Available online at http://cna-journal.com
Commun. Nonlinear Anal. 2 (2020), 100–109
Research Article

Communications in Nonlinear Analysis

Journal Homepage: www.cna-journal.com

Fixed point theorem for generalized quasi orbit type contraction mapping in redefined generalized metric spaces

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Abstract

In this manuscript, a fixed point theorem for generalized quasi orbit contractive type mappings is studied in the context of re-defined generalized metric spaces. Moreover, an appropriate example is also constructed to check the validity of the established result.

Keywords: Re-defined generalized metric space, Cauchy sequence, Orbit type contractive condition.
2010 MSC: 34A12, 47H09, 47H30.

1. Introduction and Preliminaries

Ćirić [2] introduced the notion of quasi contraction. Since then a lot of mathematicians have provided fixed point results on quasi contraction. In [7] Saluja studied common random fixed point results by using quasi contraction for two random operators in the framework of cone random metric spaces. Similarly, numerous significant results has been proved by prominent mathematicians. Alfuraidan[8] analyzed a sufficient condition for the existence of fixed point of monotone quasi-contractive mappings in metric and modular metric spaces with a graph, which extended Ran, Reurings and Jachymski fixed point theorems for monotone contraction mappings in partially ordered metric spaces and in metric space endowed with a graph to the case of quasi-contraction mappings introduced by Ciric. Bachar and Khamsi [1] studied fixed point theorem for monotone quasi contraction mappings without graph. Kiany and Amini-Hirandi [3] investigated a new fixed point theorem for generalized quasi contraction maps in generalized metric spaces. Kumam *et al.* [6] have generalized quasi contraction and investigated some fixed point theorems in metric spaces.

Jleli and Samet found a specific space known as generalized metric space [4] in which some existing spaces in literature have been unified. They have also extended some fixed point results to that space. That generalized metric space are then re-defined by Fan and Wang [9], known as re-defined generalized metric

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Received 2020-01-11

space. Fan and Wang have used concept of orbit and have derived a fixed point result using Ciric-Maiti-Pal orbit contraction in the context of re-defined generalized metric space.

Influenced by the above contributions, a fixed point result in the setting of re-defined generalized metric spaces is derived for generalized quasi orbit contractive type mappings. Some examples are also constructed to check the authenticity of the established result in these corresponding spaces.

Definition 1.1. [4] Let $X \neq \emptyset$ and define a mapping $d_g : X \times X \to [0, \infty)$. For every $\mu \epsilon X$ define a set

$$\Gamma(d_g, X, \mu) = \{\{\mu_n\}_{n\epsilon\mathbb{N}} : \lim_{n\to\infty} d_g(\mu_n, \mu_m) = 0\}$$

we call d_g a generalized metric on X, if for all $\mu, \nu, \eta \epsilon X$, d_g satisfy the following properties:

GS_1) $d_g(\mu, \nu) = o$ if and only if $\mu = \nu$;

GS_2) $d_g(\mu, \nu) = d_g(\nu, \mu)$;

GS_3) there exists an $\alpha > 0$ such that, if $\mu, \nu \epsilon X$ and $\{\mu_n\}_{n\epsilon\mathbb{N}}\epsilon\Gamma(d_g, X, \mu)$ then, $d_g(\mu, \nu) \leq \alpha \lim_{n\to\infty} \sup d_g(\mu_n, \nu)$.

Definition 1.2. [9] Suppose X be a non-empty set and let $D_r : X \times X \to [0, \infty)$. For every $\mu \epsilon X$ consider a set

$$\Lambda(D_r, X, \mu, 0) = \{\{\mu_n\}_{n\epsilon\mathbb{N}} : \lim_{n\to\infty} D_r(\mu_n, \mu_m) = 0\}$$

we call D_r a re-defined generalized metric space on X if for all $\mu, \nu, \eta \epsilon X$, D_r satisfies the conditions given below:

RGS_1) $D_r(\mu, \nu) = 0$ if and only if $\mu = \nu$;

RGS_2) $D_r(\mu, \nu) = d_1(\nu, \mu)$;

RGS_3) There exists a $\rho > 0$ such that if, $\mu, \nu \epsilon X$ and $\{\mu_n\}_{n\epsilon\mathbb{N}}\epsilon(D_r, X, \mu)$ then, $D_r(\mu, \nu) \leq \rho \upsilon \lim_{n\to\infty} \sup D_r(\mu_n, \nu)$.

In accord with the above definition we call D_r a re-defined generalized metric on X, where $\upsilon : [0, \infty) \to [0, \infty)$ is a continuous, and monotonically non-decreasing function for all $(\mu, \nu)\epsilon X \times X$ such that, $\upsilon(0) = 0$, $\upsilon(t) = t$, and $\upsilon(t) \leq \lambda t$. Here $\lambda \epsilon[0, \infty)$ and we call λ an associate number of υ.

