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Integral Type Version for Hardy and Roger Type Mapping in Dislocated Metric (d-Metric) Space

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Abstract

This work is dedicated to establish a common fixed point theorem for integral type mapping. We will introduce integral type version for Hardy and Roger type mapping using the concept of weakly compatible mappings in the context of dislocated metric (d-metric) space. Several corrolaries may be deduced from the establish result.

Keywords: Dislocated metric space, self-mapping, Cauchy sequence, Common fixed point, weakly compatible mappings, Coincidence point.
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1. Introduction and Preliminaries

Fixed point theory is one the most fruitful and applicable topic of non linear analysis, which is not only used in other mathematical theories but also in many practical problems of natural sciences and engineering too. Banach contraction principle [4] is the most popular result in the field of metric fixed point theory. This principle has many applications in several domains, such as differential equations, functional equation, integral equations, economics, wild life and many others.

In 2002, Branciari [1] obtained a fixed point theorem for a single self-mapping satisfying an analogous of Banach's contraction principle for integral type inequality in metric space. Afterwards many researchers ([3],[5], [8]) obtained fixed point results of integral type in different types of distance space.

In this article, we have established a common fixed point results of integral type contractive conditions using the concept of weakly compatible mappings in dislocated metric space introduced by Hitzler [2]. Our obtained results generalizes some well-known results of the literature.

Throughout this work $\mathbb{R}^+$ represent the set of non-negative real numbers. Now, we collect some known definitions and results from the literature which are helpful in the proof of our results.

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Definition 1.1. [3]. Let X be a non-empty set. Let $d : X \times X \to \mathbb{R}^+$ be a function satisfying the conditions for all $x, y, z \in X$,

d_1) $\int_0^{d(x,x)} \rho(t)dt = 0$;

d_2) $\int_0^{d(x,y)} \rho(t)dt = \int_0^{d(y,x)} \rho(t)dt = 0 \quad \Rightarrow \quad x = y$;

d_3) $\int_0^{d(x,y)} \rho(t)dt = \int_0^{d(y,x)} \rho(t)dt$;

d_4) $\int_0^{d(x,y)} \rho(t)dt \leq \int_0^{d(x,z)} \rho(t)dt + \int_0^{d(z,y)} \rho(t)dt$.

If d satisfies all of the above conditions then d is called a metric on X. If d satisfies the conditions from $d_2 - d_4$ then d is said to be dislocated metric (OR) shortly (d-metric) on X and if d satisfies only d_2 and d_4 then d is called dislocated quasi-metric (OR) shortly (dq-metric) on X and the pair (X, d) is called dislocated quasi-metric space.

Where $\rho : \mathbb{R}^+ \to \mathbb{R}^+$ is a Lebesque integrable mapping which is summable on each compact subset of $\mathbb{R}^+$, non-negative and such that for any $s > 0$ $\int_0^s \rho(t)dt > 0$.

Note. The above definition change to usual definition of metric space if $\rho(t) = I$.

It is clear that every metric space is dislocated metric and dislocated quasi metric space but the converse is not true. Also every dislocated metric space is dislocated quasi-metric space but the converse is not necessarily true.

The following simple but important results can be seen in [2].

Definition 1.2. A d-metric space (X, d) is said to be complete if every Cauchy sequence in X converge to a point in X.

Lemma 1.3. *Limit in d-metric space is unique.*

Theorem 1.4. *Let (X, d) be a complete d-metric space $T : X \to X$ be a contraction. Then T has a unique fixed point.*

Branciari [1] proved the following theorem in metric spaces.

