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Classification of positive solutions for nonlinear differential systems

Lianwen Wang[a,*]

[a]*School of Computer Science and Mathematics, University of Central Missouri, Warrensburg, MO 64093, USA*

Abstract

The classification of positive solutions for a class of nonlinear differential systems is investigated. Necessary and sufficient conditions are established for the existence of certain solutions. Sufficient conditions for the nonexistence of certain solutions are also discussed. In particular, some sufficient conditions for the nonexistence are optimal in some sense.

Keywords: Classification, existence, nonexistence, positive solution, nonlinear differential system.
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1. Introduction

In this paper we consider the classification of positive solutions for the system of first order nonlinear differential equations

$$\begin{aligned} x'(t) &= F(t, y(t)) \\ y'(t) &= G(t, x(t)), \end{aligned} \tag{1.1}$$

where $F, G : [a, \infty) \times \mathbb{R} \to \mathbb{R}$ are continuous functions. $rF(t, r) > 0$ and $rG(t, r) > 0$ for all $r \neq 0$ and $t \geq a$. There exist two continuous functions $a(t), b(t) : [a, \infty) \to (0, \infty)$, two continuous and increasing functions $f(r), g(r) : \mathbb{R} \to \mathbb{R}$, and two real numbers $\alpha > 1$, $\beta > 1$ such that for $t \geq a$

$$\begin{aligned} a(t)f(y) \leq F(t, y) \leq \alpha a(t)f(y), \\ b(t)g(x) \leq G(t, x) \leq \beta b(t)g(x). \end{aligned} \tag{1.2}$$

Remark 1.1. Many nonseparable functions F and G satisfy (1.2); for example, let $F(t, y) = 2ty + \sin(ty)$ where $t \geq 1$. Then $rF(t, r) > 0$ for all $r \neq 0$ and $ty \leq F(t, y) \leq 3ty$.

*Corresponding author
Email address: `lwang@ucmo.edu` (Lianwen Wang 1)

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A pair of functions $(x(t), y(t))$ is called a solution of system (1.1) with maximal existence interval $[a, \alpha_{xy})$, $a < \alpha_{xy} \le \infty$, if both $x(t)$ and $y(t)$ are differentiable and satisfy system (1.1) on $[a, \alpha_{xy})$. A solution $(x(t), y(t))$ is said to be eventually positive if there exists a $b \ge a$ such that both $x(t)$ and $y(t)$ are positive on $[b, \alpha_{xy})$. Note that there are some general conditions to guarantee that the solutions can be extended to $[a, \infty)$ [12]. We restrict our discussions to solutions of system (1.1) that can be extended to $[a, \infty)$.

Classification, existence, asymptotic behavior, and other properties of solutions of some special cases of system (1.1)–second order nonlinear differential equations–have been studied in details; see [1, 2, 4, 6, 9, 10, 11, 13] and many other publications. The investigation for nonlinear differential systems can be found in [3, 7, 8] and other literatures, but all these discussions focus on separable nonlinear differential systems

$$\begin{aligned} x'(t) &= a(t)f(y(t)) \\ y'(t) &= b(t)g(x(t)). \end{aligned} \tag{1.3}$$

In this paper, we discuss the classification of all eventually positive solutions of system (1.1) and provide results of existence and nonexistence of certain solutions. Our results completely extend all the results of [8] to the nonseparable differential system (1.1), and moreover, we establish some nonexistence theorems for certain solutions and show that some sufficient conditions for the nonexistence are optimal in some sense.

The following assumptions are imposed for the discussions:

(H1A) There exists a real number $K > 0$ such that

$$|f(uv)| \le K|f(u)||f(v)|, \ \forall u, v \in \mathbb{R}.$$

(H1B) There exists a real number $M > 0$ such that

$$|g(uv)| \le M|g(u)||g(v)|, \ \forall u, v \in \mathbb{R}.$$

(H2A) There exists a real number $r_0 > 0$ such that

$$\int_{\pm r_0}^{\pm\infty} \frac{dr}{f(g(r))} = \infty.$$

(H2B) There exists a real number $r_0 > 0$ such that

$$\int_{\pm r_0}^{\pm\infty} \frac{dr}{g(f(r))} = \infty.$$

Define four classes of solutions of system (1.1) below. We will show that all eventually positive solutions belong to one of the four classes.

