Available online at http://cna-journal.com
Commun. Nonlinear Anal. 2 (2020), 56–74
Research Article

Communications in Nonlinear Analysis

Journal Homepage: www.cna-journal.com

Some Ostrowski type inequalities for functions whose modulus of the first derivatives are log-preinvex via k-Reimann-Liouville fractional integrals

Badreddine Meftah[a,*], Safa Bouchehed[a], Hala Djebabla[a]

[a]*Department of Mathematics, Faculty of Mathematics, Computer and Material Sciences, University 8 mai 1945 Guelma, Algeria.*

Abstract

In this paper, we establish a new identity integral and then we derive some Ostrowski's inequalities for functions whose modulus of the first derivatives are log-preinvex via k-Reimann-Liouville fractional integral.

Keywords: Ostrowski inequality, k-Reimann-Liouville operator integrals, Hölder inequality, log-preinvexity.
2010 MSC: 26D10, 26D15, 26A51.

1. Introduction

In 1938, A. M. Ostrowski proved the following integral inequality

Theorem 1.1. *[29] Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I° (interior of I), and let $a, b \in I^\circ$ with $a < b$. If $|f'| \leq M$ for all $x \in (a, b)$, then*

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)\, dt \right| \leq M(b-a)\left[\frac{1}{4} + \left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right], \quad \forall x \in [a,b]. \tag{1.1}$$

Inequality (1.1) has attracted a lot of interest from researchers due to its variety of applications in numerical analysis, probability theory and other fields. Many variants, improvements, extensions and generalizations of inequality (1.1) have been discovered, for more details, we refer readers to [5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 28, 31, 32, 33, 34] and to the references therein.

*Corresponding author
Email addresses: badrimeftah@yahoo.fr (Badreddine Meftah), bouchehedsafa@yahoo.com (Safa Bouchehed), djebablahala17@gmail.com (Hala Djebabla)

Received 2020-03-05

In Alomari et al. [1] the authors gave the following Ostrowski inequalities for s-convex functions

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \le \frac{M}{b-a}\left[\frac{(x-a)^2+(b-x)^2}{1+s}\right],$$

and

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \le \frac{M}{(1+p)^{\frac{1}{p}}}\left(\frac{2}{1+s}\right)^{\frac{1}{q}}\left[\frac{(x-a)^2+(b-x)^2}{b-a}\right].$$

In [2] Alomari and Darus established the following Ostrowski inequalities for quasi-convex functions

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(u)du \right| \le \left(\frac{(b-x)^{p+1}}{(b-a)(p+1)}\right)^{\frac{1}{p}} \left(\max\left\{\left|f'(x)\right|^q, \left|f'(b)\right|^q\right\}\right)^{\frac{1}{q}}$$
$$+\left(\frac{(x-a)^{p+1}}{(b-a)(p+1)}\right)^{\frac{1}{p}} \left(\max\left\{\left|f'(x)\right|^q, \left|f'(a)\right|^q\right\}\right)^{\frac{1}{q}},$$

and

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(u)du \right| \le \frac{(b-x)^2}{2(b-a)} \left(\max\left\{\left|f'(x)\right|^q, \left|f'(b)\right|^q\right\}\right)^{\frac{1}{q}}$$
$$+\frac{(x-a)^2}{2(b-a)} \left(\max\left\{\left|f'(x)\right|^q, \left|f'(a)\right|^q\right\}\right)^{\frac{1}{q}}.$$

İşcan [4] investigate the following Ostrowski for preinvex functions

$$\left| f(x) - \frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(u)du \right| \le \frac{\eta(b,a)}{6}$$
$$\times\left\{\left(3\left(\frac{x-a}{\eta(b,a)}\right)^2 - 2\left(\frac{x-a}{\eta(b,a)}\right)^3 + 2\left(\frac{a+\eta(b,a)-x}{\eta(b,a)}\right)^3\right)\left|f'(a)\right|\right.$$
$$\left.+\left(1-3\left(\frac{x-a}{\eta(b,a)}\right)^2 + 4\left(\frac{x-a}{\eta(b,a)}\right)^3\right)\left|f'(b)\right|\right\},$$

and

$$\left| f(x) - \frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(u)du \right| \le \eta(b,a)\left(\frac{1}{2}\right)^{1-\frac{1}{q}}$$
$$\times\left\{\left(\frac{x-a}{\eta(b,a)}\right)^{2\left(1-\frac{1}{q}\right)}\left(\frac{(x-a)^2(3\eta(b,a)-2x+2a)}{6\eta^3(b,a)}\left|f'(a)\right|^q + \frac{1}{3}\left(\frac{x-a}{\eta(b,a)}\right)^3\left|f'(b)\right|^q\right)^{\frac{1}{q}}\right.$$
$$+\left(\frac{a+\eta(b,a)-x}{\eta(b,a)}\right)^{2\left(1-\frac{1}{q}\right)}\left(\frac{1}{3}\left(\frac{a+\eta(b,a)-x}{\eta(b,a)}\right)^3\left|f'(a)\right|^q\right.$$
$$\left.\left.+\left(\frac{1}{6}+\frac{(x-a)^2(2x-3\eta(b,a)-2a)}{6\eta^3(b,a)}\right)\left|f'(b)\right|^q\right)^{\frac{1}{q}}\right\}.$$

Meftah [16] gave the following Ostrowski type inequality for functions whose n^{th} order derivative are log-convex

$$\left| \int_a^b f(t)dt - \sum_{k=0}^{k=n}\left[\frac{(b-x)^{k+1}+(-1)^k(x-a)^{k+1}}{(k+1)!}\right]f^{(k)}(x) \right|$$

$$\leq \begin{cases} \frac{(x-a)^{n+1}}{(n+1)!}\left|f^{(n)}(a)\right| + \frac{(b-x)^{n+1}}{(n+1)!}\left|f^{(n)}(x)\right| \text{ if } \lambda=\tau=1 \\ \frac{(x-a)^{n+1}}{(n+1)!}\left|f^{(n)}(a)\right| + \frac{(b-x)^{n+1}}{n!}\left|f^{(n)}(x)\right| \sum_{i=1}^{\infty}\frac{(\ln\tau)^{i-1}}{(n+1)_i} \text{if } \lambda=1\neq\tau \\ \frac{(b-x)^{n+1}}{(n+1)!}\left|f^{(n)}(x)\right| + \frac{(x-a)^{n+1}}{n!}\left|f^{(n)}(a)\right| \lambda\sum_{i=1}^{\infty}\frac{(-\ln\lambda)^{i-1}}{(n+1)_i} \text{if } \lambda\neq 1=\tau \\ \frac{(x-a)^{n+1}}{n!}\left|f^{(n)}(a)\right| \lambda\sum_{i=1}^{\infty}\frac{(-\ln\lambda)^{i-1}}{(n+1)_i} + \frac{(b-x)^{n+1}}{n!}\left|f^{(n)}(x)\right| \sum_{i=1}^{\infty}\frac{(\ln\tau)^{i-1}}{(n+1)_i} \\ \text{if } \lambda\neq 1 \text{ and } \tau\neq 1, \end{cases}$$

where $\lambda = \frac{\left|f^{(n)}(x)\right|}{\left|f^{(n)}(a)\right|}$ and $\tau = \frac{\left|f^{(n)}(b)\right|}{\left|f^{(n)}(x)\right|}$. Motivated by the above results, in this paper by using a new identity we establish some new Ostrowski type inequalities for functions whose modulus of the first derivatives are log-preinvex via k-Reimann-Liouville fractional integrals.

2. Preliminaries

In this section, we recall some definitions and lemmas

Definition 2.1. [30] A function $f : I \to \mathbb{R}$ is said to be convex on I where I is an interval of $\mathbb{R}$, if

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)$$

holds for all $x, y \in I$ and all $t \in [0,1]$.

Definition 2.2. [30] A function $f : I \subset (0,+\infty) \to \mathbb{R}$ is said to be log-convex on I, if

$$f(tx + (1-t)y) \leq (f(x))^t (f(y))^{1-t}$$

holds for all $x, y \in I$ and all $t \in [0,1]$.

Definition 2.3. [36] A set $K \subset \mathbb{R}^n$ is said to be invex with respect to the map $\eta : K \times K \to \mathbb{R}^n$, if

$$x + t\eta(y,x) \in K$$

holds for all $x, y \in K$ and $t \in [0,1]$.

Definition 2.4. [36] A function $f : K \subset (0,+\infty) \to \mathbb{R}$ is said to be preinvex with respect to η, if

$$f(x + t\eta(y,x)) \leq (1-t)f(x) + tf(y)$$

holds for all $x, y \in K$ and $t \in [0,1]$.

Definition 2.5. [27] A function $f : K \to \mathbb{R}$ is said to be log-preinvex with respect to η, if

$$f(x + t\eta(y,x)) \leq (f(x))^{1-t}(f(y))^t$$

holds for all $x, y \in K$ and $t \in [0,1]$.

Definition 2.6. [25] The Riemann-Liouville integrals $J^{\alpha}_{a^+}f$ and $J^{\alpha}_{b^-}f$ of order $\alpha > 0$ with $a \geq 0$ are defined by

$$J^{\alpha}_{a^+}f(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1} f(t)dt, \qquad x > a$$

$$J^{\alpha}_{b^-}f(x) = \frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1} f(t)dt, \qquad b > x$$

respectively, for $f \in L_1[a,b]$ where $\Gamma(\alpha) = \int_0^{\infty} e^{-t}t^{\alpha-1}dt$, is the Gamma function, and $J^0_{a^+}f(x) = J^0_{b^-}f(x) = f(x)$.

Diaz et al. [3] introduced the generalized k-gamma function as

Definition 2.7. [3] If $k>0$, then k-gamma function Γ_k is defined as

$$\Gamma_k(\alpha)=\lim_{n\to\infty}\frac{n!k^n(nk)^{\frac{\alpha}{k}-1}}{(\alpha)_{n,k}},$$

where $(\alpha)_{n,k}$, is the Pochhammer k-symbol defined by

$$(\alpha)_{n,k}=x(x+k)(x+2k)...(x+(n-1)k)$$

, with $n\geq 1$.

