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On New Fixed Point Results for Some Classes of Enriched Mappings in N-Banach Spaces

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Abstract

We introduce the concepts of enriched n-contraction mapping, enriched n-Chatterjea mapping and enriched n-Kannan mapping in linear n-normed space. We prove some fixed point theorems for such mappings using Krasnoselsij iteration process in n-Banach spaces. The results presented in this paper improve the recent works of Berinde and Pacurar (J. Fixed Point Theory Appl. (2020) 22-38), Berinde and Pacurar (Preprint, arXiv:1909.03494) and Berinde and Pacurar (Preprint, arXiv:1909.02379, 2019) to linear n-normed spaces.

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1. Introduction and preliminaries

In 1989, Misiak [12] introduced the theory of n-normed spaces which is a generalization of the theory of 2-normed spaces due to Gähler [8]. Since then, many authors have studied the fixed point theory in n-normed spaces (e.g., [7], [9], [5], [6]). We will recall some main definitions related to our work as follows.

Definition 1.1. [12] Let Z be a real vector space and $n \in \mathbb{N}$. Assume that the function $\|\cdot, ..., \cdot\| : Z^n \to \mathbb{R}$ holds the following conditions:

(i) $\|z_1, ..., z_n\| = 0 \Leftrightarrow z_1, ..., z_n$ are linearly dependent,
(ii) $\|z_1, ..., z_n\|$ is invariant under any permutation,
(iii) $\|az_1, ..., z_n\| = |a| \, \|z_1, ..., z_n\|$ for all $a \in \mathbb{R}$,
(iv) $\left\|z_1 - z_1^{'}, z_2, ..., z_n\right\| \leq \|z_1, z_{2,} ..., z_n\| + \left\|z_1^{'}, z_2, ..., z_n\right\|$.

Then the pair $(Z, \|\cdot, ..., \cdot\|)$ is called an n-normed space.

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Example 1.2. [7] Let $Z = \mathbb{R}^n$ with the following Euclidean n-norm:

$$\|z_1, ..., z_n\|_E = abs\left(\begin{vmatrix} z_{11} & \cdots & z_{1n} \\ \vdots & \ddots & \vdots \\ z_{n1} & \cdots & z_{nn} \end{vmatrix}\right),$$

where $z_i = (z_{i1}, ..., z_{in}) \in \mathbb{R}^n$ for each $i = \overline{1, n}$. Then, the pair $(\mathbb{R}^n, \|z_1, ..., z_n\|)$ is an n-normed space.

Definition 1.3. [7] (i) Let Z be a n-normed space and $\{z_n\}$ a sequence in Z. We say that $\{z_n\}$ converge to some $z \in Z$ if

$$\lim_{n\to\infty} \|z_n - z, z_2, z_3, \ldots, z_n\| = 0,$$

for all $z_2, z_3, \ldots, z_n \in Z$.

(ii) Let Z be a n-normed space and $\{z_n\}$ a sequence in Z. We say that $\{z_n\}$ is a Cauchy sequence if

$$\lim_{n,m\to\infty} \|z_n - z_m, z_2, z_3, \ldots, z_n\| = 0,$$

for all $z_2, z_3, \ldots, z_n \in X$.

(iii) A n-normed space is said to be complete if every Cauchy sequence is convergent to an element of Z. A complete n-normed space Z is called n-Banach space.

Recently, some authors (see in [13], [11], [10]) introduced the n-contraction mapping, n-Chatterjea mapping and n-Kannan mapping in linear n-normed space as follows. They proved some fixed point theorems for such mappings in n-Banach space.

Definition 1.4. (cited in [13], [11]) Let $(Z, \|\cdot, ..., \cdot\|)$ be an n-normed space. Then the mapping $R : Z \to Z$ is said to be an n-contraction if there exists $a \in [0, 1)$,

$$\|Ru - Rv, z_2, z_3, \ldots, z_n\| \leq a \|u - v, z_2, z_3, \ldots, z_n\|, \tag{1.1}$$

for all $z_2, z_3, \ldots, z_n, u, v \in Z$. Also, the mapping $R : Z \to Z$ is said to be a n-nonexpansive for $a = 1$.