$$\begin{aligned} S(c,c) &= \left\{(x,y) : \lim_{t\to\infty} x(t) = c_1 > 0, \ \lim_{t\to\infty} y(t) = c_2 > 0\right\}, \\ S(c,\infty) &= \left\{(x,y) : \lim_{t\to\infty} x(t) = c_1 > 0, \ \lim_{t\to\infty} y(t) = \infty\right\}, \\ S(\infty,c) &= \left\{(x,y) : \lim_{t\to\infty} x(t) = \infty, \ \lim_{t\to\infty} y(t) = c_2 > 0\right\}, \\ S(\infty,\infty) &= \left\{(x,y) : \lim_{t\to\infty} x(t) = \infty, \ \lim_{t\to\infty} y(t) = \infty\right\}. \end{aligned}$$

Let

$$A = \int_a^\infty a(t)dt, \quad B = \int_a^\infty b(t)dt.$$

There are four possible cases for A and B: $A = \infty$ and $B = \infty$, $A = \infty$ and $B < \infty$, $A < \infty$ and $B = \infty$, and $A < \infty$ and $B < \infty$. In the following sections we will consider the classification with each of these four cases as in [8].

2. The Case $A = \infty$ And $B = \infty$

Theorem 2.1. *Suppose that $A = \infty$ and $B = \infty$. Then any eventually positive solutions of (1.1) belong to $S(\infty, \infty)$.*

Proof. Let (x, y) be an eventually positive solution of system (1.1). Then $x'(t) > 0$ and $y'(t) > 0$ for $t \geq b$. So, $x(t) \geq x(b)$ and $y(t) \geq y(b)$ for $t \geq b$. Therefore, we have

$$x(t) = x(b) + \int_b^t F(s, y(s))ds \geq \int_b^t a(s)f(y(s))ds \geq f(y(b)) \int_b^t a(s)ds,$$

and

$$y(t) = y(b) + \int_b^t G(s, x(s))ds \geq \int_b^t b(s)g(x(s))ds \geq g(x(b)) \int_b^t b(s)ds.$$

This implies that $(x, y) \in S(\infty, \infty)$. □

Remark 2.2. Theorem 2.1 extends [8] Theorem 2.1 to nonseparable differential system (1.1).

3. The Case $A = \infty$ And $B < \infty$

Theorem 3.1. *Suppose that $A = \infty$ and $B < \infty$. Then any eventually positive solutions of (1.1) belong to either $S(\infty, \infty)$ or $S(\infty, c)$.*

Proof. Let (x, y) be an eventually positive solution of system (1.1). Then $x'(t) > 0$ and $y'(t) > 0$ for $t \geq b$. So, $x(t) \geq x(b)$ and $y(t) \geq y(b)$ for $t \geq b$. Note that

$$x(t) = x(b) + \int_b^t F(s, y(s))ds \geq \int_b^t a(s)f(y(s))ds \geq f(y(b)) \int_b^t a(s)ds.$$

So, $x(t) \to \infty$ as $t \to \infty$ and $(x, y) \in S(\infty, \infty)$ or $S(\infty, c)$. □

Theorem 3.2. *Suppose that $A = \infty$ and $B < \infty$. If $S(\infty, c) \neq \emptyset$, then*

$$\int_a^\infty b(t)g\Big(f(c)\int_a^t a(s)ds\Big)dt < \infty \tag{3.1}$$

for some $c > 0$.

Proof. Let (x, y) be an eventually positive solution of system (1.1). Then $x'(t) > 0$ and $y'(t) > 0$ for $t \geq b$. So, $x(t) \geq x(b)$ and $y(t) \geq y(b)$ for $t \geq b$. Note that

$$x(t) = x(b) + \int_b^t F(s, y(s))ds \geq \int_b^t a(s)f(y(s))ds \geq f(y(b)) \int_b^t a(s)ds,$$

and

$$\begin{aligned} y(t) &= y(b) + \int_b^t G(s, x(s))ds \\ &\geq \int_b^t b(s)g(x(s))ds \\ &\geq \int_b^t b(s)g\Big(f(y(b))\int_b^t a(\xi)d\xi\Big)ds. \end{aligned}$$