Definition 2.8. [3] If $\operatorname{Re}(\alpha)>0$, then the integral form of k-gamma is given by

$$\Gamma_k(\alpha)=\int_0^\infty t^{\alpha-1}e^{-\frac{t^k}{k}}dt,$$

with the property that

$$\Gamma_k(\alpha+k)=\alpha\Gamma_k(\alpha).$$

Definition 2.9. [26] Let $f\in L[a,b]$. Then k-fractional integrals $I_{a^+}^{\alpha,k}f$ and $I_{b^-}^{\alpha,k}f$ of order $\alpha,k>0$ with $a\geq 0$ are defined as

$$I_{a+}^{\alpha,k}f(x)=\frac{1}{k\Gamma_k(\alpha)}\int_a^x(x-t)^{\frac{\alpha}{k}-1}f(t)\,dt,\quad x>a$$

and

$$I_{b-}^{\alpha,k}f(x)=\frac{1}{k\Gamma_k(\alpha)}\int_x^b(t-x)^{\frac{\alpha}{k}-1}f(t)\,dt,\ b>x.$$

Lemma 2.10. *[35] For $\alpha>0$ and $k>0$, $z>0$:*

$$J(\alpha,k)=\int_0^1(1-t)^{\alpha-1}k^tdt=\sum_{i=1}^{\infty}\frac{(\ln k)^{i-1}}{(\alpha)_i}<\infty, \tag{2.1}$$

$$H(\alpha,k,z)=\int_0^z t^{\alpha-1}k^tdt=z^\alpha k^z\sum_{i=1}^{\infty}\frac{(-z\ln k)^{i-1}}{(\alpha)_i}<\infty, \tag{2.2}$$

where $(\alpha)_i=\prod_{j=0}^{i-1}(\alpha+j)$.

3. Main results

Lemma 3.1. *Let $f:[a,a+\eta(b,a)]\to\mathbb{R}$ be a differentiable function with $a<a+\eta(b,a)$. If $f'\in L([a,a+\eta(b,a)])$, then the following equality for fractional integrals*

$$\left(\left(\tfrac{x-a}{\eta(b,a)}\right)^{\frac{\alpha}{k}}+\left(\tfrac{a+\eta(b,a)-x}{\eta(b,a)}\right)^{\frac{\alpha}{k}}\right)f(x)-\frac{\Gamma_k(\alpha+k)}{(\eta(b,a))^{\frac{\alpha}{k}}}\left(I_{x-}^{\alpha,k}f(a)+I_{x+}^{\alpha,k}f(a+\eta(b,a))\right)$$
$$=\eta(b,a)\left(\int_0^{\frac{x-a}{\eta(b,a)}}t^{\frac{\alpha}{k}}f'(a+t\eta(b,a))dt-\int_{\frac{x-a}{\eta(b,a)}}^1(1-t)^{\frac{\alpha}{k}}f'(a+t\eta(b,a))dt\right) \tag{3.1}$$

holds for all $x\in[a,a+\eta(b,a)]$.

Proof. Integrating by parts right hand side of (3.1), we get

$$
\begin{aligned}
&\eta(b,a)\left(\int_0^{\frac{x-a}{\eta(b,a)}} t^{\frac{\alpha}{k}} f'(a+t\eta(b,a))dt-\int_{\frac{x-a}{\eta(b,a)}}^1 (1-t)^{\frac{\alpha}{k}} f'(a+t\eta(b,a))dt\right)\\
&=\left(\tfrac{x-a}{\eta(b,a)}\right)^{\frac{\alpha}{k}} f(x)-\tfrac{\alpha}{k}\int_0^{\frac{x-a}{\eta(b,a)}} t^{\frac{\alpha}{k}-1} f(a+t\eta(b,a))dt\\
&\quad+\left(1-\tfrac{x-a}{\eta(b,a)}\right)^{\frac{\alpha}{k}} f(x)-\tfrac{\alpha}{k}\int_{\frac{x-a}{\eta(b,a)}}^1 (1-t)^{\frac{\alpha}{k}-1} f(a+t\eta(b,a))dt\\
&=\left(\left(\tfrac{x-a}{\eta(b,a)}\right)^{\frac{\alpha}{k}}+\left(1-\tfrac{x-a}{\eta(b,a)}\right)^{\frac{\alpha}{k}}\right)f(x)\\
&\quad-\tfrac{\alpha}{k}\left(\int_0^{\frac{x-a}{\eta(b,a)}} t^{\frac{\alpha}{k}-1} f(a+t\eta(b,a))dt+\int_{\frac{x-a}{\eta(b,a)}}^1 (1-t)^{\frac{\alpha}{k}-1} f(a+t\eta(b,a))dt\right).
\end{aligned}
\tag{3.2}
$$

Using the change of variable $u=a+t\eta(b,a)$, (3.2) becomes

$$
\begin{aligned}
&\eta(b,a)\left(\int_0^{\frac{x-a}{\eta(b,a)}} t^{\frac{\alpha}{k}} f'(a+t\eta(b,a))dt-\int_{\frac{x-a}{\eta(b,a)}}^1 (1-t)^{\frac{\alpha}{k}} f'(a+t\eta(b,a))dt\right)\\
&=\left(\left(\tfrac{x-a}{\eta(b,a)}\right)^{\frac{\alpha}{k}}+\left(1-\tfrac{x-a}{\eta(b,a)}\right)^{\frac{\alpha}{k}}\right)f(x)\\
&\quad-\tfrac{\alpha}{k(\eta(b,a))^{\frac{\alpha}{k}}}\left(\int_a^x (u-a)^{\frac{\alpha}{k}-1} f(u)du+\int_x^{a+\eta(b,a)} (a+\eta(b,a)-u)^{\frac{\alpha}{k}-1} f(u)du\right)\\
&=\left(\left(\tfrac{x-a}{\eta(b,a)}\right)^{\frac{\alpha}{k}}+\left(1-\tfrac{x-a}{\eta(b,a)}\right)^{\frac{\alpha}{k}}\right)f(x)\\
&\quad-\tfrac{\Gamma_k(\alpha+k)}{(\eta(b,a))^{\frac{\alpha}{k}}}\left(I_{x-}^{\alpha,k}f(a)+I_{x+}^{\alpha,k}f(a+\eta(b,a))\right),
\end{aligned}
$$

which is the desired result. □

Theorem 3.2. *Let $f:[a,a+\eta(b,a)]\to[0,\infty)$ be a differentiable function such that $f'\in L([a,a+\eta(b,a)])$ with $\eta(b,a)>0$. If $|f'|$ is* log*-preinvex function with respect to η such that $f'(a)\neq 0$, $f'(x)\neq 0$ and $f'(b)\neq 0$, then the following inequality for fractional integrals*

$$
\left|\left(\left(\tfrac{x-a}{\eta(b,a)}\right)^{\frac{\alpha}{k}}+\left(\tfrac{a+\eta(b,a)-x}{\eta(b,a)}\right)^{\frac{\alpha}{k}}\right)f(x)-\tfrac{\Gamma_k(\alpha+k)}{(\eta(b,a))^{\frac{\alpha}{k}}}\left(I_{x-}^{\alpha,k}f(a)+I_{x+}^{\alpha,k}f(a+\eta(b,a))\right)\right|
$$

$$\leq \begin{cases} \frac{k\eta(b,a)}{\alpha+k}\left|f'(a)\right|\left(\left(\frac{x-a}{\eta(b,a)}\right)^{\frac{\alpha}{k}+1}+\left(\frac{a+\eta(b,a)-x}{\eta(b,a)}\right)^{\frac{\alpha}{k}+1}\right) \quad \text{if } |f'(a)|=|f'(x)|=|f'(b)|, \\ \eta(b,a)\left(\frac{k}{\alpha+k}\left|f'(a)\right|\left(\frac{x-a}{\eta(b,a)}\right)^{\frac{\alpha}{k}+1}+\left|f'(b)\right|\left(1-\frac{x-a}{\eta(b,a)}\right)^{\frac{\alpha}{k}+1}\left(\frac{|f'(a)|}{|f'(b)|}\right)^{1-\frac{x-a}{\eta(b,a)}}\right. \\ \quad \left.\times\sum_{i=1}^{\infty}\frac{\left(-\left(1-\frac{x-a}{\eta(b,a)}\right)\ln\left(\frac{|f'(a)|}{|f'(b)|}\right)\right)^{i-1}}{\left(\frac{\alpha}{k}+1\right)_i}\right) \quad \text{if } |f'(a)|=|f'(x)|\neq|f'(b)|, \\ \eta(b,a)\left(\frac{k}{\alpha+k}\left|f'(b)\right|\left(1-\frac{x-a}{\eta(b,a)}\right)^{\frac{\alpha}{k}+1}+\left|f'(a)\right|\left(\frac{|f'(b)|}{|f'(a)|}\right)^{\frac{x-a}{\eta(b,a)}}\left(\frac{x-a}{\eta(b,a)}\right)^{\frac{\alpha}{k}+1}\right. \\ \quad \left.\times\sum_{i=1}^{\infty}\frac{\left(-\frac{x-a}{\eta(b,a)}\ln\frac{|f'(b)|}{|f'(a)|}\right)^{i-1}}{\left(\frac{\alpha}{k}+1\right)_i}\right) \quad \text{if } |f'(a)|\neq|f'(x)|=|f'(b)|, \\ \eta(b,a)\left(\left|f'(a)\right|\left(\frac{x-a}{\eta(b,a)}\right)^{\frac{\alpha}{k}+1}\left(\frac{|f'(x)|}{|f'(a)|}\right)^{\frac{x-a}{\eta(b,a)}}\sum_{i=1}^{\infty}\frac{\left(-\frac{x-a}{\eta(b,a)}\ln\frac{|f'(x)|}{|f'(a)|}\right)^{i-1}}{\left(\frac{\alpha}{k}+1\right)_i}\right. \\ \quad \left.+\left|f'(x)\right|\left(1-\frac{x-a}{\eta(b,a)}\right)^{\frac{\alpha}{k}+1}\left(\frac{|f'(b)|}{|f'(x)|}\right)^{\frac{x-a}{\eta(b,a)}}\sum_{i=1}^{\infty}\frac{\left(-\left(1-\frac{x-a}{\eta(b,a)}\right)\ln\frac{|f'(x)|}{|f'(b)|}\right)^{i-1}}{\left(\frac{\alpha}{k}+1\right)_i}\right) \\ \quad \text{if } |f'(a)|\neq|f'(x)|\neq|f'(b)| \end{cases}$$

holds for all $x\in[a,a+\eta(b,a)]$.