Definition 1.5. [10] Let $(Z, \|\cdot, ..., \cdot\|)$ be an n-normed space. Then the mapping $R : Z \to Z$ is said to be an n-Chatterjea if there exists $b \in [0, 1/2)$

$$\|Ru - Rv, z_2, z_3, \ldots, z_n\| \leq b \begin{bmatrix} \|u - Rv, z_2, z_3, \ldots, z_n\| \\ + \|v - Ru, z_2, z_3, \ldots, z_n\| \end{bmatrix}, \tag{1.2}$$

for all $z_2, z_3, \ldots, z_n, u, v \in Z$.

Definition 1.6. [10] Let $(Z, \|\cdot, ..., \cdot\|)$ be an n-normed space. Then the mapping $R : Z \to Z$ is said to be an n-Kannan if there exists $c \in [0, 1/2)$

$$\|Ru - Rv, z_2, z_3, \ldots, z_n\| \leq c \begin{bmatrix} \|u - Ru, z_2, z_3, \ldots, z_n\| \\ + \|v - Rv, z_2, z_3, \ldots, z_n\| \end{bmatrix}, \tag{1.3}$$

for all $z_2, z_3, \ldots, z_n, u, v \in Z$.

If $(Z, \|\cdot, ..., \cdot\|)$ is an n-Banach space and $R : Z \to Z$ is a mapping that satisfies any of the above contractive conditions (1.1), (1.2) and (1.3), then the mapping R has a fixed point in Z [10].

In 2020, Berinde and Pacurar [1] introduced the concept of enriched contraction mapping as follows. And, they proved strong convergence theorem for the Kransnoselskij iteration used to approximate the fixed points of enriched contractions. After, some authors (e.g., [3], [2], [4]) introduced the enriched Kannan mappings, enriched Chatterjea mappings and enriched nonexpansive mappings and they gave convergence results for Kransnoselskij iteration used to approximate fixed points of such mappings in Banach spaces.

Definition 1.7. [1] Let $(Z, \|\cdot\|)$ be a normed space. We say that the mapping $R : Z \to Z$ is an enriched contraction mapping if $0 \leq a < \mu + 1$ and $0 \leq \mu < \infty$ such that

$$\|\mu (u - v) + Ru - Rv\| \leq a \|u - v\|, \tag{1.4}$$

for all $u, v \in Z$. We shall call R a (μ, a)-enriched contraction because of the constants in (1.4).

Definition 1.8. [3] Let $(Z, \|\cdot\|)$ be a normed space. We say that the mapping $R : Z \to Z$ is an enriched Kannan mapping if $0 \leq c < 1/2$ and $0 \leq \mu < \infty$ such that

$$\|\mu (u - v) + Rv - Rv\| \leq c [\|u - Ru\| + \|v - Rv\|], \tag{1.5}$$

for all $u, v \in Z$. We shall call R a (μ, c)-enriched Kannan mapping because of the constants in (1.5).

Definition 1.9. [2] Let $(Z, \|\cdot\|)$ be a normed space. We say that the mapping $R : Z \to Z$ is an enriched Chatterjea mapping if $0 \leq b < 1/2$ and $0 \leq \mu < \infty$ such that

$$\|\mu (u - v) + Rv - Rv\| \leq b \left[\begin{array}{c} \|(\mu + 1) (u - v) + v - Rv\| \\ + \|(\mu + 1) (v - u) + u - Ru\| \end{array} \right], \tag{1.6}$$

for all $u, v \in Z$. We shall call R a (μ, b)-enriched Chatterjea mapping because of the constants in (1.6).

Inspired by the above studies, we define the concept of some enriched contractions in n-normed spaces. And we give some examples for class of such mappings. We also prove some fixed point theorems for enriched n-contraction mapping, enriched n-Chatterjea mapping and enriched n-Kannan mapping in n-Banach spaces.