Theorem 1.5. *Let (X, d) be a complete metric space for $\alpha \in (0, 1)$. Let $T : X \to X$ be a mapping such that for all $x, y \in X$ satisfying*

$$\int_0^{d(Tx,Ty)} \rho(t)dt \leq \alpha \cdot \int_0^{d(x,y)} \rho(t)dt.$$

Where $\rho : \mathbb{R}^+ \to \mathbb{R}^+$ is a Lebesque integrable mapping which is summable on each compact subset of $\mathbb{R}^+$, non-negative and such that for any $s > 0$ $\int_0^s \rho(t)dt > 0$. Then T has a unique fixed point in X.

Definition 1.6. [6] Let A and S be mappings on d-metric space (X, d), then A and S are said to be weakly compatible mappings if they commute at their coincident points such that $Ax = Sx$ implies $ASx = SAx$. The point $x \in X$ is called coincident point of A and S.

2. Main Results

Theorem 2.1. *Let (X, d) be a complete dislocated metric space, for $a, b, c, e, f \geq 0$ with $2(a+b+c+e+f) < 1$ and $A, B, S, T : X \to X$ be self-mappings such that for all $x, y \in X$, satisfying the conditions*

1. *$TX \subset AX$, $SX \subset AX$.*
2. *The pairs (S, A) and (T, B) are weakly compatible.*
3. $\int_0^{d(Sx,Ty)} \rho(t)dt \leq a\cdot \int_0^{d(Ax,By)} \rho(t)dt + b\cdot \int_0^{d(Ax,Sx)} \rho(t)dt + c\cdot \int_0^{d(By,Ty)} \rho(t)dt + e\cdot \int_0^{d(Ax,Ty)} \rho(t)dt + f\cdot \int_0^{d(By,Sx)} \rho(t)dt.$

where $\rho : \mathbb{R}^+ \to \mathbb{R}^+$ is a Lebesque integrable mapping which is summable on each compact subset of $\mathbb{R}^+$, non-negative and such that for any $s > 0$ $\int_0^s \rho(t)dt > 0$. Then A, B, S, T have a unique common fixed point.

Proof. Using condition (1), we define sequence x_n and y_n by the following rule

$$y_{2n} = Bx_{2n+1} = Sx_{2n}$$

and

$$y_{2n+1} = Ax_{2n+2} = Tx_{2n+1}, n = 0, 1, 2, 3......$$

Assume that $y_{2n} \neq y_{2n+1}$ for all n, then we have

$$\int_0^{d(y_{2n},y_{2n+1})} \rho(t)dt = \int_0^{d(Sx_{2n},Tx_{2n+1})} \rho(t)dt$$

By condition (2) in the theorem we have

$$\leq a \cdot \int_0^{d(Ax_{2n},Bx_{2n+1})} \rho(t)dt + b \cdot \int_0^{d(Ax_{2n},Sx_{2n})} \rho(t)dt + c \cdot \int_0^{d(Bx_{2n+2},Tx_{2n+1})} \rho(t)dt +$$

$$e \cdot \int_0^{d(Ax_{2n},Tx_{2n+1})} \rho(t)dt + f \cdot \int_0^{d(Bx_{2n+1},Sx_{2n})} \rho(t)dt.$$

Using the definition of the sequence we have

$$\leq a \cdot \int_0^{d(y_{2n-1},y_{2n})} \rho(t)dt + b \cdot \int_0^{d(y_{2n-1},y_{2n})} \rho(t)dt + c \cdot \int_0^{d(y_{2n},y_{2n+1})} \rho(t)dt +$$

$$e \cdot \int_0^{d(y_{2n-1},y_{2n+1})} \rho(t)dt + f \cdot \int_0^{d(y_{2n},y_{2n})} \rho(t)dt.$$

Simplification yields

$$\int_0^{d(y_{2n},y_{2n+1})} \rho(t)dt \leq \left(\frac{a+b+e}{1-c-e-2f}\right) \int_0^{d(y_{2n-1},y_{2n})} \rho(t)dt.$$

Let $h = \frac{a+b+e}{1-(c+e+2f)}$, so the above inequality become

$$\int_0^{d(y_{2n},y_{2n+1})} \rho(t)dt \leq h \cdot \int_0^{d(y_{2n-1},y_{2n})} \rho(t)dt.$$

