\ & b^2\left[\begin{array}{l} b(t-1) A\left(u_{n+1}, u_{n+1}, \ldots, u_{n+1}, u_{n+2}\right) \\ \quad b^2 A\left(u_{n+2}, u_{n+2}, \ldots, u_{n+2}, u_m\right) \end{array}\right. \\ =\; & b(t-1) A\left(u_n, u_n, \ldots, u_n, u_{n+1}\right)+ \\ & b^3(t-1) A\left(u_{n+1}, u_{n+1}, \ldots, u_{n+1}, u_{n+2}\right) \\ & +b^4 A\left(u_{n+2}, u_{n+2}, \ldots, u_{n+2}, u_m\right) \\ \leq\; & b(t-1) A\left(u_n, u_n, \ldots, u_n, u_{n+1}\right) \\ & +b^3(t-1) A\left(u_{n+1}, u_{n+1}, \ldots, u_{n+1}, u_{n+2}\right) \\ & +b^5(t-1) A\left(u_{n+2}, u_{n+2}, \ldots, u_{n+2}, u_{n+3}\right) \\ & +b^6 A\left(u_{n+}, u_{n+3}, \ldots, u_{n+3}, u_m\right) \\ \leq\; & b(t-1) A\left(u_n, u_n, \ldots, u_n, u_{n+1}\right) \\ & +b^3(t-1) A\left(u_{n+1}, u_{n+1}, \ldots, u_{n+1}, u_{n+2}\right) \\ & +b^5(t-1) A\left(u_{n+2}, u_{n+2}, \ldots, u_{n+2}, u_{n+3}\right)+\cdots \\ & +b^{2(m-n-1)} A\left(u_{n+(m-n-1)}, u_{n+(m-n-1)}, \ldots, u_{n+(m-n-1)}, u_m\right) \\ =\; & b(t-1) A\left(u_n, u_n, \ldots, u_n, u_{n+1}\right) \\ & +b^3(t-1) A\left(u_{n+1}, u_{n+1}, \ldots, u_{n+1}, u_{n+2}\right) \\ & +b^5(t-1) A\left(u_{n+2}, u_{n+2}, \ldots, u_{n+2}, u_{n+3}\right)+\cdots \\ & +b^{2(m-n-1)} A\left(u_{m-1}, u_{m-1}, \ldots, u_{m-1}, u_m\right). \end{aligned}$$

Applying (2.6) in above inequality, we have

$$
\begin{aligned}
A\left(u_n,u_n,\ldots,u_n,u_m\right) &\leq b\left(t-1\right)A\left(u_n,u_n,\ldots,u_n,u_{n+1}\right)\\
&\quad +b^3\left(t-1\right)A\left(u_{n+1},u_{n+1},\ldots,u_{n+1},u_{n+2}\right)\\
&\quad +b^5\left(t-1\right)A\left(u_{n+2},u_{n+2},\ldots,u_{n+2},u_{n+3}\right)+\cdots\\
&\quad +b^{2(m-n-1)}A\left(u_{m-1},u_{m-1},\ldots,u_{m-1},u_m\right)\\
&\leq b\left(t-1\right)\delta^n A\left(u_0,u_0,\ldots,u_0,u_1\right)\\
&\quad +b^3\left(t-1\right)\delta^{n+1}A\left(u_0,u_0,\ldots,u_0,u_1\right)\\
&\quad +b^5\left(t-1\right)\delta^{n+2}A\left(u_0,u_0,\ldots,u_0,u_1\right)+\cdots\\
&\quad +b^{2(m-n-1)}\delta^{m-1}A\left(u_0,u_0,\ldots,u_0,u_1\right)\\
&= b\left(t-1\right)\delta^n\left[1+b^2\delta+b^4\delta^2+\cdots+b^{2m-2n-2}\delta^{m-n-1}\right]\\
&\quad A\left(u_0,u_0,\ldots,u_0,u_1\right)\\
&\leq \frac{b\left(t-1\right)\delta^n}{1-b^2\delta}A\left(u_0,u_0,\ldots,u_0,u_1\right).
\end{aligned}
\tag{2.7}
$$

Since $\delta=\frac{\alpha+\beta+\gamma b(t-1)}{1-\beta-\gamma b^2}<1$ and $\alpha b^2+\beta b\left(b^2+1\right)+\gamma b^2\left(t-1+b\right)<1$, we obtain

$$
\begin{aligned}
\frac{\alpha b^2+\beta b^2+\gamma b^3(t-1)}{1-\beta-\gamma b^2} &< 1 \Rightarrow \frac{\alpha+\beta+\gamma b(t-1)}{1-\beta-\gamma b^2}<\frac{1}{b^2}<1\\
&\Rightarrow \delta=\frac{\alpha+\beta+\gamma b(t-1)}{1-\beta-\gamma b^2}<\frac{1}{b^2}\\
&\Rightarrow \delta b^2<1.
\end{aligned}
\tag{2.8}
$$