2. Main Results

Definition 2.1. Let $(Z, \|\cdot, ..., \cdot\|)$ be an n-normed space and U be a nonempty closed convex subset of Z. A mapping $R : U \to U$ is said to be an enriched n-contraction mapping if $0 \leq a < \mu + 1$ and $0 \leq \mu < \infty$ such that

$$\|\mu (u - v) + Ru - Rv, z_2, z_3, \ldots, z_n\| \leq a \|u - v, z_2, z_3, \ldots, z_n\| \tag{2.1}$$

for all $z_2, z_3, \ldots, z_n, u, v \in U$. We shall call R a (μ, a)-enriched n-contraction mapping because of the constants in (2.1). If we take $\mu = 0$ in (2.1), we obtain that n-contraction mapping (1.1).

Example 2.2. Let $Z = [0, 1]$ and let Z be a n-normed space. Also, let $R : Z \to Z$ be defined by $Ru = 1 - u$, for all $u \in [0, 1]$. It is clear that R is n-nonexpansive and R is not an n-contraction. We also know that R is an enriched n-contraction. Indeed, if R is an enriched n-contraction, then there exists $a \in [0, 1)$ such that

$$|u - v, z_2, z_3, \ldots, z_n| \leq a|u - v, z_2, z_3, \ldots, z_n|,$$

which implies that a contradiction. From the enriched n-contraction condition (2.1), we have that

$$|(\mu - 1)(u - v), z_2, z_3, \ldots, z_n| \leq a|u - v, z_2, z_3, \ldots, z_n|,$$

with $a \in [0, \mu + 1)$. Taking $0 < \mu < 1$ and $a = 1 - \mu$, the above inequality holds true for all $x, y \in [0, 1]$. Therefore, for any $\mu \in (0, 1)$, R is a $(\mu, 1 - \mu)$-enriched n-contraction and $Fix(T) = \{1/2\}$.

Theorem 2.3. *Let $(Z, \|\cdot, ..., \cdot\|)$ be a n-Banach space and U be a nonempty closed convex subset of Z and $R : U \to U$ an n-enriched contraction. Then*

(i) $Fix(R) = \{u^*\}$;

(ii) The Krasnoselskij iteration process $\{u_n\}$ given by

$$u_{n+1} = (1 - \gamma)u_n + \gamma Ru_n \tag{2.2}$$

converges strongly to u^*, *for any* $u_0 \in U$ *and* $0 < \gamma < 1$.

(iii) The following estimate holds

$$\|u_{n+i-1} - u^*, z_2, z_3, \ldots, z_n\| \leq \frac{\eta^i}{1-\eta} \|u_n - u_{n-1}, z_2, z_3, \ldots, z_n\|, \tag{2.3}$$

where $n = 0, 1, 2, \ldots; i = 1, 2, \ldots$ *and* $\eta = \frac{a}{1+\mu}$.

Proof. We will divide the proof into two parts.

Case 1: Assume that $\mu > 0$. In this case, let us denote $\gamma = \frac{1}{\mu+1}$. Then, from $0 < \gamma < 1$ and the n-enriched contractive (2.1), we get

$$\left\| \left(\frac{1}{\gamma} - 1 \right) + Ru - Rv, z_2, z_3, \ldots, z_n \right\| \leq a \|u - v, z_2, z_3, \ldots, z_n\|,$$

We can write the above inequality as follows

$$\|R_\gamma(u) - R_\gamma(v), z_2, z_3, \ldots, z_n\| \leq \eta \|u - v, z_2, z_3, \ldots, z_n\|, \tag{2.4}$$

where $\eta = \gamma a$ and

$$R_\gamma(u) = (1-\gamma)u + \gamma R(u). \tag{2.5}$$

Since $a \in (0, \mu+1)$, we know that $\eta \in (0,1)$ and therefore inequality (2.5) shows that R_γ is a η-contraction.