Proof. From Lemma 2.10, and properties of modulus, we have

$$\left|\left(\left(\frac{x-a}{\eta(b,a)}\right)^{\frac{\alpha}{k}}+\left(\frac{a+\eta(b,a)-x}{\eta(b,a)}\right)^{\frac{\alpha}{k}}\right)f(x)-\frac{\Gamma_k(\alpha+k)}{(\eta(b,a))^{\frac{\alpha}{k}}}\left(I_{x-}^{\alpha,k}f(a)+I_{x+}^{\alpha,k}f(a+\eta(b,a))\right)\right|$$
$$\leq\eta(b,a)\left(\int_0^{\frac{x-a}{\eta(b,a)}}t^{\frac{\alpha}{k}}\left|f'(a+t\eta(b,a))\right|dt+\int_{\frac{x-a}{\eta(b,a)}}^1(1-t)^{\frac{\alpha}{k}}\left|f'(a+t\eta(b,a))\right|dt\right). \tag{3.3}$$

Since $|f'|$ is log-preinvex, we deduce

$$\left|\left(\left(\frac{x-a}{\eta(b,a)}\right)^{\frac{\alpha}{k}}+\left(\frac{a+\eta(b,a)-x}{\eta(b,a)}\right)^{\frac{\alpha}{k}}\right)f(x)-\frac{\Gamma_k(\alpha+k)}{(\eta(b,a))^{\frac{\alpha}{k}}}\left(I_{x-}^{\alpha,k}f(a)+I_{x+}^{\alpha,k}f(a+\eta(b,a))\right)\right|$$
$$\leq\eta(b,a)\left(\int_0^{\frac{x-a}{\eta(b,a)}}t^{\frac{\alpha}{k}}\left|f'(a)\right|^{1-t}\left|f'(x)\right|^{t}dt+\int_{\frac{x-a}{\eta(b,a)}}^1(1-t)^{\frac{\alpha}{k}}\left|f'(x)\right|^{1-t}\left|f'(b)\right|^{t}dt\right)$$
$$=\eta(b,a)\left(\left|f'(a)\right|\int_0^{\frac{x-a}{\eta(b,a)}}t^{\frac{\alpha}{k}}\left(\frac{|f'(x)|}{|f'(a)|}\right)^{t}dt+\left|f'(x)\right|\int_{\frac{x-a}{\eta(b,a)}}^1(1-t)^{\frac{\alpha}{k}}\left(\frac{|f'(b)|}{|f'(x)|}\right)^{t}dt\right). \tag{3.4}$$

If $|f'(a)|=|f'(x)|=|f'(b)|$, then (3.4) gives

$$\left|\left(\left(\frac{x-a}{\eta(b,a)}\right)^{\frac{\alpha}{k}}+\left(\frac{a+\eta(b,a)-x}{\eta(b,a)}\right)^{\frac{\alpha}{k}}\right)f(x)-\frac{\Gamma_k(\alpha+k)}{(\eta(b,a))^{\frac{\alpha}{k}}}\left(I_{x-}^{\alpha,k}f(a)+I_{x+}^{\alpha,k}f(a+\eta(b,a))\right)\right|$$
$$\leq\eta(b,a)\left(\left|f'(a)\right|\int_0^{\frac{x-a}{\eta(b,a)}}t^{\frac{\alpha}{k}}dt+\left|f'(x)\right|\int_{\frac{x-a}{\eta(b,a)}}^1(1-t)^{\frac{\alpha}{k}}dt\right)$$

$$=\frac{k\eta(b,a)}{\alpha+k}\left(\left|f'(a)\right|\left(\frac{x-a}{\eta(b,a)}\right)^{\frac{\alpha}{k}+1}+\left|f'(x)\right|\left(\frac{a+\eta(b,a)-x}{\eta(b,a)}\right)^{\frac{\alpha}{k}+1}\right)$$
$$=\frac{k\eta(b,a)}{\alpha+k}\left|f'(a)\right|\left(\left(\frac{x-a}{\eta(b,a)}\right)^{\frac{\alpha}{k}+1}+\left(\frac{a+\eta(b,a)-x}{\eta(b,a)}\right)^{\frac{\alpha}{k}+1}\right). \tag{3.5}$$

If $|f'(a)|=|f'(x)|\neq|f'(b)|$, then (3.4) gives

$$\left|\left(\left(\frac{x-a}{\eta(b,a)}\right)^{\frac{\alpha}{k}}+\left(\frac{a+\eta(b,a)-x}{\eta(b,a)}\right)^{\frac{\alpha}{k}}\right)f(x)-\frac{\Gamma_k(\alpha+k)}{(\eta(b,a))^{\frac{\alpha}{k}}}\left(I_{x-}^{\alpha,k}f(a)+I_{x+}^{\alpha,k}f(a+\eta(b,a))\right)\right|$$
$$\leq\eta(b,a)\left(\left|f'(a)\right|\int_0^{\frac{x-a}{\eta(b,a)}}t^{\frac{\alpha}{k}}\left(\frac{|f'(x)|}{|f'(a)|}\right)^t dt+\left|f'(x)\right|\int_{\frac{x-a}{\eta(b,a)}}^1(1-t)^{\frac{\alpha}{k}}\left(\frac{|f'(b)|}{|f'(x)|}\right)^t dt\right)$$
$$=\eta(b,a)\left(\frac{k}{\alpha+k}\left|f'(a)\right|\left(\frac{x-a}{\eta(b,a)}\right)^{\frac{\alpha}{k}+1}+\left|f'(b)\right|\int_0^{1-\frac{x-a}{\eta(b,a)}}t^{\frac{\alpha}{k}}\left(\frac{|f'(x)|}{|f'(b)|}\right)^t dt\right)$$
$$=\eta(b,a)\left(\frac{k}{\alpha+k}\left|f'(a)\right|\left(\frac{x-a}{\eta(b,a)}\right)^{\frac{\alpha}{k}+1}\right.$$
$$\left.+\left|f'(b)\right|\left(1-\frac{x-a}{\eta(b,a)}\right)^{\frac{\alpha}{k}+1}\left(\frac{|f'(a)|}{|f'(b)|}\right)^{1-\frac{x-a}{\eta(b,a)}}\sum_{i=1}^{\infty}\frac{\left(-\left(1-\frac{x-a}{\eta(b,a)}\right)\ln\left(\frac{|f'(x)|}{|f'(b)|}\right)\right)^{i-1}}{\left(\frac{\alpha}{k}+1\right)_i}\right)$$
$$=\eta(b,a)\left(\frac{k}{\alpha+k}\left|f'(a)\right|\left(\frac{x-a}{\eta(b,a)}\right)^{\frac{\alpha}{k}+1}\right.$$
$$\left.+\left|f'(b)\right|\left(1-\frac{x-a}{\eta(b,a)}\right)^{\frac{\alpha}{k}+1}\left(\frac{|f'(a)|}{|f'(b)|}\right)^{1-\frac{x-a}{\eta(b,a)}}\sum_{i=1}^{\infty}\frac{\left(-\left(1-\frac{x-a}{\eta(b,a)}\right)\ln\left(\frac{|f'(a)|}{|f'(b)|}\right)\right)^{i-1}}{\left(\frac{\alpha}{k}+1\right)_i}\right). \tag{3.6}$$

If $|f'(a)|\neq|f'(x)|=|f'(b)|$, then (3.4) gives

$$\left|\left(\left(\frac{x-a}{\eta(b,a)}\right)^{\frac{\alpha}{k}}+\left(\frac{a+\eta(b,a)-x}{\eta(b,a)}\right)^{\frac{\alpha}{k}}\right)f(x)-\frac{\Gamma_k(\alpha+k)}{(\eta(b,a))^{\frac{\alpha}{k}}}\left(I_{x-}^{\alpha,k}f(a)+I_{x+}^{\alpha,k}f(a+\eta(b,a))\right)\right|$$
$$\leq\eta(b,a)\left(\left|f'(a)\right|\int_0^{\frac{x-a}{\eta(b,a)}}t^{\frac{\alpha}{k}}\left(\frac{|f'(x)|}{|f'(a)|}\right)^t dt+\left|f'(x)\right|\int_{\frac{x-a}{\eta(b,a)}}^1(1-t)^{\frac{\alpha}{k}}\left(\frac{|f'(b)|}{|f'(x)|}\right)^t dt\right)$$
$$=\eta(b,a)\left(\left|f'(a)\right|\int_0^{\frac{x-a}{\eta(b,a)}}t^{\frac{\alpha}{k}}\left(\frac{|f'(x)|}{|f'(a)|}\right)^t dt+\frac{k}{\alpha+k}\left|f'(x)\right|\left(1-\frac{x-a}{\eta(b,a)}\right)^{\frac{\alpha}{k}+1}\right)$$
$$=\eta(b,a)\left(\left|f'(a)\right|\left(\frac{|f'(x)|}{|f'(a)|}\right)^{\frac{x-a}{\eta(b,a)}}\left(\frac{x-a}{\eta(b,a)}\right)^{\frac{\alpha}{k}+1}\sum_{i=1}^{\infty}\frac{\left(-\frac{x-a}{\eta(b,a)}\ln\frac{|f'(x)|}{|f'(a)|}\right)^{i-1}}{\left(\frac{\alpha}{k}+1\right)_i}\right.$$
$$\left.+\frac{k}{\alpha+k}\left|f'(x)\right|\left(1-\frac{x-a}{\eta(b,a)}\right)^{\frac{\alpha}{k}+1}\right)$$
$$=\eta(b,a)\left(\left|f'(a)\right|\left(\frac{|f'(b)|}{|f'(a)|}\right)^{\frac{x-a}{\eta(b,a)}}\left(\frac{x-a}{\eta(b,a)}\right)^{\frac{\alpha}{k}+1}\sum_{i=1}^{\infty}\frac{\left(-\frac{x-a}{\eta(b,a)}\ln\frac{|f'(b)|}{|f'(a)|}\right)^{i-1}}{\left(\frac{\alpha}{k}+1\right)_i}\right.$$
$$\left.+\frac{k}{\alpha+k}\left|f'(b)\right|\left(1-\frac{x-a}{\eta(b,a)}\right)^{\frac{\alpha}{k}+1}\right). \tag{3.7}$$