Definition 1.3. [9] Consider (X, D_r, ρ, υ) be a re-defined generalized metric space and for all $\mu, \nu \epsilon X$ we have

1. A sequence $\{\mu_n\}_{n\epsilon\mathbb{N}}\epsilon X$ is convergent in X for $\mu \epsilon X$, written $\lim_{n\to\infty} \mu_n = \mu$, if and only if $\{\mu_n\}_{n\epsilon\mathbb{N}}\epsilon\Lambda(D, X, \mu, 0)$. We simply say μ_n D_r converges in X.
2. A sequence $\{\mu_n\}_{n\epsilon\mathbb{N}}\epsilon X$ is said to be Cauchy if $\lim_{n\to\infty} D_g(\mu_n, \mu_m) = 0$ where, $n \geq m > \mathbb{N}$.
3. (X, D_r, ρ, υ) is said to be a complete metric space if, every Cauchy sequence is convergent in (X, D_r, ρ, υ). We simply say the space is D_r complete.

Example 1.4. [9] Consider $X = \{1 - \frac{1}{m} : m\epsilon\mathbb{Z}^+\} \bigcup \{1, 2\}$ be the set of positive numbers. For any $(\mu, \nu)\epsilon X \times X$ the distances are defined as

$$D_r(\mu, \nu) =\mid \mu - \nu \mid, \mu, \nu \epsilon\{1 - \frac{1}{m} : m\epsilon\mathbb{Z}^+\},$$

$$D_r(1, \mu) = D_r(\mu, 1) = |\mu - 1|, \mu\epsilon\{1 - \frac{1}{m} : m\epsilon\mathbb{Z}^+\},$$

$$D_r(2, \mu) = D_r(\mu, 2) = |\mu - 2|, \mu\epsilon\{1 - \frac{1}{m} : m\epsilon\mathbb{Z}^+\},$$

$$D_r(1, 1) = D_r(2, 2) = 0, D_r(1, 2) = D_r(2, 1) = 5.$$

One can see that for the following function:

$$\upsilon(x) = \begin{cases} (x+1)^2 - 1, & \text{if } x\epsilon[0,1) \\ 3x, & \text{if } x\epsilon[1,\infty) \end{cases}$$

There exists a positive real number $\rho \geq \frac{5}{3}$ and an associated number $\lambda = 3$ of υ for which (RGS_3) is satisfied for any $\mu, \nu\epsilon X$. Accordingly, it can be seen that $\{1 - \frac{1}{m}\}_{m\epsilon\mathbb{Z}^+} \subset X$ converges to $1\epsilon X$, which implies $\{1 - \frac{1}{m}\}_{m\epsilon\mathbb{Z}^+}$ belongs to (X, D_r, ρ, υ) when $\mu = 1$ the inequality, $D_r(\mu,\nu) \leq \rho\upsilon \lim_{n\to\infty} SupD_r(\mu_n, \nu)$ holds for any $\mu, \nu\epsilon X$. Furthermore, the conditions (RGS_1), and (RGS_2) are satisfied obviously, and thus, (X, D_r, ρ, υ) is a re-defined generalized metric space.

Proposition 1.5. *[9] Every generalized metric D_g on a set $X \neq \emptyset$ is a re-defined generalized metric on X.*

Corollary 1.6. *[9] The entire standard metrics, b-metrics, and dislocated metrics on X are re-defined generalized metrics on X.*

Proposition 1.7. *[9] Let (X, D_r, ρ, υ) be a re-defined generalized metric space. Consider, a sequence $\{\mu_n\}_{n\epsilon\mathbb{N}}$ in $X \neq \emptyset$ for $\mu, \nu\epsilon X$. If $\{\mu_n\}_{n\epsilon\mathbb{N}}$ converges to μ, and $\{\mu_n\}_{n\epsilon\mathbb{N}}$ converges to ν then, $\nu = \mu$.*

2. Fixed point theorem for contractive type mapping in re-defined generalized metric metric spaces

Definition 2.1. [9] Let f be a self-mapping of a re-defined generalized metric space X for $\mu\epsilon X$

$$O(\mu, f) = \{\mu, f^2(\mu), f^3(\mu).,.,.,\}$$

is called the orbit of μ.

Theorem 2.2. *[9] Let (X, D_r, ρ, υ) be a D_r-complete re-defined generalized metric space and $g : X \to X$ be a function for which the following conditions:*

$$D_r(g(\mu), g(\nu)) \leq \kappa D_r(\mu, \nu) \;\; 0 \leq \kappa < 1,$$

and

$$Sup_n D_r(g^n(\mu), \nu) < \infty, \;\; n\epsilon N,$$

holds for any $\mu, \nu\epsilon X$ then, $\{g^n(\mu_0)\}_{n\epsilon\mathbb{N}}$ is a convergent sequence for $\mu_0\epsilon X$. In addition, if $\{g^n(\mu_0)\}_{n\epsilon\mathbb{N}}$ is convergent to ε uniquely, then g has a fixed point ε. Furthermore, if there is another fixed point ε^*, for which $Sup_n D_r(g^n(\mu_0), \varepsilon^*) < \infty$, $n\epsilon\mathbb{N}$ is true then $\varepsilon = \varepsilon^*$.