It is clear that the mappings R and R_γ have the following property:

$$Fix(R_\gamma) = Fix(R). \tag{2.6}$$

From (2.5), the Krasnoselskij iterative process $\{u_n\}$ defined by (2.2) is exactly the Picard iteration associated to R_γ, that is,

$$u_{n+1} = R_\gamma u_n. \tag{2.7}$$

If we take $u = u_n$ and $v = u_{n-1}$ in (2.4), we obtain that

$$\|R_\gamma u_n - R_\gamma u_{n-1}, z_2, z_3, \ldots, z_n\| \leq \eta \|u_n - u_{n-1}, z_2, z_3, \ldots, z_n\|$$

which implies that

$$\|u_{n+1} - u_n, z_2, z_3, \ldots, z_n\| \leq \eta \|u_n - u_{n-1}, z_2, z_3, \ldots, z_n\|. \tag{2.8}$$

From (2.8), we write the following two estimates

$$\|u_{n+m} - u_n, z_2, z_3, \ldots, z_n\| \leq \eta^n . \frac{(1-\eta^m)}{1-\eta} \|u_1 - u_0, z_2, z_3, \ldots, z_n\|, \tag{2.9}$$

and

$$\|u_{n+m} - u_n, z_2, z_3, \ldots, z_n\| \leq \eta . \frac{(1-\eta^m)}{1-\eta} \|u_n - u_{n-1}, z_2, z_3, \ldots, z_n\|, \tag{2.10}$$

Using (2.9), we get that $\{u_n\}$ is Cauchy sequence. Since $(U, \|\cdot, ..., \cdot\|)$ is an n-Banach space, there exists $u^* \in U$ such that

$$u^* = \lim_{n\to\infty} u_n. \tag{2.11}$$

Now, we will show that u^* is a fixed point of R_γ. We have

$$\begin{aligned} \|u^* - R_\gamma u^*, z_2, z_3, \ldots, z_n\| &\leq \|u^* - u_{n+1}, z_2, z_3, \ldots, z_n\| \\ &\quad + \|u_{n+1} - R_\gamma u^*, z_2, z_3, \ldots, z_n\| \\ &= \|u^* - u_{n+1}, z_2, z_3, \ldots, z_n\| \\ &\quad + \|R_\gamma u_n - R_\gamma u^*, z_2, z_3, \ldots, z_n\|. \end{aligned} \tag{2.12}$$

From (2.4), we get

$$\|R_\gamma u_n - R_\gamma u^*, z_2, z_3, \ldots, z_n\| \le \eta \|u_n - u^*, z_2, z_3, \ldots, z_n\|.$$

Using (2.12), we write

$$\begin{aligned} \|u^* - R_\gamma u^*, z_2, z_3, \ldots, z_n\| &\le \|u^* - u_{n+1}, z_2, z_3, \ldots, z_n\| \\ &\quad + \eta \|u_n - u^*, z_2, z_3, \ldots, z_n\|. \end{aligned} \tag{2.13}$$

Taking limit as $n \to \infty$ in above inequality, we get that $\|u^* - R_\gamma u^*, z_2, z_3, \ldots, z_n\| = 0$, that is, $u^* = R_\gamma u^*$. So, $u^* \in Fix(R_\gamma)$.

Next, we prove that u^* is the unique fixed point of R_γ. Assume that $v^* \neq u^*$ is another fixed point of R_γ. Then, by (2.4)

$$\|R_\gamma u^* - R_\gamma v^*, z_2, z_3, \ldots, z_n\| \le \eta \|u^* - v^*, z_2, z_3, \ldots, z_n\|$$

and

$$0 < \|u^* - v^*, z_2, z_3, \ldots, z_n\| \le \eta \|u^* - v^*, z_2, z_3, \ldots, z_n\| < \|u^* - v^*, z_2, z_3, \ldots, z_n\|$$

a contradiction. Hence $Fix(R_\gamma) = \{u^*\}$. Since $Fix(R) = Fix(R_\gamma)$, we have that $Fix(R) = \{u^*\}$.

(ii) Let $\{u_n\}$ be the Krasnoselskij iteration as in (2.2). From (2.11), we know that $\{u_n\}$ converges strongly to u^*. Thus, claim (ii) is proven.