Since (U,A) is a cone A_b-metric space with normal constant K, using (2.7) we have

$$\left\|A\left(u_n,u_n,\ldots,u_n,u_m\right)\right\|\leq K\frac{b\left(t-1\right)\delta^n}{1-b^2\delta}\left\|A\left(u_0,u_0,\ldots,u_0,u_1\right)\right\|.$$

From the inequality (2.5), we know that $0\leq\delta<1$. Let $\left\|A\left(u_0,u_0,\ldots,u_0,u_1\right)\right\|>0$. If we take limit as $m,n\to\infty$ in above inequality, we get

$$\lim_{n,m\to\infty}\left\|A\left(u_n,u_n,\ldots,u_n,u_m\right)\right\|=0.$$

This implies that

$$\lim_{n,m\to\infty}A\left(u_n,u_n,\ldots,u_n,u_m\right)=0.$$

Thus $\{u_n\}$ is a Cauchy sequence in U. Also, suppose that $\left\|A\left(u_0,u_0,\ldots,u_0,u_1\right)\right\|=0$, then $A\left(u_n,u_n,\ldots,u_n,u_m\right)=0$ for all $m>n$ and $\{u_n\}$ is a Cauchy sequence in U. Since (U,A) is a complete metric space, the sequence $\{u_n\}$ converges to $u^*\in X$.

Now, we show that u^* is a fixed point of T. From (2.1), we get

$$\begin{aligned}
A(u^*,u^*,\dots,u^*,Tu^*) &\le b(t-1)A(u^*,u^*,\dots,u^*,Tu_n)+b^2A(Tu_n,Tu_n,\dots,Tu_n,Tu^*) \\
&\le b(t-1)A(u^*,u^*,\dots,u^*,u_{n+1}) \\
&\quad +b^2\alpha A(u_n,u_n,\dots,u_n,u^*) \\
&\quad +b^2\beta\left[A(u_n,u_n,\dots,u_n,Tu_n)+A(u^*,u^*,\dots,u^*,Tu^*)\right] \\
&\quad +b^2\gamma\left[A(u_n,u_n,\dots,u_n,Tu^*)+A(u^*,u^*,\dots,u^*,Tu_n)\right] \\
&= b(t-1)A(u^*,u^*,\dots,u^*,u_{n+1})+b^2\alpha A(u_n,u_n,\dots,u_n,u^*) \\
&\quad +b^2\beta A(u_n,u_n,\dots,u_n,u_{n+1})+b^2\beta A(u^*,u^*,\dots,u^*,Tu^*) \\
&\quad +b^2\gamma A(u_n,u_n,\dots,u_n,Tu^*)+b^2\gamma A(u^*,u^*,\dots,u^*,u_{n+1}) \\
&\le \left[b^2(t-1)+b^3\gamma\right]A(u_{n+1},u_{n+1},\dots,u_{n+1},u^*) \\
&\quad +b^2\alpha A(u_n,u_n,\dots,u_n,u^*)+b^2\beta A(u_n,u_n,\dots,u_n,u_{n+1}) \\
&\quad +b^2\beta A(u^*,u^*,\dots,u^*,Tu^*) \\
&\quad +b^2\gamma\left[\begin{array}{c} b(t-1)A(u_n,u_n,\dots,u_n,u^*) \\ +b^2A(u^*,u^*,\dots,u^*,Tu^*)\end{array}\right] \\
&= \left[b^2(t-1)+b^3\gamma\right]A(u_{n+1},u_{n+1},\dots,u_{n+1},u^*) \\
&\quad +b^2\beta A(u_n,u_n,\dots,u_n,u_{n+1}) \\
&\quad +\left[b^2\alpha+b^3\gamma(t-1)\right]A(u_n,u_n,\dots,u_n,u^*) \\
&\quad +\left(b^2\beta+b^4\gamma\right)A(u^*,u^*,\dots,u^*,Tu^*)
\end{aligned} \tag{2.9}$$

From (2.9), we get

$$\begin{aligned}
\left[1-\left(b^2\beta+b^4\gamma\right)\right]A(u^*,u^*,\dots,u^*,Tu^*) &\le \left[b^2(t-1)+b^3\gamma\right]A(u_{n+1},u_{n+1},\dots,u_{n+1},u^*) \\
&\quad +b^2\beta A(u_n,u_n,\dots,u_n,u_{n+1}) \\
&\quad +\left[b^2\alpha+b^3\gamma(t-1)\right]A(u_n,u_n,\dots,u_n,u^*).
\end{aligned}$$