In the case where $|f'(a)| \neq |f'(b)| \neq |f'(x)|$, then applying Lemma 3.1 for (3.4) we get

$$\begin{aligned}
&\left|\left(\left(\tfrac{x-a}{\eta(b,a)}\right)^{\frac{\alpha}{k}}+\left(\tfrac{a+\eta(b,a)-x}{\eta(b,a)}\right)^{\frac{\alpha}{k}}\right)f(x)-\tfrac{\Gamma_k(\alpha+k)}{(\eta(b,a))^{\frac{\alpha}{k}}}\left(I_{x-}^{\alpha,k}f(a)+I_{x+}^{\alpha,k}f(a+\eta(b,a))\right)\right| \\
\leq&\eta(b,a)\left(|f'(a)|\int_0^{\frac{x-a}{\eta(b,a)}}t^{\frac{\alpha}{k}}\left(\tfrac{|f'(x)|}{|f'(a)|}\right)^t dt+|f'(x)|\int_{\frac{x-a}{\eta(b,a)}}^1(1-t)^{\frac{\alpha}{k}}\left(\tfrac{|f'(b)|}{|f'(x)|}\right)^t dt\right) \\
=&\eta(b,a)\left(|f'(a)|\int_0^{\frac{x-a}{\eta(b,a)}}t^{\frac{\alpha}{k}}\left(\tfrac{|f'(x)|}{|f'(a)|}\right)^t dt+|f'(b)|\int_0^{1-\frac{x-a}{\eta(b,a)}}t^{\frac{\alpha}{k}}\left(\tfrac{|f'(x)|}{|f'(b)|}\right)^t dt\right) \\
=&\eta(b,a)\left(|f'(a)|\left(\tfrac{x-a}{\eta(b,a)}\right)^{\frac{\alpha}{k}+1}\left(\tfrac{|f'(x)|}{|f'(a)|}\right)^{\frac{x-a}{\eta(b,a)}}\sum_{i=1}^{\infty}\frac{\left(-\frac{x-a}{\eta(b,a)}\ln\frac{|f'(x)|}{|f'(a)|}\right)^{i-1}}{\left(\frac{\alpha}{k}+1\right)_i}\right. \\
&\left.+|f'(x)|\left(1-\tfrac{x-a}{\eta(b,a)}\right)^{\frac{\alpha}{k}+1}\left(\tfrac{|f'(b)|}{|f'(x)|}\right)^{\frac{x-a}{\eta(b,a)}}\sum_{i=1}^{\infty}\frac{\left(-\left(1-\frac{x-a}{\eta(b,a)}\right)\ln\frac{|f'(x)|}{|f'(b)|}\right)^{i-1}}{\left(\frac{\alpha}{k}+1\right)_i}\right).
\end{aligned}\tag{3.8}$$

From (3.5)-(3.8), we obtain the desired result. □

Corollary 3.3. *In Theorem 3.2, if we choose $\eta(b,a)=b-a$, we have*

$$\begin{aligned}
&\left|\left(\left(\tfrac{x-a}{b-a}\right)^{\frac{\alpha}{k}}+\left(\tfrac{b-x}{b-a}\right)^{\frac{\alpha}{k}}\right)f(x)-\tfrac{\Gamma_k(\alpha+k)}{(b-a)^{\frac{\alpha}{k}}}\left(I_{x-}^{\alpha,k}f(a)+I_{x+}^{\alpha,k}f(b)\right)\right| \\
\leq&\begin{cases}
\frac{k(b-a)}{\alpha+k}|f'(a)|\left(\left(\frac{x-a}{b-a}\right)^{\frac{\alpha}{k}+1}+\left(\frac{b-x}{b-a}\right)^{\frac{\alpha}{k}+1}\right)\ \text{if } |f'(a)|=|f'(x)|=|f'(b)|, \\
(b-a)\left(\frac{k}{\alpha+k}|f'(a)|\left(\frac{x-a}{b-a}\right)^{\frac{\alpha}{k}+1}+|f'(b)|\left(\frac{b-x}{b-a}\right)^{\frac{\alpha}{k}+1}\left(\frac{|f'(a)|}{|f'(b)|}\right)^{\frac{b-x}{b-a}}\right. \\
\quad\left.\times\sum_{i=1}^{\infty}\frac{\left(-\frac{b-x}{b-a}\ln\left(\frac{|f'(a)|}{|f'(b)|}\right)\right)^{i-1}}{\left(\frac{\alpha}{k}+1\right)_i}\right)\ \text{if } |f'(a)|=|f'(x)|\neq|f'(b)|, \\
(b-a)\left(\frac{k}{\alpha+k}|f'(b)|\left(\frac{b-x}{b-a}\right)^{\frac{\alpha}{k}+1}+|f'(a)|\left(\frac{|f'(b)|}{|f'(a)|}\right)^{\frac{x-a}{b-a}}\left(\frac{x-a}{b-a}\right)^{\frac{\alpha}{k}+1}\right. \\
\quad\left.\times\sum_{i=1}^{\infty}\frac{\left(-\frac{x-a}{b-a}\ln\frac{|f'(b)|}{|f'(a)|}\right)^{i-1}}{\left(\frac{\alpha}{k}+1\right)_i}\right)\ \text{if } |f'(a)|\neq|f'(x)|=|f'(b)|, \\
(b-a)\left(|f'(a)|\left(\frac{x-a}{b-a}\right)^{\frac{\alpha}{k}+1}\left(\frac{|f'(x)|}{|f'(a)|}\right)^{\frac{x-a}{b-a}}\sum_{i=1}^{\infty}\frac{\left(-\frac{x-a}{b-a}\ln\frac{|f'(x)|}{|f'(a)|}\right)^{i-1}}{\left(\frac{\alpha}{k}+1\right)_i}\right. \\
\quad\left.+|f'(x)|\left(\frac{b-x}{b-a}\right)^{\frac{\alpha}{k}+1}\left(\frac{|f'(b)|}{|f'(x)|}\right)^{\frac{x-a}{b-a}}\sum_{i=1}^{\infty}\frac{\left(-\frac{b-x}{b-a}\ln\frac{|f'(x)|}{|f'(b)|}\right)^{i-1}}{\left(\frac{\alpha}{k}+1\right)_i}\right) \\
\quad\text{if } |f'(a)|\neq|f'(x)|\neq|f'(b)|.
\end{cases}
\end{aligned}$$

Corollary 3.4. *In Theorem 3.2, if we taking $k=1$, we obtain*

$$\left|\left(\left(\tfrac{x-a}{\eta(b,a)}\right)^{\alpha}+\left(\tfrac{a+\eta(b,a)-x}{\eta(b,a)}\right)^{\alpha}\right)f(x)-\tfrac{\Gamma(\alpha+1)}{(\eta(b,a))^{\alpha}}\left(J_{x-}^{\alpha}f(a)+J_{x+}^{\alpha}f(a+\eta(b,a))\right)\right|$$

$$\leq \begin{cases} \frac{\eta(b,a)}{\alpha+1}\left|f'(a)\right|\left(\left(\frac{x-a}{\eta(b,a)}\right)^{\alpha+1}+\left(\frac{a+\eta(b,a)-x}{\eta(b,a)}\right)^{\alpha+1}\right) \text{ if } \left|f'(a)\right|=\left|f'(x)\right|=\left|f'(b)\right|, \\ \eta(b,a)\left(\frac{1}{\alpha+1}\left|f'(a)\right|\left(\frac{x-a}{\eta(b,a)}\right)^{\alpha+1}+\left|f'(b)\right|\left(1-\frac{x-a}{\eta(b,a)}\right)^{\alpha+1}\left(\frac{|f'(a)|}{|f'(b)|}\right)^{1-\frac{x-a}{\eta(b,a)}}\right. \\ \left.\times\sum_{i=1}^{\infty}\frac{\left(-\left(1-\frac{x-a}{\eta(b,a)}\right)\ln\left(\frac{|f'(a)|}{|f'(b)|}\right)\right)^{i-1}}{(\alpha+1)_i}\right) \text{ if } \left|f'(a)\right|=\left|f'(x)\right|\neq\left|f'(b)\right|, \\ \eta(b,a)\left(\frac{1}{\alpha+1}\left|f'(b)\right|\left(1-\frac{x-a}{\eta(b,a)}\right)^{\alpha+1}+\left|f'(a)\right|\left(\frac{|f'(b)|}{|f'(a)|}\right)^{\frac{x-a}{\eta(b,a)}}\left(\frac{x-a}{\eta(b,a)}\right)^{\alpha+1}\right. \\ \left.\times\sum_{i=1}^{\infty}\frac{\left(-\frac{x-a}{\eta(b,a)}\ln\frac{|f'(b)|}{|f'(a)|}\right)^{i-1}}{(\alpha+1)_i}\right) \text{ if } \left|f'(a)\right|\neq\left|f'(x)\right|=\left|f'(b)\right|, \\ \eta(b,a)\left(\left|f'(a)\right|\left(\frac{x-a}{\eta(b,a)}\right)^{\alpha+1}\left(\frac{|f'(x)|}{|f'(a)|}\right)^{\frac{x-a}{\eta(b,a)}}\sum_{i=1}^{\infty}\frac{\left(-\frac{x-a}{\eta(b,a)}\ln\frac{|f'(x)|}{|f'(a)|}\right)^{i-1}}{(\alpha+1)_i}\right. \\ \left.+\left|f'(x)\right|\left(1-\frac{x-a}{\eta(b,a)}\right)^{\alpha+1}\left(\frac{|f'(b)|}{|f'(x)|}\right)^{\frac{x-a}{\eta(b,a)}}\sum_{i=1}^{\infty}\frac{\left(-\left(1-\frac{x-a}{\eta(b,a)}\right)\ln\frac{|f'(x)|}{|f'(b)|}\right)^{i-1}}{(\alpha+1)_i}\right) \\ \text{if } \left|f'(a)\right|\neq\left|f'(x)\right|\neq\left|f'(b)\right|. \end{cases}$$

Moreover choosing $\eta(b,a)=b-a$ *it yields*

$$\left|\left(\left(\frac{x-a}{b-a}\right)^{\alpha}+\left(\frac{b-x}{b-a}\right)^{\alpha}\right)f(x)-\frac{\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left(J_{x-}^{\alpha}f(a)+J_{x+}^{\alpha}f(b)\right)\right|$$

$$\leq \begin{cases} \frac{(b-a)}{\alpha+1}\left|f'(a)\right|\left(\left(\frac{x-a}{b-a}\right)^{\alpha+1}+\left(\frac{b-x}{b-a}\right)^{\alpha+1}\right) \text{ if } \left|f'(a)\right|=\left|f'(x)\right|=\left|f'(b)\right|, \\ (b-a)\left(\frac{1}{\alpha+1}\left|f'(a)\right|\left(\frac{x-a}{b-a}\right)^{\alpha+1}+\left|f'(b)\right|\left(\frac{b-x}{b-a}\right)^{\alpha+1}\left(\frac{|f'(a)|}{|f'(b)|}\right)^{\frac{b-x}{b-a}}\right. \\ \left.\times\sum_{i=1}^{\infty}\frac{\left(-\frac{b-x}{b-a}\ln\left(\frac{|f'(a)|}{|f'(b)|}\right)\right)^{i-1}}{(\alpha+1)_i}\right) \text{ if } \left|f'(a)\right|=\left|f'(x)\right|\neq\left|f'(b)\right|, \\ (b-a)\left(\frac{1}{\alpha+1}\left|f'(b)\right|\left(\frac{b-x}{b-a}\right)^{\alpha+1}+\left|f'(a)\right|\left(\frac{|f'(b)|}{|f'(a)|}\right)^{\frac{x-a}{b-a}}\left(\frac{x-a}{b-a}\right)^{\alpha+1}\right. \\ \left.\times\sum_{i=1}^{\infty}\frac{\left(-\frac{x-a}{b-a}\ln\frac{|f'(b)|}{|f'(a)|}\right)^{i-1}}{(\alpha+1)_i}\right) \text{ if } \left|f'(a)\right|\neq\left|f'(x)\right|=\left|f'(b)\right|, \\ (b-a)\left(\left|f'(a)\right|\left(\frac{x-a}{b-a}\right)^{\alpha+1}\left(\frac{|f'(x)|}{|f'(a)|}\right)^{\frac{x-a}{b-a}}\sum_{i=1}^{\infty}\frac{\left(-\frac{x-a}{b-a}\ln\frac{|f'(x)|}{|f'(a)|}\right)^{i-1}}{(\alpha+1)_i}\right. \\ \left.+\left|f'(x)\right|\left(\frac{b-x}{b-a}\right)^{\alpha+1}\left(\frac{|f'(b)|}{|f'(x)|}\right)^{\frac{x-a}{b-a}}\sum_{i=1}^{\infty}\frac{\left(-\frac{b-x}{b-a}\ln\frac{|f'(x)|}{|f'(b)|}\right)^{i-1}}{(\alpha+1)_i}\right) \\ \text{if } \left|f'(a)\right|\neq\left|f'(x)\right|\neq\left|f'(b)\right|. \end{cases}$$