Therefore,

$$\int_b^\infty b(t)g\Big(f(y(b))\int_b^t a(s)ds\Big)dt < \infty.$$

□

Theorem 3.3. *Suppose that* $A = \infty$ *and* $B < \infty$*. If*

$$\int_a^\infty b(t)g\Big(\alpha f(c)\int_a^t a(s)ds\Big)dt < \infty \tag{3.2}$$

for some $c > 0$*, then* $S(\infty, c) \neq \emptyset$*.*

Proof. Take a large $T > a$ such that

$$\int_a^\infty b(t)g\Big(\alpha f(c)\int_a^t a(s)ds\Big)dt < \frac{c}{2\beta}.$$

Let $CB[T,\infty)$ be the Banach space of all bounded and continuous functions defined on $[T,\infty)$ with the supremum norm and let X be a subset of $CB[T,\infty)$ defined as

$$X = \{y \in CB[T,\infty) : \frac{c}{2} \le y(t) \le c,\ t \ge T\}.$$

Clearly, X is a convex and bounded subset of $CB[T,\infty)$. Define an operator $J : X \to CB[T,\infty)$ as

$$(Jy)(t) = c - \int_t^\infty G\Big(s, \int_T^s F(\xi, y(\xi))d\xi\Big)ds, t \ge T.$$

In the following we will show that J maps X into X, it is continuous, and JX is precompact.

First of all, J maps X into X because for any $y \in X$

$$\begin{aligned} c \ge (Jy)(t) &\ge c - \int_T^\infty G\Big(t, \int_T^t F(\xi, y(\xi))d\xi\Big)dt \\ &\ge c - \beta\int_T^\infty b(t)g\Big(\alpha f(c)\int_a^t a(s)ds\Big)dt \\ &\ge c/2. \end{aligned}$$

Let y_n, $y \in X$ such that $\|y_n - y\| \to 0$ as $n \to \infty$. Since F and G are continuous, for each $s \ge T$, we have

$$G\Big(s, \int_T^s F(\xi, y_n(\xi))d\xi\Big) - G\Big(s, \int_T^s F(\xi, y(\xi))d\xi\Big) \to 0, n \to \infty.$$

Also,

$$\Big|G\Big(s, \int_T^s F(\xi, y_n(\xi))d\xi\Big) - G\Big(s, \int_T^s F(\xi, y(\xi))d\xi\Big)\Big| \le 2\beta b(s)g\Big(\alpha f(c)\int_T^s a(\xi)d\xi\Big).$$

By the Lebesgue's Dominated Convergence Theorem, we have

$$\begin{aligned} \lim_{n\to\infty}\|Jy_n - Jy\| &= \lim_{n\to\infty}\sup_{t\ge T}|(Jy_n)(t) - (Jy)(t)\| \\ &\le \lim_{n\to\infty}\int_T^\infty \Big|G\Big(t, \int_T^t F(s, y_n(s))ds\Big) - G\Big(t, \int_T^t F(s, y(s))ds\Big)\Big|dt \\ &= 0. \end{aligned}$$

Thus, J is continuous.

JX is equicontinuous because for any $t_1, t_2 \ge T$ and $t_2 > t_1$

$$\begin{aligned} |(Jy)(t_2) - (Jy)(t_1)| &= \int_{t_1}^{t_2} G\Big(t, \int_T^t F(s, y(s))ds\Big)dt \\ &\le \beta\int_{t_1}^{t_2} b(t)g\Big(\alpha f(c)\int_T^t a(s)ds\Big)dt. \end{aligned}$$

Also JX is uniformly bounded. The precompactness of JX follows from Arzelà–Ascoli Theorem.

By Schauder's fixed-point theorem, J has a fixed point in X, let it be $\bar{y}$. Define

$$\bar{x} = \int_T^t F(s, \bar{y}(s))ds.$$

It is easy to check that $(\bar{x}, \bar{y})$ is a class $S(\infty, c)$ solution of system (1.1). □

Combining Theorem 3.2 and Theorem 3.3, we have

Corollary 3.4. *Suppose that $A = \infty$ and $B < \infty$. Then $S(\infty, c) \neq \emptyset$ for system (1.3) if and only if*

$$\int_a^\infty b(t)g\Big(f(c)\int_a^t a(s)ds\Big)dt < \infty$$

for some $c > 0$.