Lemma 2.3. *[9] Let (X, D_r, ρ, υ) be a D_r-complete re-defined generalized metric space and $h : X \to X$ be a contraction in which the following definitions:*

$$D_r(h(\mu), h(\nu)) \leq \kappa D_r(\nu, \mu) \;\; 0 \leq \kappa < 1,$$

and

$$Sup_n D_r(h^n(\mu), \mu) < \infty, \;\; n\epsilon\mathbb{N},$$

holds for any $\mu, \nu\epsilon X$ then,

$$\delta(D_r, h, \mu) = Sup\{D_r(h^i(\mu), h^j(\nu)), i, j\epsilon\mathbb{N}\} < \infty.$$

In this case it has been shown that $\{h^n(\mu_0)\}_{n\epsilon\mathbb{N}}$ is a convergent sequence for $x_0\epsilon X$. In addition, if $\{f^n(\mu_0)\}_{n\epsilon\mathbb{N}}$ is uniquely convergent to ι then, h has a fixed point ι. Furthermore, if another fixed point of h exists, say ι^* for which $Sup_n D_r(h^n(\mu_0), \iota^*) < \infty$ is true then, $D_r(\iota, \iota^*) < \infty$.

Corollary 2.4. *[9] Suppose (X, D_r, ρ, υ) be a D_r-complete re-defined generalized metric space and a self mapping $g : X \to X$ for which*

$$D_r(g(\mu), g(\nu)) \leq \kappa D_r(\mu, \nu) \;\; 0 \leq \kappa < 1,$$

and

$$\delta(D_r, g, \mu) = Sup\{D_r(g^i(\mu), g^j(\mu)), i, j \epsilon \mathbb{N}\} < \infty,$$

holds for any $\mu, \nu \epsilon X$ then, g has a fixed point ι and $\{g^n(x_0)\}_{n\epsilon\mathbb{N}}$ converges to ι. Moreover, if g has another fixed point ι^* for which $Sup_n D_g(f^n(\mu_0), \zeta^*) < \infty$ is true then, $\iota = \iota^*$.

Corollary 2.5. *[9] Let (X, D_r, ρ, υ) be a D_r-complete re-defined generalized metric space and a mapping $g : X \to X$ which satisfies the following statements:*

$$D_r(g(\mu), g(\nu)) \leq \kappa D_r(\mu, \nu) \;\; 0 \leq \kappa < 1,$$

and

$$Sup_n D_r(g^n(x), x) < \infty, \;\; n\epsilon\mathbb{N},$$

for any $\mu, \nu \epsilon X$ then, g has a fixed point ι and $\{g^n(\mu_0)\}_{n\epsilon\mathbb{N}}$ uniquely converges to ι. In addition, if g has another fixed point ι^* for which $Sup_n D_r(g^n(\mu_0), \iota^*) < \infty$ is true then, $\iota = \iota^*$

Definition 2.6. [6] Suppose $g : X \to X$ be a contraction on a re-defined generalized metric space (X, D_r, ρ, υ), $X \neq \emptyset$ is said to be g-orbitally complete. If there exists an element $\mu \epsilon X$ such that for any element $a, b \epsilon \overline{R(\mu, g)}$ and $a, b \notin \overline{R(\mu, f)} \backslash R(\mu, g)$. Where $R(\mu, g)$ is orbit of μ

Definition 2.7. [3] Suppose $g : X \to X$ be a contraction on a metric space (X, d) and $\mu, \nu \epsilon X$ then, g is called a quasi type contraction if there is a number $\beta \epsilon [0, 1)$ such that,

$$d(g(a), g(b)) \leq \beta max\big\{d(a, b) + d(a, g(a)), d(b, g(b)) + d(a, g(b)) + d(b, g(a))\big\}. \tag{2.1}$$

the contraction (2.1) has been generalized to the following contraction by Poom Kumam *et al.*.

Definition 2.8. [6] Consider $g : X \to X$ be a contraction mapping on metric space X. The mapping g is called a generalized quasi-contraction if there exists a number $\beta \epsilon [0, 1)$ such that for all $\mu, \nu \epsilon X$

$$\begin{aligned} d(g(a), g(b)) \leq \beta max\big\{&d(a, b), d(a, g(a)), d(b, g(b)), d(a, g(b)), d(b, g(a)), d(g^2(a), a), \\ &d(g^2(a), g(a)), d(g^2(a), b), d(g^2(a), g(b))\big\}. \end{aligned} \tag{2.2}$$

3. Main Results

Before Proceeding to our main work we need to extend the contraction (2.1) to re-defined generalized metric spaces.