(iii) If we take limit as $m \to \infty$ in (2.9) and (2.10), we get that

$$\|u_n - u^*, z_2, z_3, \ldots, z_n\| \le .\frac{\eta^n}{1-\eta} \|u_1 - u_0, z_2, z_3, \ldots, z_n\|, \tag{2.14}$$

and

$$\|u_n - u^*, z_2, z_3, \ldots, z_n\| \le \frac{\eta}{1-\eta} \|u_n - u_{n-1}, z_2, z_3, \ldots, z_n\|, \tag{2.15}$$

respectively, where $\eta = \frac{a}{\mu+1}$. If we combine with (2.14) and (2.15), we have that the unifying error estimate (2.3).

Case 2: Assume that $\mu = 0$. In this case, $\gamma = 1$, $\eta = a$ and we proceed like in but with R $(= R_1)$ instead of R_γ, Krasnoselskij iteration (2.2) reduces in fact to the simple Picard iteration associated to R,

$$u_{n+1} = Ru_n, n \ge 0.$$

□

Definition 2.4. Let $(Z, \|\cdot, \ldots, \cdot\|)$ be an n-normed space and U be a nonempty closed convex subset of Z. A mapping $R : Z \to Z$ is said to be an enriched n-Chatterjea mapping if $0 \le \mu < \infty$ and $0 \le b < 1/2$ such that

$$\begin{aligned} &\|\mu(u-v) + Ru - Rv, z_2, z_3, \ldots, z_n\| \\ \le\ & b \begin{bmatrix} \|(\mu+1)(u-v) + v - Rv, z_2, z_3, \ldots, z_n\| \\ + \|(\mu+1)(v-u) + u - Ru, z_2, z_3, \ldots, z_n\| \end{bmatrix} \end{aligned} \tag{2.16}$$

for all $z_2, z_3, \ldots, z_n, u, v \in U$. We shall also call R a (μ, b)-enriched n-Chatterjea mapping because of the constants in (2.16). If we take $\mu = 0$ in (2.16), we obtain that n-Chatterjea mapping (1.2).

Example 2.5. Let $Z = [0, 1]$ and let Z be a n-normed space. Also, let $R : Z \to Z$ be defined by $Ru = 1 - u$, for all $u \in [0, 1]$. We know that R is not an n-Chatterjea mapping but is an enriched n-Chatterjea mapping. Indeed, if R is an n-Chatterjea mapping, then there exists $b \in [0, 1/2)$ such that

$$\begin{aligned} |u - v, z_2, z_3, \ldots, z_n| &\le b\left[|u - 1 + v, z_2, z_3, \ldots, z_n| + |v - 1 + u, z_2, z_3, \ldots, z_n|\right] \\ &= 2b|u + v - 1, z_2, z_3, \ldots, z_n|. \end{aligned}$$

This is a contradiction for $u = 0$ and $v = 1$. The enriched n-Chatterjea condition (2.16) is in this case equivalent to

$$|(\mu-1)(u-v), z_2, z_3, \dots, z_n| \leq b[|(\mu+1)u-(\mu-1)v-1, z_2, z_3, \dots, z_n| + |(\mu+1)v-(\mu-1)u-1, z_2, z_3, \dots, z_n|]. \tag{2.17}$$

We consider the following inequality

$$\begin{aligned} 2\mu|u-v, z_2, z_3, \dots, z_n| &= |[(\mu+1)u-(\mu-1)v-1, z_2, z_3, \dots, z_n] \\ &\quad -[(\mu+1)v-(\mu-1)u-1, z_2, z_3, \dots, z_n]| \\ &\leq |(\mu+1)u-(\mu-1)v-1, z_2, z_3, \dots, z_n| \\ &\quad +|(\mu+1)v-(\mu-1)u-1, z_2, z_3, \dots, z_n|, \end{aligned}$$

in order to obtain (2.17). It is necessary to have $\frac{|\mu-1|}{2\mu} \leq b$, for a certain $b \in [0, 1/2)$.