Theorem 3.5. *Let* $f:[a,a+\eta(b,a)]\to[0,\infty)$ *be a differentiable function such that* $f'\in L([a,a+\eta(b,a)])$ *with* $\eta(b,a)>0$*. If* $|f'|^q$ *is* log*-preinvex function where* $q>1$ *with* $\frac{1}{p}+\frac{1}{q}=1$*, then the following inequality for fractional integrals*

$$\left|\left(\left(\frac{x-a}{\eta(b,a)}\right)^{\frac{\alpha}{k}}+\left(\frac{a+\eta(b,a)-x}{\eta(b,a)}\right)^{\frac{\alpha}{k}}\right)f(x)-\frac{\Gamma_k(\alpha+k)}{(\eta(b,a))^{\frac{\alpha}{k}}}\left(I_{x-}^{\alpha,k}f(a)+I_{x+}^{\alpha,k}f(a+\eta(b,a))\right)\right|$$

$$\leq \begin{cases} \eta(b,a)\left(\frac{k}{\alpha p+k}\right)^{\frac{1}{p}}\left(|f'(a)|\left(\frac{x-a}{\eta(b,a)}\right)^{\frac{\alpha}{k}+1}+|f'(x)|\left(1-\frac{x-a}{\eta(b,a)}\right)^{\frac{\alpha}{k}+1}\right) \\ \quad \text{if } |f'(a)|=|f'(x)|=|f'(b)|, \\ \eta(b,a)\left(\frac{k}{\alpha p+k}\right)^{\frac{1}{p}}\left(|f'(a)|\left(\frac{x-a}{\eta(b,a)}\right)^{\frac{\alpha}{k}+1}+|f'(a)|^{-\frac{x-a}{\eta(b,a)}}\left(1-\frac{x-a}{\eta(b,a)}\right)^{\frac{\alpha}{k}+\frac{1}{p}}\right. \\ \times\left.\left(\frac{|f'(b)|^q|f'(a)|^{q\frac{x-a}{\eta(b,a)}}-|f'(a)|^q|f'(b)|^{q\frac{x-a}{\eta(b,a)}}}{\ln|f'(b)|^q-\ln|f'(a)|^q}\right)^{\frac{1}{q}}\right) \quad \text{if } |f'(a)|=|f'(x)|\neq|f'(b)|, \\ \eta(b,a)\left(\frac{k}{\alpha p+k}\right)^{\frac{1}{p}}\left(|f'(a)|^{1-\frac{x-a}{\eta(b,a)}}\left(\frac{x-a}{\eta(b,a)}\right)^{\frac{\alpha}{k}+\frac{1}{p}}\left(\frac{|f'(b)|^{q\frac{x-a}{\eta(b,a)}}-|f'(a)|^{q\frac{x-a}{\eta(b,a)}}}{\ln|f'(b)|^q-\ln|f'(a)|^q}\right)^{\frac{1}{q}}\right. \\ \left.+|f'(b)|\left(1-\frac{x-a}{\eta(b,a)}\right)^{\frac{\alpha}{k}+1}\right) \quad \text{if } |f'(a)|\neq|f'(x)|=|f'(b)|, \\ \eta(b,a)\left(\frac{k}{\alpha p+k}\right)^{\frac{1}{p}}\left(|f'(a)|^{1-\frac{x-a}{\eta(b,a)}}\left(\frac{x-a}{\eta(b,a)}\right)^{\frac{\alpha}{k}+\frac{1}{p}}\left(\frac{|f'(x)|^{q\frac{x-a}{\eta(b,a)}}-|f'(a)|^{q\frac{x-a}{\eta(b,a)}}}{\ln|f'(x)|^q-\ln|f'(a)|^q}\right)^{\frac{1}{q}}\right. \\ \left.+|f'(x)|^{-\frac{x-a}{\eta(b,a)}}\left(1-\frac{x-a}{\eta(b,a)}\right)^{\frac{\alpha}{k}+\frac{1}{p}}\left(\frac{|f'(b)|^q|f'(x)|^{q\frac{x-a}{\eta(b,a)}}-|f'(x)|^q|f'(b)|^{q\frac{x-a}{\eta(b,a)}}}{\ln|f'(b)|^q-\ln|f'(x)|^q}\right)^{\frac{1}{q}}\right) \\ \quad \text{if } |f'(a)|\neq|f'(x)|\neq|f'(b)| \end{cases}$$

holds for all $x \in [a, a+\eta(b,a)]$.

Proof. From Lemma 2.10, properties of modulus, and Hölder's inequality, we have

$$\left|\left(\left(\frac{x-a}{\eta(b,a)}\right)^{\frac{\alpha}{k}}+\left(\frac{a+\eta(b,a)-x}{\eta(b,a)}\right)^{\frac{\alpha}{k}}\right)f(x)-\frac{\Gamma_k(\alpha+k)}{(\eta(b,a))^{\frac{\alpha}{k}}}\left(I_{x-}^{\alpha,k}f(a)+I_{x+}^{\alpha,k}f(a+\eta(b,a))\right)\right|$$

$$\leq\eta(b,a)\left(\left(\int_0^{\frac{x-a}{\eta(b,a)}}t^{\frac{\alpha p}{k}}dt\right)^{\frac{1}{p}}\left(\int_0^{\frac{x-a}{\eta(b,a)}}\left|f'(a+t\eta(b,a))\right|^q dt\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\int_{\frac{x-a}{\eta(b,a)}}^1(1-t)^{\frac{\alpha p}{k}}dt\right)^{\frac{1}{p}}\left(\int_{\frac{x-a}{\eta(b,a)}}^1\left|f'(a+t\eta(b,a))\right|^q dt\right)^{\frac{1}{q}}\right)$$

$$=\eta(b,a)\left(\frac{k}{\alpha p+k}\right)^{\frac{1}{p}}\left(\left(\frac{x-a}{\eta(b,a)}\right)^{\frac{\alpha}{k}+\frac{1}{p}}\left(\int_0^{\frac{x-a}{\eta(b,a)}}\left|f'(a+t\eta(b,a))\right|^q dt\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(1-\frac{x-a}{\eta(b,a)}\right)^{\frac{\alpha}{k}+\frac{1}{p}}\left(\int_{\frac{x-a}{\eta(b,a)}}^1\left|f'(a+t\eta(b,a))\right|^q dt\right)^{\frac{1}{q}}\right). \quad (3.9)$$

Since $|f'|^q$ is log-preinvex, from (3.9) we have

$$\left|\left(\left(\frac{x-a}{\eta(b,a)}\right)^{\frac{\alpha}{k}}+\left(\frac{a+\eta(b,a)-x}{\eta(b,a)}\right)^{\frac{\alpha}{k}}\right)f(x)-\frac{\Gamma_k(\alpha+k)}{(\eta(b,a))^{\frac{\alpha}{k}}}\left(I_{x-}^{\alpha,k}f(a)+I_{x+}^{\alpha,k}f(a+\eta(b,a))\right)\right|$$

$$\leq \eta(b,a)\left(\frac{k}{\alpha p+k}\right)^{\frac{1}{p}}\left(\left|f'(a)\right|\left(\frac{x-a}{\eta(b,a)}\right)^{\frac{\alpha}{k}+\frac{1}{p}}\left(\int\limits_{0}^{\frac{x-a}{\eta(b,a)}}\left(\frac{|f'(x)|^{q}}{|f'(a)|^{q}}\right)^{t}dt\right)^{\frac{1}{q}}\right.$$

$$\left.+\left|f'(x)\right|\left(1-\frac{x-a}{\eta(b,a)}\right)^{\frac{\alpha}{k}+\frac{1}{p}}\left(\int\limits_{\frac{x-a}{\eta(b,a)}}^{1}\left(\frac{|f'(b)|^{q}}{|f'(x)|^{q}}\right)^{t}dt\right)^{\frac{1}{q}}\right). \tag{3.10}$$

If $|f'(a)|=|f'(b)|=|f'(x)|$, then (3.10) gives

$$\left|\left(\left(\frac{x-a}{\eta(b,a)}\right)^{\frac{\alpha}{k}}+\left(\frac{a+\eta(b,a)-x}{\eta(b,a)}\right)^{\frac{\alpha}{k}}\right)f(x)-\frac{\Gamma_k(\alpha+k)}{(\eta(b,a))^{\frac{\alpha}{k}}}\left(I_{x-}^{\alpha,k}f(a)+I_{x+}^{\alpha,k}f(a+\eta(b,a))\right)\right|$$

$$\leq \eta(b,a)\left(\frac{k}{\alpha p+k}\right)^{\frac{1}{p}}\left(\left|f'(a)\right|\left(\frac{x-a}{\eta(b,a)}\right)^{\frac{\alpha}{k}+1}+\left|f'(x)\right|\left(1-\frac{x-a}{\eta(b,a)}\right)^{\frac{\alpha}{k}+1}\right). \tag{3.11}$$