Definition 3.1. Suppose $g : X \to X$ be a contraction on a re-defined generalized metric space X. The mapping g is called a generalized quasi orbit-contraction if there exists a number $\beta \epsilon [0, 1)$ such that for all $\xi, \varsigma \epsilon X$.

$$\begin{aligned} D_r(f(\xi), f(\varsigma)) \leq \beta max\big\{&D_r(\xi, \varsigma), D_r(\xi, f(\xi)), D_r(\varsigma, f(\varsigma)), D_r(\xi, f(\varsigma)), D_r(\varsigma, f(\xi)), D_r(f^2(\xi), \xi), \\ &D_r(f^2(\xi), f(\xi)), D_r(f^2(\xi), \varsigma), D_r(f^2(\xi), f(\varsigma))\big\}, \end{aligned} \tag{3.1}$$

holds. Now, we proceed to our main result.

Theorem 3.2. *Consider $f : X \to X$ be a generalized quasi orbit mapping of contractive type on a D_r-complete re-defined generalized metric space (X, D_r, ρ, υ), satisfying the condition:*

$$D_r(f(\mu), f(\nu)) \leq \beta max\big\{D_r(\mu,\nu), D_r(\mu, f(\mu)), D_r(\nu, f(\nu)), D_r(\mu, f(\nu)), D_r(\nu, f(\mu)), D_r(f^2(\mu), \mu), D_r(f^2(\mu), f(\mu)), D_r(f^2(\mu), \nu), D_r(f^2(\mu), f(\nu))\big\}. \tag{3.2}$$

for all $\mu, \nu \epsilon X$. λ be an associated number of υ, $O(\mu, f)$ be the orbit of μ, and $\mu_n = f^n(\mu)\epsilon O(\mu, f)$ then we have the following results:

If $Sup_n D_r(f^n(\mu), \mu) < \infty$, for all $\mu \epsilon X$ then, $\{f^n(\mu_0)\}_{n\epsilon\mathbb{N}}$ is convergent to ζ uniquely, and if $\rho\lambda\beta < 1$ then, f contains a fixed point δ. In addition, if there exists a fixed point other than δ say, δ^* for which $Sup_n D_r(f^n(\mu_0), \delta^*) < \infty$ for $n\epsilon\mathbb{N}$ is true then, $\delta = \delta^*$.

Proof. For $\mu\epsilon O(\mu, f)$ the condition $Sup_m D_r(f^m(\mu), \mu) < \infty$ implies

$$Sup_m D_r(f^m(\mu_n), \mu_n) = Sup_n D_r(f^{m+n}(\mu), f^n(\mu)) < \infty,$$

for every fixed $n\epsilon\mathbb{N}$ and all $m\epsilon\mathbb{N}$. Hence,

$$Sup_{m_1,m_2} D_r(f^{m_1}(\mu_n), f^{m_2}(\mu_n)) = Sup_{m_1,m_2} D_r(f^{n+m_1}(\mu), f^{n+m_2}(\mu)) < \infty,$$

for all $n, m_1, m_2\epsilon\mathbb{N}$. Denote $Sup_{m_1,m_2} D_r(f^{m_1}(\mu_n), f^{m_2}(\mu_n))$ by $\gamma_n(\mu, f)$ $n\epsilon\mathbb{N}$ and all $m_1, m_2\epsilon\mathbb{N}$. Because f is a generalized quasi orbit-contraction mapping then, for every fixed $n\epsilon\mathbb{N}$ and all $m_1, m_2\epsilon\mathbb{N}$, we have

$$\begin{aligned}
D_r(f^{m_1}(\mu_{n+1}), f^{m_2}(\mu_{n+1})) = D_r(f(\mu_{n+m_1}), f(\mu_{n+m_2})),\\
&\leq \beta max\big\{D_r(\mu_{n+m_1}, \mu_{n+m_2}), D_r(\mu_{n+m_1}, f(\mu_{n+m_1})),\\
&\quad D_r(\mu_{n+m_2}, f(\mu_{n+m_2})), D_r(\mu_{n+m_1}, f(\mu_{n+m_2})),\\
&\quad D_r(\mu_{n+m_2}, f(\mu_{n+m_1})), D_r(f^2(\mu_{n+m_1}), \mu_{n+m_1}),\\
&\quad D_r(f^2(\mu_{n+m_1}), \mu_{n+m_2}), D_r(f^2(\mu_{n+m_1}), \mu_{n+m_2}),\\
&\quad D_r(f^2(\mu_{n+m_1}), f(\mu_{n+m_2}))\big\}.\\
&\leq \beta max\big\{D_r(f^{m_1}(\mu_n), f^{m_2}(\mu_n)), D_r(f^{m_1}(\mu_n), f^{m_1+1}(\mu_n)),\\
&\quad D_r(f^{m_2}(\mu_n), f^{m_2+1}(\mu_n)), D_r(f^{m_1}(\mu_n), f^{m_2+1}(\mu_n)),\\
&\quad D_r(f^{m_2}(\mu_n), f^{m_1+1}(\mu_n)), D_r(f^{m_1+2}(\mu_n), f^{m_1}(\mu_n)),\\
&\quad D_r(f^{m_1+2}(\mu_n), f^{m_2}(\mu_n)), D_\mu(f^{m_1+2}(\mu_n), f^{m_2}(\mu_n)),\\
&\quad D_r(f^{m_1+2}(\mu_n), f^{m_2+1}(\mu_n))\big\}.
\end{aligned}$$