The only possibility is to have $\mu < 1$ when, by taking $\frac{1-\mu}{2\mu} = b$. We also obtain that $\mu = \frac{1}{b+2}$. *Therefore, R is a $(\frac{1}{b+2}, b)$ -enriched n-Chatterjea mapping for any $b \in [0, 1/2)$ and $Fix(R) = \{1/2\}$.*

Theorem 2.6. *Let $(Z, \|\cdot, ..., \cdot\|)$ be an n-Banach space and U be a nonempty closed convex subset of Z. Assume that the $R : U \to U$ an enriched n-Chatterjea mapping. Then*

(i) $Fix(R) = \{u^*\}$;

(ii) The Krasnoselskij iteration process $\{u_n\}$ given by

$$u_{n+1} = (1-\gamma)u_n + \gamma Ru_n \tag{2.18}$$

converges strongly to u^*, for any $u_0 \in U$ and $0 < \gamma < 1$.

(iii) The following estimate holds

$$\|u_{n+i-1} - u^*, z_2, z_3, \dots, z_n\| \leq \frac{\eta^i}{1-\eta} \|u_n - u_{n-1}, z_2, z_3, \dots, z_n\|, \tag{2.19}$$

where $n = 0, 1, 2, \dots; i = 1, 2, \dots$ and $\eta = \frac{b}{1-b}$.

Proof. We will use similar method at the proof of Theorem 2.3. For any $\gamma \in (0, 1)$ consider the averaged mapping R, given by

$$R(u) = (1-\gamma)u + R(u), \forall u \in U. \tag{2.20}$$

It easy to see that R possesses the following important property:

$$Fix(R_\gamma) = Fix(R).$$

If $\mu > 0$ in (2.16), then let us take $\gamma = \frac{1}{\mu+1}$. Obviously, we have $0 < \gamma < 1$ and thus the contractive condition (1.2) reduces

$$\left\| \left(\frac{1}{\gamma} - 1 \right) + Ru - Rv, z_2, z_3, \dots, z_n \right\| \leq b \left[\begin{array}{c} \left\| \frac{1}{\gamma}(u-v) + v + Rv, z_2, z_3, \dots, z_n \right\| \\ + \left\| \frac{1}{\gamma}(v-u) + u - Ru, z_2, z_3, \dots, z_n \right\| \end{array} \right]$$

which is equaivalent to

$$\|(1-\gamma) + \gamma(Ru - Rv), z_2, z_3, \dots, z_n\| \leq b \left[\begin{array}{c} \|(u-v) + \gamma(v - Rv), z_2, z_3, \dots, z_n\| \\ + \|(v-u) + \gamma(u - Ru), z_2, z_3, \dots, z_n\| \end{array} \right].$$

The above inequality can be written as follows

$$\|R_\gamma(u) - R_\gamma(v), z_2, z_3, \dots, z_n\| \leq b \left[\begin{array}{c} \|u - R_\gamma v, z_2, z_3, \dots, z_n\| \\ + \|v - R_\gamma u, z_2, z_3, \dots, z_n\| \end{array} \right] \tag{2.21}$$

with $b \in [0, 1/2)$.

The above inequality shows that R is a Chatterjea n-contraction in the sense of (1.2).

According to (2.20), the iterative process $\{u_n\}$ defined by (2.18) is the Picard iteration associated to R_γ, that is,

$$u_{n+1} = R_\gamma u_n, n \geq 0.$$

Taking $u = u_n$ and $v = u_{n-1}$ in (2.21), we obtain that

$$\|R_\gamma u_n - R_\gamma u_{n-1}, z_2, z_3, \ldots, z_n\| \leq b \left[\begin{array}{c} \|u_n - R_\gamma u_{n-1}, z_2, z_3, \ldots, z_n\| \\ + \|u_{n-1} - R_\gamma u_n, z_2, z_3, \ldots, z_n\| \end{array} \right]$$

which implies that

$$\|u_{n+1} - u_n, z_2, z_3, \ldots, z_n\| \leq b \left[\begin{array}{c} \|u_n - u_n, z_2, z_3, \ldots, z_n\| \\ + \|u_{n-1} - u_{n+1}, z_2, z_3, \ldots, z_n\| \end{array} \right]. \tag{2.22}$$