If $|f'(a)|=|f'(x)|\neq|f'(b)|$, then (3.10) gives

$$\left|\left(\left(\frac{x-a}{\eta(b,a)}\right)^{\frac{\alpha}{k}}+\left(\frac{a+\eta(b,a)-x}{\eta(b,a)}\right)^{\frac{\alpha}{k}}\right)f(x)-\frac{\Gamma_k(\alpha+k)}{(\eta(b,a))^{\frac{\alpha}{k}}}\left(I_{x-}^{\alpha,k}f(a)+I_{x+}^{\alpha,k}f(a+\eta(b,a))\right)\right|$$

$$\leq \eta(b,a)\left(\frac{k}{\alpha p+k}\right)^{\frac{1}{p}}\left(\left|f'(a)\right|\left(\frac{x-a}{\eta(b,a)}\right)^{\frac{\alpha}{k}+1}\right.$$

$$\left.+\left|f'(a)\right|^{-\frac{x-a}{\eta(b,a)}}\left(1-\frac{x-a}{\eta(b,a)}\right)^{\frac{\alpha}{k}+\frac{1}{p}}\left(\frac{|f'(b)|^{q}|f'(a)|^{q\frac{x-a}{\eta(b,a)}}-|f'(a)|^{q}|f'(b)|^{q\frac{x-a}{\eta(b,a)}}}{\ln|f'(b)|^{q}-\ln|f'(a)|^{q}}\right)^{\frac{1}{q}}\right). \tag{3.12}$$

If $|f'(a)|\neq|f'(x)|=|f'(b)|$, then (3.10) gives

$$\left|\left(\left(\frac{x-a}{\eta(b,a)}\right)^{\frac{\alpha}{k}}+\left(\frac{a+\eta(b,a)-x}{\eta(b,a)}\right)^{\frac{\alpha}{k}}\right)f(x)-\frac{\Gamma_k(\alpha+k)}{(\eta(b,a))^{\frac{\alpha}{k}}}\left(I_{x-}^{\alpha,k}f(a)+I_{x+}^{\alpha,k}f(a+\eta(b,a))\right)\right|$$

$$\leq \eta(b,a)\left(\frac{k}{\alpha p+k}\right)^{\frac{1}{p}}\left(\left|f'(a)\right|^{1-\frac{x-a}{\eta(b,a)}}\left(\frac{x-a}{\eta(b,a)}\right)^{\frac{\alpha}{k}+\frac{1}{p}}\left(\frac{|f'(b)|^{q\frac{x-a}{\eta(b,a)}}-|f'(a)|^{q\frac{x-a}{\eta(b,a)}}}{\ln|f'(b)|^{q}-\ln|f'(a)|^{q}}\right)^{\frac{1}{q}}\right.$$

$$\left.+\left|f'(b)\right|\left(1-\frac{x-a}{\eta(b,a)}\right)^{\frac{\alpha}{k}+1}\right). \tag{3.13}$$

If $|f'(a)|\neq|f'(x)|\neq|f'(b)|$, then (3.10) gives

$$\left|\left(\left(\frac{x-a}{\eta(b,a)}\right)^{\frac{\alpha}{k}}+\left(\frac{a+\eta(b,a)-x}{\eta(b,a)}\right)^{\frac{\alpha}{k}}\right)f(x)-\frac{\Gamma_k(\alpha+k)}{(\eta(b,a))^{\frac{\alpha}{k}}}\left(I_{x-}^{\alpha,k}f(a)+I_{x+}^{\alpha,k}f(a+\eta(b,a))\right)\right|$$

$$\leq \eta(b,a)\left(\frac{k}{\alpha p+k}\right)^{\frac{1}{p}}\left(\left|f'(a)\right|^{1-\frac{x-a}{\eta(b,a)}}\left(\frac{x-a}{\eta(b,a)}\right)^{\frac{\alpha}{k}+\frac{1}{p}}\left(\frac{|f'(x)|^{q\frac{x-a}{\eta(b,a)}}-|f'(a)|^{q\frac{x-a}{\eta(b,a)}}}{\ln|f'(x)|^{q}-\ln|f'(a)|^{q}}\right)^{\frac{1}{q}}\right.$$

$$\left.+\left|f'(x)\right|^{-\frac{x-a}{\eta(b,a)}}\left(1-\frac{x-a}{\eta(b,a)}\right)^{\frac{\alpha}{k}+\frac{1}{p}}\left(\frac{|f'(b)|^{q}|f'(x)|^{q\frac{x-a}{\eta(b,a)}}-|f'(x)|^{q}|f'(b)|^{q\frac{x-a}{\eta(b,a)}}}{\ln|f'(b)|^{q}-\ln|f'(x)|^{q}}\right)^{\frac{1}{q}}\right). \tag{3.14}$$

The desired result follows from (3.11)-(3.14). □

Corollary 3.6. *In Theorem 3.5, if we choose $\eta(b,a)=b-a$, we have*

$$\left|\left(\left(\tfrac{x-a}{b-a}\right)^{\frac{\alpha}{k}}+\left(\tfrac{b-x}{b-a}\right)^{\frac{\alpha}{k}}\right)f(x)-\tfrac{\Gamma_k(\alpha+k)}{(b-a)^{\frac{\alpha}{k}}}\left(I_{x-}^{\alpha,k}f(a)+I_{x+}^{\alpha,k}f(b)\right)\right|$$
$$\leq\begin{cases}(b-a)\left(\frac{k}{\alpha p+k}\right)^{\frac{1}{p}}\left(|f'(a)|\left(\frac{x-a}{b-a}\right)^{\frac{\alpha}{k}+1}+|f'(x)|\left(\frac{b-x}{b-a}\right)^{\frac{\alpha}{k}+1}\right)\\ \qquad \text{if } |f'(a)|=|f'(x)|=|f'(b)|,\\ (b-a)\left(\frac{k}{\alpha p+k}\right)^{\frac{1}{p}}\left(|f'(a)|\left(\frac{x-a}{b-a}\right)^{\frac{\alpha}{k}+1}+|f'(a)|^{-\frac{x-a}{b-a}}\left(\frac{b-x}{b-a}\right)^{\frac{\alpha}{k}+\frac{1}{p}}\right.\\ \times\left.\left(\frac{|f'(b)|^q|f'(a)|^{q\frac{x-a}{b-a}}-|f'(a)|^q|f'(b)|^{q\frac{x-a}{b-a}}}{\ln|f'(b)|^q-\ln|f'(a)|^q}\right)^{\frac{1}{q}}\right)\ \text{if } |f'(a)|=|f'(x)|\neq|f'(b)|,\\ (b-a)\left(\frac{k}{\alpha p+k}\right)^{\frac{1}{p}}\left(|f'(a)|^{\frac{b-x}{b-a}}\left(\frac{x-a}{b-a}\right)^{\frac{\alpha}{k}+\frac{1}{p}}\left(\frac{|f'(b)|^{q\frac{x-a}{b-a}}-|f'(a)|^{q\frac{x-a}{b-a}}}{\ln|f'(b)|^q-\ln|f'(a)|^q}\right)^{\frac{1}{q}}\right.\\ \left.+|f'(b)|\left(\frac{b-x}{b-a}\right)^{\frac{\alpha}{k}+1}\right)\ \text{if } |f'(a)|\neq|f'(x)|=|f'(b)|,\\ (b-a)\left(\frac{k}{\alpha p+k}\right)^{\frac{1}{p}}\left(|f'(a)|^{\frac{b-x}{b-a}}\left(\frac{x-a}{b-a}\right)^{\frac{\alpha}{k}+\frac{1}{p}}\left(\frac{|f'(x)|^{q\frac{x-a}{b-a}}-|f'(a)|^{q\frac{x-a}{b-a}}}{\ln|f'(x)|^q-\ln|f'(a)|^q}\right)^{\frac{1}{q}}\right.\\ \left.+|f'(x)|^{-\frac{x-a}{b-a}}\left(\frac{b-x}{b-a}\right)^{\frac{\alpha}{k}+\frac{1}{p}}\left(\frac{|f'(b)|^q|f'(x)|^{q\frac{x-a}{b-a}}-|f'(x)|^q|f'(b)|^{q\frac{x-a}{b-a}}}{\ln|f'(b)|^q-\ln|f'(x)|^q}\right)^{\frac{1}{q}}\right)\\ \qquad \text{if } |f'(a)|\neq|f'(x)|\neq|f'(b)|.\end{cases}$$

Corollary 3.7. *In Theorem 3.5, if we taking $k=1$, we obtain*

$$\left|\left(\left(\tfrac{x-a}{\eta(b,a)}\right)^{\alpha}+\left(\tfrac{a+\eta(b,a)-x}{\eta(b,a)}\right)^{\alpha}\right)f(x)-\tfrac{\Gamma(\alpha+1)}{(\eta(b,a))^{\alpha}}\left(J_{x-}^{\alpha}f(a)+J_{x+}^{\alpha}f(a+\eta(b,a))\right)\right|$$
$$\leq\begin{cases}\eta(b,a)\left(\frac{1}{\alpha p+1}\right)^{\frac{1}{p}}\left(|f'(a)|\left(\frac{x-a}{\eta(b,a)}\right)^{\alpha+1}+|f'(x)|\left(1-\frac{x-a}{\eta(b,a)}\right)^{\alpha+1}\right)\\ \qquad \text{if } |f'(a)|=|f'(x)|=|f'(b)|,\\ \eta(b,a)\left(\frac{1}{\alpha p+1}\right)^{\frac{1}{p}}\left(|f'(a)|\left(\frac{x-a}{\eta(b,a)}\right)^{\alpha+1}+|f'(a)|^{-\frac{x-a}{\eta(b,a)}}\left(1-\frac{x-a}{\eta(b,a)}\right)^{\alpha+\frac{1}{p}}\right.\\ \times\left.\left(\frac{|f'(b)|^q|f'(a)|^{q\frac{x-a}{\eta(b,a)}}-|f'(a)|^q|f'(b)|^{q\frac{x-a}{\eta(b,a)}}}{\ln|f'(b)|^q-\ln|f'(a)|^q}\right)^{\frac{1}{q}}\right)\ \text{if } |f'(a)|=|f'(x)|\neq|f'(b)|,\\ \eta(b,a)\left(\frac{1}{\alpha p+1}\right)^{\frac{1}{p}}\left(|f'(a)|^{1-\frac{x-a}{\eta(b,a)}}\left(\frac{x-a}{\eta(b,a)}\right)^{\alpha+\frac{1}{p}}\left(\frac{|f'(b)|^{q\frac{x-a}{\eta(b,a)}}-|f'(a)|^{q\frac{x-a}{\eta(b,a)}}}{\ln|f'(b)|^q-\ln|f'(a)|^q}\right)^{\frac{1}{q}}\right.\\ \left.+|f'(b)|\left(1-\frac{x-a}{\eta(b,a)}\right)^{\alpha+1}\right)\ \text{if } |f'(a)|\neq|f'(x)|=|f'(b)|,\\ \eta(b,a)\left(\frac{1}{\alpha p+1}\right)^{\frac{1}{p}}\left(|f'(a)|^{1-\frac{x-a}{\eta(b,a)}}\left(\frac{x-a}{\eta(b,a)}\right)^{\alpha+\frac{1}{p}}\left(\frac{|f'(x)|^{q\frac{x-a}{\eta(b,a)}}-|f'(a)|^{q\frac{x-a}{\eta(b,a)}}}{\ln|f'(x)|^q-\ln|f'(a)|^q}\right)^{\frac{1}{q}}\right.\\ \left.+|f'(x)|^{-\frac{x-a}{\eta(b,a)}}\left(1-\frac{x-a}{\eta(b,a)}\right)^{\alpha+\frac{1}{p}}\left(\frac{|f'(b)|^q|f'(x)|^{q\frac{x-a}{\eta(b,a)}}-|f'(x)|^q|f'(b)|^{q\frac{x-a}{\eta(b,a)}}}{\ln|f'(b)|^q-\ln|f'(x)|^q}\right)^{\frac{1}{q}}\right)\\ \qquad \text{if } |f'(a)|\neq|f'(x)|\neq|f'(b)|.\end{cases}$$