Note that

$$Sup_{m_1,m_2} D_r(f^{m_1+1}(\mu_n), f^{m_2+1}(\mu_n)) = Sup_{m_1,m_2} D_r(f^{m_1}(\mu_n), f^{m_2}(\mu_n)),$$

then,

$$Sup_{m_1,m_2} D_r(f^{m_1}(\mu_{n+1}), f^{m_2}(\mu_{n+1})) \leq \beta Sup_{m_1,m_2} D_r(f^{m_1}(\mu_n), f^{m_2}(\mu_n)),$$

that is

$$\gamma_{n+1}(\mu, f) \leq \beta\gamma_n(\mu, f)$$

for all $n\epsilon\mathbb{N}$. Subsequently,

$$D_r(f^n(\mu), f^{n+m}(\mu_n)) \leq \beta\gamma_n(\mu, f) \leq \beta^2\gamma_{n-1}(\mu, f) \leq ... \leq \beta^n\gamma_0(\mu, f)$$

Because $\beta\epsilon[0,1)$ and $\gamma_O(\mu, f)$ is bounded therefore, $D_r(f^n(\mu), f^{n+m}(\mu_n)) \to 0$ when $n \to \infty$, which implies that $\{f^n(\mu_0)\}_{n\epsilon\mathbb{N}}$ is a Cauchy sequence. While on the other side, X is D_r-complete then the limit δ of the sequence $\{f^n(\mu_0)\}_{n\epsilon\mathbb{N}}$ is in X.

On words, we show that δ is a fixed point of f. If $D_r(\delta, f(\delta)) = 0$ then from (RGS_3) it is obvious that, $f(\delta) = \delta$.

If $D_r(\delta, f(\delta)) > 0$ using (RGS_3) one can get

$$D_r(\delta, f(\delta)) \leq \rho\upsilon \lim_{n\to\infty} Sup D_r(f^n(\delta), \delta), \tag{3.3}$$

$$\begin{aligned} D_r(f^n(\delta_0), f(\delta)) \leq \beta max\big\{ & D_r(f^{n-1}(\delta_0), \delta), D_r(f^{n-1}(\delta_0), f^n(\delta)), D_r(\delta, f(\delta)), D_r(f^{n-1}(\delta_0), f(\delta)), \\ & D_r(\delta, f^n(\delta_0)), D_r(f^{n+1}(\delta_0), f^{n-1}(\delta_0)), D_r(f^{n+1}(\delta_0), \delta), D_r(f^{n+1}(\delta_0), \delta), \\ & D_r(f^{n+1}(\delta_0), f(\delta))\big\}. \end{aligned}$$

is true for any positive number n. Taking upper limits on the both sides

$$\begin{aligned} \lim_{n\to\infty} Sup D_r(f^n(\delta_0), f(\delta)) \leq \beta max\big\{ & \lim_{n\to\infty} Sup D_r(f^{n-1}(\delta_0), \delta), \lim_{n\to\infty} Sup D_r(f^{n-1}(\delta_0), f^n(\delta_0)), \\ & D_r(\delta, f(\delta)), \lim_{n\to\infty} Sup D_r(f^{n-1}(\delta_0), f(\delta)), \\ & \lim_{n\to\infty} Sup D_r(\delta, f^n(\delta_0)), \lim_{n\to\infty} Sup D_r(f^{n+1}(\delta_0)), f^{n-1}(\delta_0)), \\ & \lim_{n\to\infty} Sup D_r(f^{n+1}(\delta_0), \delta), \lim_{n\to\infty} Sup D_r(f^{n+1}(\delta_0), \delta), \\ & \lim_{n\to\infty} Sup D_r(f^{n+1}(\delta_0), f(\delta))\big\}. \end{aligned}$$

Because $\{f^n(\delta)\}_{n\epsilon\mathbb{N}}$ is a Cauchy Sequence and is convergent to δ. Consequently,

$$\begin{aligned} & \lim_{n\to\infty} Sup D_r(f^{n-1}(\delta_0), \delta) = \lim_{n\to\infty} Sup D_r(f^{n-1}(\delta_0), f^n(\delta_0)) \\ & = \lim_{n\to\infty} Sup D_r(f^{n-1}(\delta_0), f(\delta)) = \lim_{n\to\infty} Sup D_r(\delta, f^n(\delta_0)) \\ & = \lim_{n\to\infty} Sup D_r(f^{n+1}(\delta_0), \delta) = \lim_{n\to\infty} Sup D_r(f^{n+1}(\delta_0)), f^{n-1}(\delta_0)), \\ & = \lim_{n\to\infty} Sup D_r(f^{n+1}(\delta_0), \delta) = \lim_{n\to\infty} Sup D_r(f^{n+1}(\delta_0), f(\delta)) = 0, \end{aligned}$$

hence

$$\lim_{n\to\infty} Sup D_r(f^n(\delta_0), f(\delta)) \leq \beta max\big\{D_r(\delta, f(\delta))\big\},$$

hence we get

$$\beta\, max\big\{D_r(\delta, f(\delta))\big\} = \beta\, D_r(\delta, f(\delta)).$$