Corollary 3.5. *Suppose that $A = \infty$, $B < \infty$, and (H1B) hold. Then $S(\infty, c) \neq \emptyset$ for system (1.1) if and only if*

$$\int_a^\infty b(t)g\Big(\int_a^t a(s)ds\Big)dt < \infty. \tag{3.3}$$

Proof. We will show that (3.1), (3.2), and (3.3) are equivalent under assumption (H1B). Indeed, if (3.2) is satisfied, then (3.1) is satisfied since

$$\int_a^\infty b(t)g\Big(f(c)\int_a^t a(s)ds\Big)dt \le \int_a^\infty b(t)g\Big(\alpha f(c)\int_a^t a(s)ds\Big)dt.$$

If (3.1) is true, so is (3.3) because

$$\begin{aligned}
&\int_a^\infty b(t)g\Big(\int_a^t a(s)ds\Big)dt \\
&= \int_a^\infty b(t)g\Big(\frac{1}{f(c)}f(c)\int_a^t a(s)ds\Big)dt \\
&\le Mg\Big(\frac{1}{f(c)}\Big)\int_a^\infty b(t)g\Big(f(c)\int_a^t (s)ds\Big)dt.
\end{aligned}$$

If (3.3) holds, so does (3.2) since

$$\int_a^\infty b(t)g\Big(\alpha f(c)\int_a^t a(s)ds\Big)dt \le Mg(\alpha f(c))\int_a^\infty b(t)g\Big(\int_a^t a(s)ds\Big)dt.$$

□

The next result provides the condition for the emptiness of class $S(\infty, \infty)$.

Theorem 3.6. *Suppose that $A = \infty$, $B < \infty$, (H1A), and (H2A) are satisfied. In addition, let (3.3) hold. Then $S(\infty, \infty) = \emptyset$.*

Proof. Let (x, y) be an eventually positive solution of system (1.1) that belongs to class $S(\infty,\infty)$. Then $x'(t) > 0$ and $y'(t) > 0$ for $t \geq b$. Note that

$$\begin{aligned} y(t) &= y(b) + \int_b^t G(s, x(s))ds \\ &\leq y(b) + \beta \int_b^t b(s)g(x(s))ds \\ &\leq y(b) + \beta g(x(t)) \int_b^t b(s)ds \\ &= g(x(t))\Big(\frac{y(b)}{g(x(t))} + \beta \int_b^t b(s)ds\Big) \\ &\leq g(x(t))\Big(\frac{y(b)}{g(x(b))} + \beta \int_b^t b(s)ds\Big). \end{aligned}$$

Choose $L > 1$ such that

$$\frac{y(b)}{g(x(b))} + \beta \int_b^t b(s)ds \leq L \int_b^t b(s)ds,$$

we have

$$y(t) \leq Lg(x(t)) \int_b^t b(s)ds.$$

Applying (H1A) we have

$$\begin{aligned} x'(t) = F(t, y(t)) &\leq \alpha a(t) f\Big(Lg(x(t))\Big) \\ &\leq \alpha K^2 f(L)a(t)f(g(x(t)))f\Big(\int_b^t b(s)ds\Big). \end{aligned}$$

Then

$$\frac{x'(t)}{f(g(x(t)))} \leq \alpha K^2 f(L)a(t)f(g(x(t)))f\Big(\int_b^t b(s)ds\Big).$$

Integrating from b to t yields

$$\int_{x(b)}^{x(t)} \frac{dr}{f(g(r))} \leq \alpha K^2 f(L) \int_b^t a(s)f\Big(\int_b^s b(\sigma)d\sigma\Big)ds.$$

Note that $\lim_{t\to\infty} x(t) = \infty$, taking the limit as $t \to \infty$ we have

$$\int_{x(b)}^{\infty} \frac{dr}{f(g(r))} \leq \alpha K^2 f(L) \int_b^{\infty} a(t)f\Big(\int_b^t b(s)ds\Big)dt < \infty,$$

which contradicts (H2A). Therefore, $S(\infty,\infty) = \emptyset$. □

Remark 3.7. (H2A) in Theorem 3.6 is sharp. For example, consider the differential system on $t \geq 1$

$$\begin{aligned} x'(t) &= \frac{1}{t^{\frac{1}{3}}} y^{\frac{1}{3}}(t) \\ y'(t) &= \frac{1}{t^5} x^5(t). \end{aligned} \tag{3.4}$$

Here, $a(t) = \frac{1}{t^{\frac{1}{3}}}$, $b(t) = \frac{1}{t^5}$, $f(r) = r^{\frac{1}{3}}$, and $g(r) = r^5$. Clearly, $A = \infty$ and $B < \infty$. Moreover,

$$\int_1^{\infty} \frac{dr}{f(g(r))} = \int_1^{\infty} \frac{dr}{r^{\frac{5}{3}}} < \infty,$$

and

$$\int_1^\infty b(t)g\Big(\int_1^t a(s)ds\Big)dt < \Big(\frac{3}{2}\Big)^5 \int_1^\infty \frac{dt}{t^{\frac{5}{3}}} < \infty.$$

However, $(x,y)=(t,t)$ is a $S(\infty,\infty)$ solution of system (3.4).