Using (2.22), we obtain

$$\|u_{n+1} - u_n, z_2, z_3, \ldots, z_n\| \leq b \left[\begin{array}{c} \|u_{n-1} - u_n, z_2, z_3, \ldots, z_n\| \\ + \|u_n - u_{n+1}, z_2, z_3, \ldots, z_n\| \end{array} \right]$$

which yields

$$\|u_{n+1} - u_n, z_2, z_3, \ldots, z_n\| \leq \frac{b}{1-b} \|u_{n-1} - u_n, z_2, z_3, \ldots, z_n\|.$$

Since $0 < b < \frac{1}{2}$, by denoting $\eta = \frac{b}{1-b}$, we have $0 < \gamma < 1$ and therefore the sequence $\{x_n\}$ satisfies

$$\|u_{n+1} - u_n, z_2, z_3, \ldots, z_n\| \leq \eta \|u_{n-1} - u_n, z_2, z_3, \ldots, z_n\|. \tag{2.23}$$

From (2.23), we obtain the following two estimates

$$\|u_{n+m} - u_n, z_2, z_3, \ldots, z_n\| \leq \eta^n \cdot \frac{1-\eta^m}{1-\eta} \cdot \|u_1 - u_0, z_2, z_3, \ldots, z_n\|, \tag{2.24}$$

and

$$\|u_{n+m} - u_n, z_2, z_3, \ldots, z_n\| \leq \eta \cdot \frac{1-\eta^m}{1-\eta} \cdot \|u_n - u_{n-1}, z_2, z_3, \ldots, z_n\|. \tag{2.25}$$

Using (2.24), we have that $\{u_n\}$ is a Cauchy sequence. Since $(Z, \|\cdot, ..., \cdot\|)$ is an n-Banach space, there exists to $u^* \in U$ such that

$$u^* = \lim_{n\to\infty} u_n. \tag{2.26}$$

Now, we will prove that u^* is a fixed point of R_γ. We have

$$\begin{aligned} \|u^* - R_\gamma u^*, z_2, z_3, \ldots, z_n\| &\leq \|u^* - u_{n+1}, z_2, z_3, \ldots, z_n\| \\ &\quad + \|u_{n+1} - R_\gamma u^*, z_2, z_3, \ldots, z_n\| \\ &= \|u^* - u_{n+1}, z_2, z_3, \ldots, z_n\| \\ &\quad + \|R_\gamma u_n - R_\gamma u^*, z_2, z_3, \ldots, z_n\|. \end{aligned} \tag{2.27}$$

From (2.21), we write

$$\|R_\gamma u_n - R_\gamma u^*, z_2, z_3, \ldots, z_n\| \leq b \left[\begin{array}{c} \|u_n - R_\gamma u^*, z_2, z_3, \ldots, z_n\| \\ + \|u^* - R_\gamma u_n, z_2, z_3, \ldots, z_n\| \end{array} \right]$$

and therefore, by (2.27) one obtains

$$\begin{aligned} (1-b) \|u^* - R_\gamma u^*, z_2, z_3, \ldots, z_n\| &\leq (1+b) \|u^* - u_{n+1}, z_2, z_3, \ldots, z_n\| \\ &\quad + b \|u_n - u^*, z_2, z_3, \ldots, z_n\|. \end{aligned}$$

This implies that

$$\begin{aligned} \|u^* - R_\gamma u^*, z_2, z_3, \ldots, z_n\| &\leq \frac{1+b}{1-b} \|u^* - u_{n+1}, z_2, z_3, \ldots, z_n\| \\ &+\eta \|u_n - u^*, z_2, z_3, \ldots, z_n\| . \end{aligned} \tag{2.28}$$

Now, by taking limit as $n \to \infty$ in (2.28) we get $\|u^* - R_\gamma u^*, z_2, z_3, \ldots, z_n\| = 0$, that is, $u^* = R_\gamma u^*$. So, $u^* \in Fix(R_\gamma)$.