Combining it with inequality (3.5), and (RGS_3) one can get

$$\begin{aligned} D_r(\delta, f(\delta)) & \leq \rho\upsilon \lim_{n\to\infty} Sup D_r(f^n(\delta_0), \delta), \\ & \leq \rho\lambda\beta\, D_r(\delta, f(\delta)), \;\; since \;\; \upsilon(t) \leq \lambda t \end{aligned}$$

it is followed from $\rho\lambda\beta < 1$ that $D_r(\delta, f(\delta)) = 0$, which violates the condition $D_r(\delta, f(\delta)) > 0$, so by (RGS_1)

$$f(\delta) = \delta.$$

Which completes the proof that f has a fixed point.

Now to show that f has a unique fixed point. On the contrary, suppose that the fixed point of f is not unique so, f has at least one more fixed point say, ζ^* with $\delta \neq \delta^*$ such that $Sup_n D_r(f^n(\delta_0), \delta^*) < \infty$ for $n\epsilon\mathbb{N}$ we need to show that $\delta = \delta^*$. By (RGS_3), we have

$$D_r(\delta, \delta^*) \leq \rho \upsilon \lim_{n\to\infty} Sup D_r(f^n(\delta_0), \delta^*).$$

Because $\upsilon(t)$ is continuous when $t > 0$ and $Sup_n D_r(f^n(\delta_0), \delta^*) < \infty$ therefore,

$$\rho \upsilon \lim_{n\to\infty} Sup D_r(f^n(\delta_0), \delta^*) < \infty.$$

that is $D_r(\delta, \delta^*) < \infty$. On the other hand,

$$\begin{aligned} D_r(\delta, \delta^*) &= D_r(f(\delta), f(\delta^*)) \\ &\leq \beta max\big\{D_r(\delta, \delta^*), D_r(\delta, f(\delta)), D_r(\delta^*, f(\delta^*)), D_r(\delta, f(\delta^*)), D_r(\delta^*, f(\delta)), \\ &\quad D_r(f^2(\delta), \delta), D_r(f^2(\delta), f(\delta)), D_r(f^2(\delta), \delta^*), D_r(f^2(\delta), f(\delta^*))\big\}. \end{aligned}$$

As $f(\delta^*) \to \delta^*$ by hypothesis of fixed point, and using (RGS_1) by simple calculation one can obtain

$$D_r(\delta, \delta^*) \leq \beta max\big\{D_r(\delta, \delta^*), D_r(\delta, \delta^*), D_r(\delta, \delta^*), D_r(\delta, \delta^*), D_r(\delta, \delta^*)\big\} \tag{3.4}$$

Which implies that $D_r(\delta, \delta^*) \leq \beta D_r(\delta, \delta^*) < \infty$ thus, the inequality

$$D_r(\delta, \delta^*) \leq \beta D_r(\delta, \delta^*)$$

holds. Consequently, $D_r(\delta, \delta^*) = D_r(f(\delta), f(\delta^*)) \leq \beta D_r(\delta, \delta^*) < \infty$, which further implies that, $D_r(\delta, \delta^*) = 0$. Now using (RGS_1) the result can be obtained easily

$$\delta = \delta^*.$$

Finally, the above calculation proves that, f has really a unique fixed point. □

Onwards, an example is provided to check the validity of the theorem.

Example 3.3. For $X = [1, 2]$ then, for any $\nu, \mu\epsilon X$ the metrics between the numbers are given below;

$$D_r(\mu, \nu) = \mid \mu - \nu \mid, \mu, \nu\epsilon X;$$

$$D_r(1, \mu) = D_r(\mu, 1) = |\mu - 1|, \mu\epsilon X;$$
$$D_r(2, \nu) = D_r(\nu, 2) = |\nu - 2|, \nu\epsilon X;$$
$$D_r(1, 1) = D_r(2, 2) = D_r(\mu, \mu) = D_r(\nu, \nu)0, D_r(1, 2) = D_r(2, 1) = 1;$$

Conspicuously, every Cauchy sequence $\{1 - \frac{1}{n} : n\epsilon\mathbb{Z}^+\}$ converges to $1\epsilon X$, while all the others are subsequences of $\{1 - \frac{1}{n} : n\epsilon\mathbb{Z}^+\}$ thus, X is a complete metric space. Further, we show that X is a re-defined generalized metric space. By (RGS_3) one can observe that for

$$\upsilon(x) = \begin{cases} (x+1)^2 - 1, & \text{if } x\epsilon[0, 1) \\ 3x, & \text{if } x\epsilon[1, \infty) \end{cases}$$

there exists a real number $\rho = \frac{5}{3}$ and an associated number $\lambda = 3$ for which, (RGS_3) satisfies for any $\mu, \nu\epsilon X$. Accordingly, it can be observed that every sequence in X is constant and converges in X and therefore, belongs to $\Lambda(D_r, X, \mu, 0)$. When $\mu = 1$, the inequality $D_r(\mu, \nu) \leq \rho \upsilon \lim_{n\to\infty} Sup D_r(\mu_n, \nu)$ holds, for any $\mu, \nu\epsilon X$. Moreover, the conditions (RGS_1), and (RGS_2) are trivially satisfied. Which shows that X is a re-defined generalized metric space.