Remark 3.8. Theorem 3.1 extends [8] Theorem 3.1 to system (1.1). By Corollary 3.4, Theorem 3.2 and Theorem 3.3 extend [8] Theorem 3.2 to system (1.1).

4. The Case $A<\infty$ And $B=\infty$

Because of the symmetric feature of x and y in system (1.1), with the same arguments in the previous section, we have the following results.

Theorem 4.1. *Suppose that $A<\infty$ and $B=\infty$. Then any eventually positive solutions of (1.1) belong to either $S(\infty,\infty)$ or $S(c,\infty)$.*

Theorem 4.2. *Suppose that $A<\infty$ and $B=\infty$. If $S(c,\infty)\neq\emptyset$, then*

$$\int_a^\infty a(t)f\Big(g(c)\int_a^t b(s)ds\Big)dt < \infty$$

for some $c>0$.

Theorem 4.3. *Suppose that $A<\infty$ and $B=\infty$. If*

$$\int_a^\infty a(t)f\Big(\beta g(c)\int_a^t b(s)ds\Big)dt < \infty$$

for some $c>0$, then $S(c,\infty)\neq\emptyset$.

Combining Theorem 4.2 and Theorem 4.3, we obtain

Corollary 4.4. *Suppose that $A<\infty$ and $B=\infty$. Then $S(c,\infty)\neq\emptyset$ for system (1.3) if and only if*

$$\int_a^\infty a(t)f\Big(g(c)\int_a^t b(s)ds\Big)dt < \infty \tag{4.1}$$

for some $c>0$.

Corollary 4.5. *Suppose that $A<\infty$, $B=\infty$, and (H1A) hold. Then $S(c,\infty)\neq\emptyset$ for system (1.1) if and only if*

$$\int_a^\infty a(t)f\Big(\int_a^t b(s)ds\Big)dt < \infty. \tag{4.2}$$

Theorem 4.6. *Suppose that $A<\infty$, $B=\infty$, (H1B), and (H2B) are satisfied. In addition, let (4.2) hold. Then $S(\infty,\infty)=\emptyset$.*

Remark 4.7. (H2B) in Theorem 4.6 is sharp. This can be explained from Remark 3.7 by switching x and y in the example.

Remark 4.8. Theorem 4.1 extends [8] Theorem 4.1 to system (1.1). By Corollary 4.4, Theorem 4.1 and Theorem 4.2 extend [8] Theorem 4.2 to system (1.1).

5. The Case $A < \infty$ And $B < \infty$

Theorem 5.1. *Suppose that $A < \infty$ and $B < \infty$. Then all eventually positive solutions of (1.1) belong to either $S(\infty, \infty)$ or $S(c, c)$.*

Proof. Let (x, y) be an eventually positive solution of system (1.1). Then $x'(t) > 0$ and $y'(t) > 0$ for $t \geq b$. If $\lim_{t\to\infty} x(t) = c_1 > 0$, then $x(t) \leq c_1$ for $t \geq b$ and

$$\begin{aligned} y(t) &= y(b) + \int_b^t G(s, x(s))ds \\ &\leq y(b) + \beta \int_b^t b(s)g(x(s))ds \\ &\leq y(b) + \beta g(c_1) \int_b^t b(s)ds \\ &\leq y(b) + \beta g(c_1)B < \infty, \end{aligned}$$

which implies that $\lim_{t\to\infty} y(t) = c_2 > 0$. Similarly, if $\lim_{t\to\infty} y(t) = c_2 > 0$, then $\lim_{t\to\infty} x(t) = c_1 > 0$. □

Theorem 5.2. *The solution class $S(c, c) \neq \emptyset$ if and only if $A < \infty$ and $B < \infty$.*

Proof. Let (x, y) be an eventually positive class $S(c, c)$ solution of system (1.1). Then $x'(t) > 0$ and $y'(t) > 0$ for $t \geq b$, also, $\lim_{t\to\infty} x(t) = c_1 > 0$ and $\lim_{t\to\infty} y(t) = c_2 > 0$. In view of

$$\begin{aligned} x(t) &= x(b) + \int_b^t F(s, y(s))ds \\ &\geq \int_b^t a(s)f(y(s))ds \\ &\geq f(y(b)) \int_b^t a(s)ds, \end{aligned}$$

we have $A < \infty$. The proof of $B < \infty$ is similar.