Moreover if we take $\eta(b,a)=b-a$, we get

$$\left|\left(\left(\tfrac{x-a}{b-a}\right)^{\alpha}+\left(\tfrac{b-x}{b-a}\right)^{\alpha}\right)f(x)-\tfrac{\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left(J_{x-}^{\alpha}f(a)+J_{x+}^{\alpha}f(b)\right)\right|$$

$$\leq \begin{cases} (b-a)\left(\frac{1}{\alpha p+1}\right)^{\frac{1}{p}}\left(|f'(a)|\left(\frac{x-a}{b-a}\right)^{\alpha+1}+|f'(x)|\left(\frac{b-x}{b-a}\right)^{\alpha+1}\right) \\ \text{if } |f'(a)|=|f'(x)|=|f'(b)|, \\ (b-a)\left(\frac{1}{\alpha p+1}\right)^{\frac{1}{p}}\left(|f'(a)|\left(\frac{x-a}{b-a}\right)^{\alpha+1}+|f'(a)|^{-\frac{x-a}{b-a}}\left(\frac{b-x}{b-a}\right)^{\alpha+\frac{1}{p}}\right. \\ \left.\times\left(\frac{|f'(b)|^q|f'(a)|^{q\frac{x-a}{b-a}}-|f'(a)|^q|f'(b)|^{q\frac{x-a}{b-a}}}{\ln|f'(b)|^q-\ln|f'(a)|^q}\right)^{\frac{1}{q}}\right) \text{ if } |f'(a)|=|f'(x)|\neq|f'(b)|, \\ (b-a)\left(\frac{1}{\alpha p+1}\right)^{\frac{1}{p}}\left(|f'(a)|^{\frac{b-x}{b-a}}\left(\frac{x-a}{b-a}\right)^{\alpha+\frac{1}{p}}\left(\frac{|f'(b)|^{q\frac{x-a}{b-a}}-|f'(a)|^{q\frac{x-a}{b-a}}}{\ln|f'(b)|^q-\ln|f'(a)|^q}\right)^{\frac{1}{q}}\right. \\ \left.+|f'(b)|\left(\frac{b-x}{b-a}\right)^{\alpha+1}\right) \text{ if } |f'(a)|\neq|f'(x)|=|f'(b)|, \\ (b-a)\left(\frac{1}{\alpha p+1}\right)^{\frac{1}{p}}\left(|f'(a)|^{\frac{b-x}{b-a}}\left(\frac{x-a}{b-a}\right)^{\alpha+\frac{1}{p}}\left(\frac{|f'(x)|^{q\frac{x-a}{b-a}}-|f'(a)|^{q\frac{x-a}{b-a}}}{\ln|f'(x)|^q-\ln|f'(a)|^q}\right)^{\frac{1}{q}}\right. \\ \left.+|f'(x)|^{-\frac{x-a}{b-a}}\left(\frac{b-x}{b-a}\right)^{\alpha+\frac{1}{p}}\left(\frac{|f'(b)|^q|f'(x)|^{q\frac{x-a}{b-a}}-|f'(x)|^q|f'(b)|^{q\frac{x-a}{b-a}}}{\ln|f'(b)|^q-\ln|f'(x)|^q}\right)^{\frac{1}{q}}\right) \\ \text{if } |f'(a)|\neq|f'(x)|\neq|f'(b)|. \end{cases}$$

Theorem 3.8. *Let $f:[a,a+\eta(b,a)]\to[0,\infty)$ be a differentiable function such that $f'\in L([a,a+\eta(b,a)])$ with $\eta(b,a)>0$. If $|f'|^q$ is* log-preinvex *function where $q\geq 1$, then the following inequality for fractional integrals*

$$\left|\left(\left(\frac{x-a}{\eta(b,a)}\right)^{\frac{\alpha}{k}}+\left(\frac{a+\eta(b,a)-x}{\eta(b,a)}\right)^{\frac{\alpha}{k}}\right)f(x)-\frac{\Gamma_k(\alpha+k)}{(\eta(b,a))^{\frac{\alpha}{k}}}\left(I_{x-}^{\alpha,k}f(a)+I_{x+}^{\alpha,k}f(a+\eta(b,a))\right)\right|$$

$$\leq \begin{cases} \eta(b,a)\left(\frac{k}{\alpha+k}\right)|f'(a)|\left(\left(\frac{x-a}{\eta(b,a)}\right)^{\frac{\alpha}{k}+1}+\left(1-\frac{x-a}{\eta(b,a)}\right)^{\frac{\alpha}{k}+1}\right) \\ \text{if } |f'(a)|=|f'(x)|=|f'(b)|, \\ \eta(b,a)\left(\frac{k}{\alpha+k}\right)^{1-\frac{1}{q}}\left(|f'(a)|\left(\frac{x-a}{\eta(b,a)}\right)^{\frac{\alpha}{k}+1}\left(\frac{k}{\alpha+k}\right)^{\frac{1}{q}}\right. \\ \left.+\frac{|f'(a)|^{1-\frac{x-a}{\eta(b,a)}}}{|f'(b)|^{-\frac{x-a}{\eta(b,a)}}}\left(1-\frac{x-a}{\eta(b,a)}\right)^{\frac{\alpha}{k}+1}\left(\sum_{i=1}^{\infty}\frac{\left(-\left(1-\frac{x-a}{\eta(b,a)}\right)\ln\frac{|f'(a)|^q}{|f'(b)|^q}\right)^{i-1}}{\left(\frac{\alpha}{k}+1\right)_i}\right)^{\frac{1}{q}}\right) \\ \text{if } |f'(a)|=|f'(x)|\neq|f'(b)|, \\ \eta(b,a)\left(\frac{k}{\alpha+k}\right)^{1-\frac{1}{q}}\left(\frac{|f'(x)|^{\frac{x-a}{\eta(b,a)}}}{|f'(a)|^{\frac{x-a}{\eta(b,a)}-1}}\left(\frac{x-a}{\eta(b,a)}\right)^{\frac{\alpha}{k}+1}\left(\sum_{i=1}^{\infty}\frac{\left(-\frac{x-a}{\eta(b,a)}\ln\frac{|f'(x)|^q}{|f'(a)|^q}\right)^{i-1}}{\left(\frac{\alpha}{k}+1\right)_i}\right)^{\frac{1}{q}}\right. \\ \left.+|f'(b)|\left(1-\frac{x-a}{\eta(b,a)}\right)^{\frac{\alpha}{k}+1}\left(\frac{k}{\alpha+k}\right)^{\frac{1}{q}}\right) \\ \text{if } |f'(a)|\neq|f'(x)|=|f'(b)|, \\ \eta(b,a)\left(\frac{k}{\alpha+k}\right)^{1-\frac{1}{q}}\left(\frac{|f'(x)|^{\frac{x-a}{\eta(b,a)}}}{|f'(a)|^{\frac{x-a}{\eta(b,a)}-1}}\left(\frac{x-a}{\eta(b,a)}\right)^{\frac{\alpha}{k}+1}\left(\sum_{i=1}^{\infty}\frac{\left(-\frac{x-a}{\eta(b,a)}\ln\frac{|f'(x)|^q}{|f'(a)|^q}\right)^{i-1}}{\left(\frac{\alpha}{k}+1\right)_i}\right)^{\frac{1}{q}}\right. \\ \left.+\frac{|f'(x)|^{1-\frac{x-a}{\eta(b,a)}}}{|f'(b)|^{-\frac{x-a}{\eta(b,a)}}}\left(1-\frac{x-a}{\eta(b,a)}\right)^{\frac{\alpha}{k}+1}\left(\sum_{i=1}^{\infty}\frac{\left(-\left(1-\frac{x-a}{\eta(b,a)}\right)\ln\frac{|f'(x)|^q}{|f'(b)|^q}\right)^{i-1}}{\left(\frac{\alpha}{k}+1\right)_i}\right)^{\frac{1}{q}}\right) \\ \text{if } |f'(a)|\neq|f'(x)|\neq|f'(b)| \end{cases}$$

holds for all $x \in [a, a+\eta(b,a)]$.