Now, consider the mapping

$$f(1-\frac{1}{n}) = 1 - \frac{1}{2(n+1)} \;\; n\epsilon\mathbb{Z}^+ \;\; f(1)=1, \;\; f(2)=1;$$

without any tedious calculation one can get

$$D_r(f(1-\frac{1}{n}), f(1-\frac{1}{m})) = \frac{|n-m|}{2(n+1)(m+1)}, \;\; m,n\epsilon\mathbb{Z}^+;$$

$$D_r(f(1),f(1)) = D_r(f(1),f(2)) = D_r(f(2),f(2)) = 0;$$

$$D_r(f(2), f(1-\frac{1}{n})) = D_r(f(1-\frac{1}{n}), f(2)) = \frac{1}{2(n+1)}, \;\; n\epsilon\mathbb{Z}^+;$$

$$D_r(f(1), f(1-\frac{1}{m})) = D_r(f(2), f(1-\frac{1}{m})) = \frac{1}{2(m+1)}, \;\; m\epsilon\mathbb{Z}^+.$$

Then, we have

$$D_r(2,2) = D_r(1,1) = D_r(1,f(2)) = D_r(f^2(\mu), f(2)) = 0;$$

$$D_r(2,f(1)) = D_r(2,1) = D_r(2,f(2)) = D_r(f^2(2),2) = 1;$$

$$D_r(f(1-\frac{1}{m}), f(1)) = D_r(f(1-\frac{1}{m}), f(2)) = \frac{1}{2(m+1)};$$

$$D_r(f(1-\frac{1}{m}), f(1-\frac{1}{n})), = \frac{|n-m|}{2(n+1)(m+1)}, \;\; m,n\epsilon\mathbb{Z}^+;$$

$$D_r(f^2(1-\frac{1}{m}), f(1)) = D_r(f^2(1-\frac{1}{m}), f(2)) = \frac{1}{4(m+1)},$$

as one can see that,

$D_r(\mu,\nu) = 0, 1, \frac{|m-n|}{2(n+1)(m+1)}, \; , \frac{1}{2(m+1)}, \frac{1}{4(m+1)}$, for $m,n\epsilon\mathbb{Z}^+$ in either of the case appeared in generalized quasi contraction in the setting of re-defined generalized metric spaces.

One can prove that if $0 \leq \beta < 1$ is chosen the inequality (3.2) will hold for any $\mu,\nu\epsilon X$. If we take $\mu = 1-\frac{1}{n}$ for any fixed $n\epsilon\mathbb{N}$ then, $n\to\infty$,

$$D_r(f^n(\mu),\mu) = 1 - \frac{1}{2^m(n+1)+2m} \to 1.$$

In addition, it is clear that,

$$f^n(1-\frac{1}{n}) = \frac{|2^m(n+1)+2m-n|}{(2^m(n+1)+2m)n} < \infty \;\; n\epsilon\mathbb{Z}^+$$

for fixed $n\epsilon\mathbb{Z}^+$ thus, $sup_n D_r(f^n(\mu),\mu) < \infty$, $n\epsilon\mathbb{Z}^+$.

Likely, one can prove that whether $\mu = 1$ or $\mu = 2$ the condition

$$f^n(\mu) \to 1, \qquad Sup_n D_r(f^n(\mu),\mu) < \infty, \qquad and \qquad \rho\lambda\beta < 1,$$

holds. All the conditions of Theorem (3.2) are satisfied by the above example. As it has been witnessed that there is really a unique fixed point $1\epsilon X$.

Remark 3.4. It is compulsory to note in Theorem (3.2) the re-defined generalized metric space (X, D_r, ρ, υ) needs not to be always complete but, can be at least, f-orbitally complete. If the contraction (3.2) of Theorem (3.2) holds, only at points in the closure of the orbit of μ for some point $\mu\epsilon X$ instead for all $\mu,\nu\epsilon X$. Under such circumstances, the fixed point of generalized quasi orbit contractive mapping needs not to be unique. In such a condition, the following theorem can be stated.

Theorem 3.5. *Let $f : X \to X$ be a generalized quasi contractive type mapping on a f-orbitally complete re-defined generalized metric space (X, D_r, ρ, υ) satisfying the condition:*

$$\begin{aligned} D_r(f(a), f(b)) \leq \beta max\big\{&D_r(a,b), D_r(a, f(a)), D_r(b, f(b)), D_r(a, f(b)), D_r(b, f(a)), D_r(f^2(a), a),\\ &D_r(f^2(a), f(a)), D_r(f^2(a), b), D_r(f^2(a), f(b))\big\}. \end{aligned} \tag{3.5}$$

for all $a, b\epsilon\overline{O(\mu, f)}$, λ is an associated number of υ, $O(\mu, f)$ is the orbit of μ, and $\mu_n = f^n(\mu)\epsilon O(\mu, f)$ then, the followed results can be obtained:

If $Sup_n D_r(f^n(\mu), \mu) < \infty$ for all $\mu\epsilon X$ then, $\{f^n(\mu_0)\}_{n\epsilon\mathbb{N}}$ is uniquely convergent to δ and $\rho\lambda\beta < 1$ then f contains a fixed point δ.