This implies that

$$\begin{aligned}
A(u^*,u^*,\dots,u^*,Tu^*) &\le \frac{1}{1-(b^2\beta+b^4\gamma)}\left\{\begin{array}{c}\left[b^2(t-1)+b^3\gamma\right] \\ A(u_{n+1},u_{n+1},\dots,u_{n+1},u^*)\end{array}\right. \\
&\quad +b^2\beta A(u_n,u_n,\dots,u_n,u_{n+1}) \\
&\quad \left.+\left[b^2\alpha+b^3\gamma(t-1)\right]A(u_n,u_n,\dots,u_n,u^*)\right\}.
\end{aligned} \tag{2.10}$$

Using (2.8), we have

$$\frac{\alpha+\beta+\gamma b(t-1)}{1-\beta-\gamma b^2}<\frac{1}{b^2}.$$

This implies that

$$\frac{1}{b^2}>\frac{\alpha+\beta+\gamma b(t-1)}{1-(\beta+\gamma b^2)}>\alpha+\beta+\gamma b(t-1)>\beta+\gamma b^2 \tag{2.11}$$

for $b<t$.

It follows from (2.11) that

$$1-\left(b^2\beta+b^4\gamma\right)>0.$$

Using (2.10), we get

$$\begin{aligned}
\|A(u^*,u^*,\dots,u^*,Tu^*)\| &\le K\frac{1}{1-(b^2\beta+b^4\gamma)}\left\{\begin{array}{c}\left[b^2(t-1)+b^3\gamma\right] \\ \|A(u_{n+1},u_{n+1},\dots,u_{n+1},u^*)\|\end{array}\right. \\
&\quad +b^2\beta\|A(u_n,u_n,\dots,u_n,u_{n+1})\| \\
&\quad \left.+\left[b^2\alpha+b^3\gamma(t-1)\right]\|A(u_n,u_n,\dots,u_n,u^*)\|\right\}.
\end{aligned} \tag{2.12}$$

If we take limit for $n \to \infty$ in above inequality (2.12), we obtain

$$\lim_{n\to\infty} \|A(u^*, u^*, \ldots, u^*, Tu^*)\| = 0.$$

This implies that $A(u^*, u^*, \ldots, u^*, Tu^*) = 0$, that is, $Tu^* = u^*$. Therefore u^* is a fixed point of the mapping T. Finally, we show that the uniqueness of fixed point of T. Let v^* be another fixed point of T. That is, $Tu^* = u^*$ and $Tv^* = v^*$. Using (2.1), we obtain

$$\begin{aligned} A(u^*, u^*, \ldots, u^*, v^*) &= A(Tu^*, Tu^*, \ldots, Tu^*, Tv^*) \leq \alpha A(u^*, u^*, \ldots, u^*, v^*) \\ &\quad +\beta [A(u^*, u^*, \ldots, u^*, Tu^*) + A(v^*, v^*, \ldots, v^*, Tv^*)] \\ &\quad +\gamma [A(u^*, u^*, \ldots, u^*, Tv^*) + A(v^*, v^*, \ldots, v^*, Tu^*)] \\ &= \alpha A(u^*, u^*, \ldots, u^*, v^*) + \gamma \begin{bmatrix} A(u^*, u^*, \ldots, u^*, v^*) + \\ A(v^*, v^*, \ldots, v^*, u^*) \end{bmatrix} \\ &= [\alpha + \gamma(1+b)] A(u^*, u^*, \ldots, u^*, v^*) \end{aligned}$$

then,

$$[1 - \alpha - \gamma(1+b)] A(u^*, u^*, \ldots, u^*, v^*) \leq 0,$$

where $1-\alpha-\gamma(1+b) > 0$, as $\alpha b^2+\beta\left(b^2+1\right)+\gamma b^2 [1+b(t-1)] < 1$. This implies that $A(u^*, u^*, \ldots, u^*, v^*) = 0 \Longrightarrow u^* = v^*$ and hence, T has a unique fixed point in U. □

If we take $\beta = \gamma = 0$ in Theorem 2.2, we obtain the following result (see Theorem 3.1 in [13]). Theorem 2.2 is also generalize Lemma 4.1 in [14] and some results in [16] from A_b-metric space to cone A_b-metric space.