Also

$$\int_0^{d(2n-1,y_{2n})} \rho(t)dt \leq h \cdot \int_0^{d(y_{2n-2},y_{2n-1})} \rho(t)dt.$$

So

$$\int_0^{d(y_{2n},y_{2n+1})} \rho(t)dt \leq h^2 \cdot \int_0^{d(y_{2n-2},y_{2n-1})} \rho(t)dt.$$

Proceeding in such away one can get

$$\int_0^{d(y_n,y_{n+1})} \rho(t)dt \leq h^n \cdot \int_0^{d(y_0,y_1)} \rho(t)dt.$$

Since $h < 1$ and taking limit $n \to \infty$, we have $h^n \to 0$. Hence

$$\int_0^{d(y_n,y_{n+1})} \rho(t)dt \to 0.$$

Which implies that $d(y_n, y_{n+1}) \to 0$ as $n \to \infty$. Hence $\{y_n\}$ is a Cauchy sequence in complete d-metric space X. So there must exists $z \in X$ such that

$$\lim_{n\to\infty} y_n = z.$$

Therefore the subsequences $Sx_{2n} \to z, Bx_{2n} \to z, Tx_{2n+1} \to z\ and Ax_{2n+2} \to z$. Since $TX \subset AX$, there exist a point $u \in X$ such that $Au = z$. So

$$\int_0^{d(Su,z)} \rho(t)dt = \int_0^{d(Su,Tx_{2n+1})} \rho(t)dt$$

$$\leq a \cdot \int_0^{d(Au,Bx_{2n+1})} \rho(t)dt + b \cdot \int_0^{d(Au,Su)} \rho(t)dt + c \cdot \int_0^{d(Bx_{2n+1},Tx_{2n+1})} \rho(t)dt +$$

$$e \cdot \int_0^{d(Au,Tx_{2n+1})} \rho(t)dt + f \cdot \int_0^{d(Bx_{2n+1},Su)} \rho(t)dt.$$

$$\leq a \cdot \int_0^{d(z,Bx_{2n+1})} \rho(t)dt + b \cdot \int_0^{d(z,Su)} \rho(t)dt + c \cdot \int_0^{d(Bx_{2n+1},Tx_{2n+1})} \rho(t)dt +$$

$$e \cdot \int_0^{d(z,Tx_{2n+1})} \rho(t)dt + f \cdot \int_0^{d(Bx_{2n+1},Su)} \rho(t)dt.$$

Taking limit $n \to \infty$ and after simplification

$$\int_0^{d(Su,z)} \rho(t)dt \leq [2(a+c+e)+(b+f)] \int_0^{d(Su,z)} \rho(t)dt.$$

This is possible only if $Su = Au = z$.
Again since $SX \subset BX$ there exist a point $v \in X$ such that $z = Bv$.

$$\int_0^{d(z,Tv)} \rho(t)dt = \int_0^{d(Su,Tv)} \rho(t)dt$$

$$\leq a \cdot \int_0^{d(Au,Bv)} \rho(t)dt + b \cdot \int_0^{d(Au,Su)} \rho(t)dt + c \cdot \int_0^{d(Bv,Tv)} \rho(t)dt +$$

$$e \cdot \int_0^{d(Au,Tv)} \rho(t)dt + f \cdot \int_0^{d(Bv,Su)} \rho(t)dt.$$

$$\leq a \cdot \int_0^{d(z,z)} \rho(t)dt + b \cdot \int_0^{d(z,z)} \rho(t)dt + c \cdot \int_0^{d(z,Tv)} \rho(t)dt +$$

$$e \cdot \int_0^{d(z,Tv)} \rho(t)dt + f \cdot \int_0^{d(z,z)} \rho(t)dt.$$

$$\int_0^{d(z,Tv)} \rho(t)dt \leq [2(a+b+f) + (c+e)] \int_0^{d(z,Tv)} \rho(t)dt.$$