Conversely, for two real numbers $c > 0$ and $d > 0$, we have

$$\int_a^\infty a(t)f(2c)dt < \infty, \quad \int_a^\infty b(t)g(2d)dt < \infty.$$

Pick $T > b$ large enough such that

$$\int_T^\infty a(t)f(2c)dt < \frac{d}{\alpha}, \quad \int_T^\infty b(t)g(2d)dt < \frac{c}{\beta}.$$

Let $CB[T, \infty) \times CB[T, \infty)$ be the space of all continuous and bounded function pairs with the usual pointwise ordering $\leq$. Define a subset of $CB[T, \infty) \times CB[T, \infty)$ as

$$X = \{(x, y) \in CB[T, \infty) \times CB[T, \infty) : d \leq x(t) \leq 2d, c \leq y(t) \leq 2c, t \geq T\}.$$

Clearly, for any subset Ω of X, $\inf \Omega \in X$ and $\sup \Omega \in X$. Consider an operator $J : X \to CB[T, \infty) \times CB[T, \infty)$ with

$$\begin{aligned} (Jx)(t) &= d + \int_T^t F(s, y(s))ds \\ (Jy)(t) &= c + \int_T^t G(s, x(s))ds. \end{aligned}$$

The operator J satisfies all the assumptions of Knaster's fixed-point theorem [5]: J maps X into X and preserve the order. Indeed, if $(x, y) \in X$, then

$$d \le (Jx)(t) \le d + \alpha \int_T^\infty a(t) f(y(t)) dt \le d + \alpha \int_T^\infty a(t) f(2c) dt \le 2d,$$

and

$$c \le (Jy)(t) \le c + \beta \int_T^\infty b(t) g(x(t)) dt \le c + \beta \int_T^\infty b(t) g(2d) dt \le 2c.$$

By Knaster's fixed-point theorem, J has a fixed-point in X, let it be $(\bar{x}, \bar{y}) \in X$. Then

$$\bar{x}'(t) = d + \int_T^t F(s, \bar{y}(s)) ds$$
$$\bar{y}'(t) = c + \int_T^t G(s, \bar{x}(s)) ds.$$

It is easy to check that $(\bar{x}, \bar{y})$ is a class $S(c, c)$ solution of system (1.1). □

Theorem 5.3. *Suppose that $A < \infty$ and $B < \infty$. Then the solution class $S(\infty, \infty) = \emptyset$ if one of the following two conditions is satisfied:*
(1): (H1A) and (H2A),
(2): (H1B) and (H2B).

Proof. Note that $A < \infty$ and $B < \infty$ imply (3.3) and (4.2). The rest proof is similar to that of Theorem 3.6 and Theorem 4.6. □

Remark 5.4. (H2A) and (H2B) are sharp in Theorem 5.3. For example, consider the differential system on $t \ge 1$

$$\begin{aligned} x'(t) &= \frac{1}{t^3} y^3(t) \\ y'(t) &= \frac{1}{t^5} x^5(t). \end{aligned} \tag{5.1}$$

Here, $a(t) = \frac{1}{t^3}$, $b(t) = \frac{1}{t^5}$, $f(r) = r^3$, and $g(r) = r^5$. Clearly, $A < \infty$ and $B < \infty$. Moreover,

$$\int_1^\infty \frac{dr}{f(g(r))} = \int_1^\infty \frac{dr}{g(f(r))} = \int_1^\infty \frac{dr}{r^{15}} < \infty.$$

However, $(x, y) = (t, t)$ is a class $S(\infty, \infty)$ solution of system (5.1).

Remark 5.5. Theorem 5.1 and Theorem 5.2 extend Theorem 5.1 and Theorem 5.2 in [8] to nonseparable differential system (1.1), respectively.

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