Corollary 2.3. *[13] Let (U, A) be a complete cone A_b-metric space and let W be a normal cone with normal constant K. Suppose the mapping $T : U \to U$ satisfies the following condition:*

$$A(Tu, Tu, \ldots, Tu, Tv) \leq \alpha A(u, u, \ldots, u, v)$$

for all $u, v \in U$, where $0 \leq \alpha < \frac{1}{b^2}$. Then T has a unique fixed point in T and Picard iteration $\{u_n\}$ defined by $u_{n+1} = Tu_n$ converges to a fixed point of T.

If we take $\alpha = \gamma = 0$ in Theorem 2.2, we have the following result.

Corollary 2.4. *Let (U, A) be a complete cone A_b-metric space and let W be a normal cone with normal constant K. Suppose the mapping $T : U \to U$ satisfies the following condition:*

$$A(Tu, Tu, \ldots, Tu, Tv) \leq \beta [A(u, u, \ldots, u, Tu) + A(v, v, \ldots, v, Tv)]$$

for all $u, v \in U$, where $0 \leq \beta < \frac{1}{b^2+1}$. Then T has a unique fixed point in T and Picard iteration $\{u_n\}$ defined by $u_{n+1} = Tu_n$ converges to a fixed point of T.

Remark 2.5. Corollary 2.4 expands the Theorem 3.2 in [13], relaxed the contraction condition from $0 \leq \beta < \min\left\{\frac{1}{2}, \frac{1}{(t-1)b^2}\right\}$ to $0 \leq \beta < \frac{1}{b^2+1}$. Clearly, Kannan type contraction mapping in above corollary is not depend on t-dimension.

Putting $\alpha = \beta = 0$ in Theorem 2.2, we obtain the following result (see Theorem 3.3 in [13]).

Corollary 2.6. *[13] Let (U, A) be a complete cone A_b-metric space and let W be a normal cone with normal constant K. Suppose the mapping $T : U \to U$ satisfies the following condition:*

$$A(Tu, Tu, \ldots, Tu, Tv) \leq \gamma A(u, u, \ldots, u, Tv) + A(v, v, \ldots, v, Tu)$$

for all $u, v \in U$, where $0 \leq \gamma < \frac{1}{b^2[1+b(t-1)]}$. Then T has a unique fixed point in T and Picard iteration $\{u_n\}$ defined by $u_{n+1} = Tu_n$ converges to a fixed point of T.

Based on the above corollaries and remark, we can say that our main Theorem 2.2 combines Theorem 3.1-3.3 in [13] under a single theorem and generalizes some results.

Now, using Theorem 2.2, we give the following corollaries without proofs.

Corollary 2.7. *Let (U, A) be a complete cone A_b-metric space and let W be a normal cone with normal constant K. Suppose the mapping $T : U \to U$ satisfies the following condition:*

$$\begin{aligned} A(Tu, Tu, \ldots, Tu, Tv) \quad &\leq \quad \alpha A(u, u, \ldots, u, v) + \beta A(u, u, \ldots, u, Tu) \\ &\quad\; +\theta A(v, v, \ldots, v, Tv) \end{aligned}$$

for all $u, v \in U$, where $0 \leq \alpha b^2 + (\beta + \theta)\left(\frac{b^2+1}{2}\right) < 1$. Then T has a unique fixed point in T and Picard iteration $\{u_n\}$ defined by $u_{n+1} = Tu_n$ converges to a fixed point of T.

Corollary 2.8. *Let (U, A) be a complete cone A_b-metric space and let W be a normal cone with normal constant K. Suppose the mapping $T : U \to U$ satisfies the following condition:*

$$\begin{aligned} A(Tu, Tu, \ldots, Tu, Tv) \quad &\leq \quad \alpha A(u, u, \ldots, u, v) + \beta A(u, u, \ldots, u, Tu) \\ &\quad\; +\theta A(v, v, \ldots, v, Tv) + \gamma A(u, u, \ldots, u, Tv) \\ &\quad\; +\eta A(v, v, \ldots, v, Tu) \end{aligned}$$

for all $u, v \in U$, where $0 \leq \alpha b^2 + (\beta + \theta)\left(\frac{b^2+1}{2}\right) + \left(\frac{\gamma+\eta}{2}\right) b^2 \left[1 + b\,(t-1)\right] < 1$. Then T has a unique fixed point in T and Picard iteration $\{u_n\}$ defined by $u_{n+1} = Tu_n$ converges to a fixed point of T.

Remark 2.9. Above Corollaries 2.3-2.8 also generalize Banach contraction principle [3], Kannan contraction prenciple [11], Chatterjea contraction principle [5], Reich contraction principle [12], Ciri´c contraction prenciple [6] and Hardy-Rogers contraction prenciple [8] from to ordinary complete metric space (U, d) to complete cone A_b-metric space.

Conflict of Interest

The authors have no conflict of interest regarding the publication of this article.

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