This is possible only if $d(z, Tv) = 0$, so we get $Tv = z$. Hence $Su = Au = Tv = Bv = z$.
Since the pair (S, A) are weakly compatible, so $ASu = SAu$ implies $Sz = Az$.
Now to show that z is fixed point of S.

$$\int_0^{d(Sz,z)} \rho(t)dt = \int_0^{d(Sz,Tv)} \rho(t)dt$$

$$\leq a \cdot \int_0^{d(Az,Bv)} \rho(t)dt + b \cdot \int_0^{d(Az,Sz)} \rho(t)dt + c \cdot \int_0^{d(Bv,Tv)} \rho(t)dt +$$

$$e \cdot \int_0^{d(Az,Tv)} \rho(t)dt + f \cdot \int_0^{d(Bv,Sz)} \rho(t)dt.$$

$$\leq a \cdot \int_0^{d(Sz,z)} \rho(t)dt + b \cdot \int_0^{d(Sz,Sz)} \rho(t)dt + c \cdot \int_0^{d(z,z)} \rho(t)dt +$$

$$e \cdot \int_0^{d(Sz,z)} \rho(t)dt + f \cdot \int_0^{d(z,Sz)} \rho(t)dt.$$

$$\int_0^{d(Sz,z)} \rho(t)dt \leq (a + 2b + 2c + e + f) \int_0^{d(Sz,z)} \rho(t)dt.$$

The above inequality is possible if $d(Sz, z) = 0$ implies that $Sz = Az = z$.

Again the pair (T, B) are weakly compatible, so $TBv = BTv$ implies $Tz = Bz$.
Now to show that z is fixed point of T. Consider

$$\int_0^{d(z,Tz)} \rho(t)dt = \int_0^{d(Sz,Tz)} \rho(t)dt$$

$$\leq a \cdot \int_0^{d(Az,Bz)} \rho(t)dt + b \cdot \int_0^{d(Az,Sz)} \rho(t)dt + c \cdot \int_0^{d(Bz,Tz)} \rho(t)dt +$$

$$e \cdot \int_0^{d(Az,Tz)} \rho(t)dt + f \cdot \int_0^{d(Bz,Sz)} \rho(t)dt.$$

$$\leq a \cdot \int_0^{d(z,Tz)} \rho(t)dt + b \cdot \int_0^{d(z,z)} \rho(t)dt + c \cdot \int_0^{d(Tz,Tz)} \rho(t)dt +$$

$$e \cdot \int_0^{d(z,Tz)} \rho(t)dt + f \cdot \int_0^{d(Tz,z)} \rho(t)dt.$$

$$\int_0^{d(z,Tz)} \rho(t)dt \leq (a + 2b + 2c + e + f) \int_0^{d(z,Tz)} \rho(t)dt.$$

The above inequality is possible if $d(z, Tz) = 0$ implies that $Sz = Bz = Az = Tz = z$. Thus we obtained a common fixed point for mappings A, B, S and T.

Uniqueness. Now consider that u, v are two distinct fixed points of A, B, S and T then again by given condition (3) in the theorem we have

$$\int_0^{d(u,v)} \rho(t)dt = \int_0^{d(Su,Tv)} \rho(t)dt$$

$$\leq a \cdot \int_0^{d(Au,Bv)} \rho(t)dt + b \cdot \int_0^{d(Au,Su)} \rho(t)dt + c \cdot \int_0^{d(Bv,Tv)} \rho(t)dt +$$

$$e \cdot \int_0^{d(Au,Tv)} \rho(t)dt + f \cdot \int_0^{d(Bv,Su)} \rho(t)dt.$$

Now using the fact that u, v are fixed points of A, B, S and T and then simplifying We get the following inequality

$$\int_0^{d(u,v)} \rho(t)dt \leq (a + 2b + 2c + e + f) \int_0^{d(u,v)} \rho(t)dt.$$

Since $a + 2b + 2c + e + f < 1$, so the a above inequality is possible if $d(u, v) = 0$ similarly we can show that $d(v, u) = 0$ which implies that $u = v$. Hence fixed point of mappings A, B, S and T is unique. □

Theorem(2.1) yields the following corollaries.