Next, we prove that u^* is the unique fixed point of R_γ. Assume that $v^* \neq u^*$ is another fixed point of R_γ. Then, by (2.21)

$$\|R_\gamma u^* - R_\gamma v^*, z_2, z_3, \ldots, z_n\| \leq b \left[\begin{array}{c} \|u^* - R_\gamma v^*, z_2, z_3, \ldots, z_n\| \\ + \|v^* - R_\gamma u^*, z_2, z_3, \ldots, z_n\| \end{array} \right]$$

which implies that

$$\|u^* - v^*, z_2, z_3, \ldots, z_n\| \leq 2b \|u^* - v^*, z_2, z_3, \ldots, z_n\| .$$

This is a contradiction. Hence $Fix(R_\gamma) = \{u^*\}$ and since $Fix(R) = Fix(R_\gamma)$, we have that $Fix(R) = \{u^*\}$.

(ii) Let $\{u_n\}$ be the Krasnoselskij iterasyon (2.18), with $u_0 \in U$ arbitrary. From (2.26), we know that $\{u_n\}$ converges strongly to u^*. Thus, claim (ii) is proven.

(iii) If we take limit as $m \to \infty$ in (2.24) and (2.25), we have that

$$\|u_n - u^*, z_2, z_3, \ldots, z_n\| \leq \frac{\eta^n}{1-\eta} \|u_1 - u_0, z_2, z_3, \ldots, z_n\| , \tag{2.29}$$

and

$$\|u_n - u^*, z_2, z_3, \ldots, z_n\| \leq \frac{\eta}{1-\eta} \|u_n - u_{n-1}, z_2, z_3, \ldots, z_n\| , \tag{2.30}$$

respectively, where $\eta = \frac{b}{1-b}$. If we merge (2.29) and (2.30), we obtain that the error estimate (2.19).

The remaining case $k = 0$ is similar to $k \neq 0$ with the only difference that in this case $\gamma = 1$ and hence we work with $T = T_1$, when Krasnoselskij iteration (2.18) reduces to the simple Picard iteration

$$u_{n+1} = Ru_n, n \geq 0.$$

□

Finally, we consider the definition of enriched n-Kannan mapping in n-normed space. We will also give a fixed point theorem without proof for this mapping in n-Banach space.

Definition 2.7. Let $(Z, \|\cdot, ..., \cdot\|)$ be an n-normed space and U be a nonempty closed convex subset of X. A mapping $R : U \to U$ is said to be an enriched n-Kannan mapping if $0 \leq \mu < \infty)$ and $0 \leq \gamma < 1/2)$ such that

$$\|\mu (u - v) + Ru - Rv, z_2, z_3, \ldots, z_n\| \leq c \left[\begin{array}{c} \|u - Ru, z_2, z_3, \ldots, z_n\| \\ + \|v - Rv, z_2, z_3, \ldots, z_n\| \end{array} \right] \tag{2.31}$$

for all $z_2, z_3, \ldots, z_n, u, v \in U$. We shall call Ra (μ, c)-enriched n-Kannan mapping because of the constants in (2.31). If we take $\mu = 0$ in (2.31), we obtain that n-Kannan mapping (1.3).

Theorem 2.8. *Let $(Z, \|\cdot, ..., \cdot\|)$ be an n-Banach space and U be a nonempty closed convex subset of Z. Assume that the $R : U \to U$ an enriched n-Kannan mapping. Then*

(i) $Fix(R) = \{u^*\}$;

(ii) The Krasnoselskij iteration process $\{u_n\}$ given by

$$u_{n+1} = (1 - \gamma)u_n + \gamma Ru_n$$

converges strongly to u^*, for any $u_0 \in U$ and $0 < \gamma < 1$.

(iii) The following estimate holds

$$\|u_{n+i-1} - u^*, z_2, z_3, \ldots, z_n\| \leq \frac{\eta^i}{1-\eta} \|u_n - u_{n-1}, z_2, z_3, \ldots, z_n\| ,$$

where $n = 0, 1, 2, \ldots; i = 1, 2, \ldots$ *and* $\eta = \frac{c}{1+\mu}$.

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