Proof. This theorem has already been proved above. Except, the fixed point needs not to be unique in this case. □

An example can be provided to verify the result.

Example 3.6. Let $X = \big\{2 - \frac{1}{a}, 4 - \frac{1}{b}, 9 - \frac{1}{c}, a, b, c\epsilon\mathbb{Z}^+\big\} \bigcup \big\{2, 4, 8\big\}$. Consider the subspace $X_1 = \big\{2 - \frac{1}{a}, 4 - \frac{1}{b}, 9 - \frac{1}{c}, a, b, c\epsilon\mathbb{Z}^+\big\}$ and define the distances:

$$\begin{aligned} D_r(l,p) &= |l - p| \;\; l, p\,\epsilon X;\\ D_r(l,2) = D_r(2,l) &= |l-2|, l\,\epsilon X_1;\\ D_r(l,4) = D_r(4,l) &= |l-4|, l\,\epsilon X_1;\\ D_r(l,9) = D_r(9,l) &= |l-9|, l\,\epsilon X_1; \end{aligned}$$

$D_r(2,2) = D_r(4,4) = D_r(9,9) = 0, D_r(2,4) = D_r(4,2) = 2, D_r(2,9) = D_r(9,2) = 7, D_r(4,9) = D_r(9,4) = 5.$
One can verify that X is a re-defined generalized metric space if we choose $\rho = 3$ and

$$\upsilon(x) = \begin{cases} e^t - 1, & \text{if } x\epsilon[0,1)\\ (e-1)t, & \text{if } x\epsilon[1,\infty) \end{cases}$$

with an associated number $\lambda = (e - 1)$ the inequality $D_r(l,p) \leq \rho\upsilon \lim\limits_{n\to\infty} Sup D_r(l_n, p)$ holds, for any $l, p\epsilon X$. In addition, the inequalities (RGS_1), and (RGS_2) are obviously satisfied. However, it is clear that X is not a complete re-defined generalized metric space since, the Cauchy sequence $\big\{9 - \frac{1}{c}\big\}_{c\epsilon\mathbb{Z}+} \to 9 \notin X$.

Define the mapping $f(2) = 4, f(4) = 4, f(8) = 2, f(2 - \frac{1}{a}) = 2 - \frac{1}{2(a+1)}, f(4 - \frac{1}{b}) = 4 - \frac{1}{2(b+1)}, f(9 - \frac{1}{c}) = 4 - \frac{1}{2(c+1)}$.

Clearly, one can easily verify that f is not a contractive type mapping. In fact, one can get
$D_r(f(2), f(4 - \frac{1}{b})) = 2 - \frac{1}{2(b+1)} > D_g(2, 4 - \frac{1}{b}) = 2 - \frac{1}{b}$.

However, it can be verified that f is a generalized quasi orbit contractive type mapping. Consider the orbit;

$$O(2 - \frac{1}{n}, f) = \{2 - \frac{1}{n}, f(2 - \frac{1}{n}, f^2(2 - \frac{1}{n}), ..., f^m(2 - \frac{1}{n})), ...\}$$

for every fixed $m\epsilon\mathbb{Z}^+$, where,

$$f^m(2 - \frac{1}{n}) = 2 - \frac{1}{2^n(n+1) + 2n}.$$

In terms of υ, ρ, and λ discussed above, one can choose that for $0 < \beta < 1$ such that $\rho\lambda\beta < 1$ when $l, p\epsilon\overline{O(2 - \frac{1}{n}, f)}$ and $l, p \notin \overline{O(2 - \frac{1}{n}, f)}\backslash O(2 - \frac{1}{n}, f)$ while at the same instant inequality (3.5) is satisfied. Thus, it is satisfactory to verify that the orbit $O(2 - \frac{1}{n}, f)$ satisfies the condition $Sup_n D_r(f^n(2 - \frac{1}{n}), 2 - \frac{1}{n}) < \infty$. Furthermore, one can observe that $f^m(2 - \frac{1}{n}) \to 2$ which shows that 2 is a fixed point of f. Likely, if one consider $f^m(4 - \frac{1}{n}) \to 4$, which shows that 4 is a fixed point of f. Similarly, the rest of the orbits satisfy the conditions of theorem (3.5). Through which the arguments that under certain circumstances will a fixed point of f needs not to be unique, is explained.

Conclusion

In the present study, a fixed point result for quasi generalized orbit contraction in the setting of re-defined generalized metric space is obtained. Moreover, a condition is found when a fixed point of generalized quasi orbit type contraction will not be unique, in the setting of re-defined generalized metric spaces.

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