Putting $A = B = I$ in the above theorem we get the following corollary.

Corollary 2.2. . *Let (X, d) be a complete dislocated metric space, for $a, b, c, e, f \geq 0$ with $2(a+b+c+e+f) < 1$ and $S, T : X \to X$ be self-mappings such that for all $x, y \in X$, satisfying the conditions*

$$\int_0^{d(Sx,Ty)} \rho(t)dt \leq a \cdot \int_0^{d(x,y)} \rho(t)dt + b \cdot \int_0^{d(x,Sx)} \rho(t)dt + c \cdot \int_0^{d(y,Ty)} \rho(t)dt +$$

$$e \cdot \int_0^{d(x,Ty)} \rho(t)dt + f \cdot \int_0^{d(y,Sx)} \rho(t)dt$$

. *where $\rho : \mathbb{R}^+ \to \mathbb{R}^+$ is a Lebesque integrable mapping which is summable on each compact subset of $\mathbb{R}^+$, non-negative and such that for any $s > 0$ $\int_0^s \rho(t)dt > 0$. Then S and T have a unique common fixed point.*

In the above corollary if $S = T$, then we get the corollary as follow.

Corollary 2.3. *Let (X, d) be a complete dislocated metric space, for $a, b, c, e, f \geq 0$ with $2(a+b+c+e+f) < 1$ and $T : X \to X$ be a self-mapping such that for all $x, y \in X$, satisfying the conditions*

$$\int_0^{d(Tx,Ty)} \rho(t)dt \leq a \cdot \int_0^{d(x,y)} \rho(t)dt + b \cdot \int_0^{d(x,Tx)} \rho(t)dt + c \cdot \int_0^{d(y,Ty)} \rho(t)dt +$$

$$e \cdot \int_0^{d(x,Ty)} \rho(t)dt + f \cdot \int_0^{d(y,Tx)} \rho(t)dt$$

. *where $\rho : \mathbb{R}^+ \to \mathbb{R}^+$ is a Lebesque integrable mapping which is summable on each compact subset of $\mathbb{R}^+$, non-negative and such that for any $s > 0$ $\int_0^s \rho(t)dt > 0$. Then T has a unique fixed point.*

If we take $A = T$ and $B = S$ in Corollary 2.2, we get the following result.

Corollary 2.4. *Let (X, d) be a complete dislocated metric space, for $a, b, c, e, f \geq 0$ with $2(a+b+c+e+f) < 1$ and $S, T : X \to X$ be self-mappings such that for all $x, y \in X$, satisfying the conditions*

$$\int_0^{d(Sx,Ty)} \rho(t)dt \leq a \cdot \int_0^{d(Tx,Sy)} \rho(t)dt + b \cdot \int_0^{d(Tx,Sx)} \rho(t)dt + c \cdot \int_0^{d(Sy,Ty)} \rho(t)dt +$$

$$e \cdot \int_0^{d(Tx,Ty)} \rho(t)dt + f \cdot \int_0^{d(Sy,Sx)} \rho(t)dt$$

. *where $\rho : \mathbb{R}^+ \to \mathbb{R}^+$ is a Lebesque integrable mapping which is summable on each compact subset of $\mathbb{R}^+$, non-negative and such that for any $s > 0$ $\int_0^s \rho(t)dt > 0$. Then S and T have a unique common fixed point.*

Remark. Our established theorem and corollaries deduced from it generalize and extend many fixed point results in the literature.

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