Proof. From Lemma 2.10, properties of modulus, and power mean inequality, we have

$$\begin{aligned}
&\left|\left(\left(\tfrac{x-a}{\eta(b,a)}\right)^{\frac{\alpha}{k}}+\left(\tfrac{a+\eta(b,a)-x}{\eta(b,a)}\right)^{\frac{\alpha}{k}}\right)f(x)-\tfrac{\Gamma_k(\alpha+k)}{(\eta(b,a))^{\frac{\alpha}{k}}}\left(I_{x-}^{\alpha,k}f(a)+I_{x+}^{\alpha,k}f(a+\eta(b,a))\right)\right|\\
\leq&\eta(b,a)\left(\left(\int_0^{\frac{x-a}{\eta(b,a)}}t^{\frac{\alpha}{k}}dt\right)^{1-\frac{1}{q}}\left(\int_0^{\frac{x-a}{\eta(b,a)}}t^{\frac{\alpha}{k}}\left|f'(a+t\eta(b,a))\right|^q dt\right)^{\frac{1}{q}}\right.\\
&\left.+\left(\int_{\frac{x-a}{\eta(b,a)}}^1(1-t)^{\frac{\alpha}{k}}dt\right)^{1-\frac{1}{q}}\left(\int_{\frac{x-a}{\eta(b,a)}}^1(1-t)^{\frac{\alpha}{k}}\left|f'(a+t\eta(b,a))\right|^q dt\right)^{\frac{1}{q}}\right)\\
=&\eta(b,a)\left(\tfrac{k}{\alpha+k}\right)^{1-\frac{1}{q}}\left(\left(\tfrac{x-a}{\eta(b,a)}\right)^{\left(\frac{\alpha}{k}+1\right)\left(1-\frac{1}{q}\right)}\left(\int_0^{\frac{x-a}{\eta(b,a)}}t^{\frac{\alpha}{k}}\left|f'(a+t\eta(b,a))\right|^q dt\right)^{\frac{1}{q}}\right.\\
&\left.+\left(1-\tfrac{x-a}{\eta(b,a)}\right)^{\left(\frac{\alpha}{k}+1\right)\left(1-\frac{1}{q}\right)}\left(\int_{\frac{x-a}{\eta(b,a)}}^1(1-t)^{\frac{\alpha}{k}}\left|f'(a+t\eta(b,a))\right|^q dt\right)^{\frac{1}{q}}\right). \qquad (3.15)
\end{aligned}$$

Since $|f'|^q$ is log-preinvex, from (3.15) we have

$$\begin{aligned}
&\left|\left(\left(\tfrac{x-a}{\eta(b,a)}\right)^{\frac{\alpha}{k}}+\left(\tfrac{a+\eta(b,a)-x}{\eta(b,a)}\right)^{\frac{\alpha}{k}}\right)f(x)-\tfrac{\Gamma_k(\alpha+k)}{(\eta(b,a))^{\frac{\alpha}{k}}}\left(I_{x-}^{\alpha,k}f(a)+I_{x+}^{\alpha,k}f(a+\eta(b,a))\right)\right|\\
\leq&\eta(b,a)\left(\tfrac{k}{\alpha+k}\right)^{1-\frac{1}{q}}\left(\left|f'(a)\right|\left(\tfrac{x-a}{\eta(b,a)}\right)^{\left(\frac{\alpha}{k}+1\right)\left(1-\frac{1}{q}\right)}\left(\int_0^{\frac{x-a}{\eta(b,a)}}t^{\frac{\alpha}{k}}\left(\tfrac{|f'(x)|^q}{|f'(a)|^q}\right)^t dt\right)^{\frac{1}{q}}\right.\\
&\left.+\left|f'(x)\right|\left(1-\tfrac{x-a}{\eta(b,a)}\right)^{\left(\frac{\alpha}{k}+1\right)\left(1-\frac{1}{q}\right)}\left(\int_{\frac{x-a}{\eta(b,a)}}^1(1-t)^{\frac{\alpha}{k}}\left(\tfrac{|f'(b)|^q}{|f'(x)|^q}\right)^t dt\right)^{\frac{1}{q}}\right). \qquad (3.16)
\end{aligned}$$

If $|f'(a)| = |f'(b)| = |f'(x)|$, then (3.16) gives

$$\begin{aligned}
&\left|\left(\left(\tfrac{x-a}{\eta(b,a)}\right)^{\frac{\alpha}{k}}+\left(\tfrac{a+\eta(b,a)-x}{\eta(b,a)}\right)^{\frac{\alpha}{k}}\right)f(x)-\tfrac{\Gamma_k(\alpha+k)}{(\eta(b,a))^{\frac{\alpha}{k}}}\left(I_{x-}^{\alpha,k}f(a)+I_{x+}^{\alpha,k}f(a+\eta(b,a))\right)\right|\\
\leq&\eta(b,a)\left(\tfrac{k}{\alpha+k}\right)\left|f'(a)\right|\left(\left(\tfrac{x-a}{\eta(b,a)}\right)^{\frac{\alpha}{k}+1}+\left(1-\tfrac{x-a}{\eta(b,a)}\right)^{\frac{\alpha}{k}+1}\right). \qquad (3.17)
\end{aligned}$$

If $|f'(a)| = |f'(x)| \neq |f'(b)|$, then (3.16) gives

$$\left|\left(\left(\tfrac{x-a}{\eta(b,a)}\right)^{\frac{\alpha}{k}}+\left(\tfrac{a+\eta(b,a)-x}{\eta(b,a)}\right)^{\frac{\alpha}{k}}\right)f(x)-\tfrac{\Gamma_k(\alpha+k)}{(\eta(b,a))^{\frac{\alpha}{k}}}\left(I_{x-}^{\alpha,k}f(a)+I_{x+}^{\alpha,k}f(a+\eta(b,a))\right)\right|$$

$$\leq \eta(b,a)\left(\frac{k}{\alpha+k}\right)^{1-\frac{1}{q}}\left(\left|f'(a)\right|\left(\frac{x-a}{\eta(b,a)}\right)^{\frac{\alpha}{k}+1}\left(\frac{k}{\alpha+k}\right)^{\frac{1}{q}}\right.$$
$$\left.+\frac{|f'(a)|^{1-\frac{x-a}{\eta(b,a)}}}{|f'(b)|^{-\frac{x-a}{\eta(b,a)}}}\left(1-\frac{x-a}{\eta(b,a)}\right)^{\frac{\alpha}{k}+1}\left(\sum_{i=1}^{\infty}\frac{\left(-\left(1-\frac{x-a}{\eta(b,a)}\right)\ln\frac{|f'(a)|^q}{|f'(b)|^q}\right)^{i-1}}{\left(\frac{\alpha}{k}+1\right)_i}\right)^{\frac{1}{q}}\right). \tag{3.18}$$

If $|f'(a)| \neq |f'(x)| = |f'(b)|$, then (3.16) gives

$$\left|\left(\left(\frac{x-a}{\eta(b,a)}\right)^{\frac{\alpha}{k}}+\left(\frac{a+\eta(b,a)-x}{\eta(b,a)}\right)^{\frac{\alpha}{k}}\right)f(x)-\frac{\Gamma_k(\alpha+k)}{(\eta(b,a))^{\frac{\alpha}{k}}}\left(I_{x-}^{\alpha,k}f(a)+I_{x+}^{\alpha,k}f(a+\eta(b,a))\right)\right|$$
$$\leq \eta(b,a)\left(\frac{k}{\alpha+k}\right)^{1-\frac{1}{q}}\left(\frac{|f'(b)|^{\frac{x-a}{\eta(b,a)}}}{|f'(a)|^{\frac{x-a}{\eta(b,a)}-1}}\left(\frac{x-a}{\eta(b,a)}\right)^{\frac{\alpha}{k}+1}\left(\sum_{i=1}^{\infty}\frac{\left(-\frac{x-a}{\eta(b,a)}\ln\frac{|f'(b)|^q}{|f'(a)|^q}\right)^{i-1}}{\left(\frac{\alpha}{k}+1\right)_i}\right)^{\frac{1}{q}}\right.$$
$$\left.+\left|f'(b)\right|\left(1-\frac{x-a}{\eta(b,a)}\right)^{\frac{\alpha}{k}+1}\left(\frac{k}{\alpha+k}\right)^{\frac{1}{q}}\right). \tag{3.19}$$

If $|f'(a)| \neq |f'(b)| \neq |f'(x)|$, then (3.16) gives

$$\left|\left(\left(\frac{x-a}{\eta(b,a)}\right)^{\frac{\alpha}{k}}+\left(\frac{a+\eta(b,a)-x}{\eta(b,a)}\right)^{\frac{\alpha}{k}}\right)f(x)-\frac{\Gamma_k(\alpha+k)}{(\eta(b,a))^{\frac{\alpha}{k}}}\left(I_{x-}^{\alpha,k}f(a)+I_{x+}^{\alpha,k}f(a+\eta(b,a))\right)\right|$$
$$\leq \eta(b,a)\left(\frac{k}{\alpha+k}\right)^{1-\frac{1}{q}}\left(\frac{|f'(x)|^{\frac{x-a}{\eta(b,a)}}}{|f'(a)|^{\frac{x-a}{\eta(b,a)}-1}}\left(\frac{x-a}{\eta(b,a)}\right)^{\frac{\alpha}{k}+1}\left(\sum_{i=1}^{\infty}\frac{\left(-\frac{x-a}{\eta(b,a)}\ln\frac{|f'(x)|^q}{|f'(a)|^q}\right)^{i-1}}{\left(\frac{\alpha}{k}+1\right)_i}\right)^{\frac{1}{q}}\right.$$
$$\left.+\frac{|f'(x)|^{1-\frac{x-a}{\eta(b,a)}}}{|f'(b)|^{-\frac{x-a}{\eta(b,a)}}}\left(1-\frac{x-a}{\eta(b,a)}\right)^{\frac{\alpha}{k}+1}\left(\sum_{i=1}^{\infty}\frac{\left(-\left(1-\frac{x-a}{\eta(b,a)}\right)\ln\frac{|f'(x)|^q}{|f'(b)|^q}\right)^{i-1}}{\left(\frac{\alpha}{k}+1\right)_i}\right)^{\frac{1}{q}}\right). \tag{3.20}$$

The desired result follows from (3.17)-(3.20). □

Corollary 3.9. *In Theorem 3.8, if we choose $\eta(b,a) = b-a$, we have*

$$\left|\left(\left(\frac{x-a}{b-a}\right)^{\frac{\alpha}{k}}+\left(\frac{b-x}{b-a}\right)^{\frac{\alpha}{k}}\right)f(x)-\frac{\Gamma_k(\alpha+k)}{(b-a)^{\frac{\alpha}{k}}}\left(I_{x-}^{\alpha,k}f(a)+I_{x+}^{\alpha,k}f(b)\right)\right|$$

$$\leq \begin{cases} (b-a)\left(\frac{k}{\alpha+k}\right)|f'(a)|\left(\left(\frac{x-a}{b-a}\right)^{\frac{\alpha}{k}+1}+\left(\frac{b-x}{b-a}\right)^{\frac{\alpha}{k}+1}\right) \\ \quad \textit{if } |f'(a)|=|f'(x)|=|f'(b)|, \\ (b-a)\left(\frac{k}{\alpha+k}\right)^{1-\frac{1}{q}}\left(|f'(a)|\left(\frac{x-a}{b-a}\right)^{\frac{\alpha}{k}+1}\left(\frac{k}{\alpha+k}\right)^{\frac{1}{q}}\right. \\ \left.+\frac{|f'(a)|^{\frac{b-x}{b-a}}}{|f'(b)|^{-\frac{x-a}{b-a}}}\left(\frac{b-x}{b-a}\right)^{\frac{\alpha}{k}+1}\left(\sum_{i=1}^{\infty}\frac{\left(-\frac{b-x}{b-a}\ln\frac{|f'(a)|^q}{|f'(b)|^q}\right)^{i-1}}{\left(\frac{\alpha}{k}+1\right)_i}\right)^{\frac{1}{q}}\right) \\ \quad \textit{if } |f'(a)|=|f'(x)|\neq|f'(b)|, \\ (b-a)\left(\frac{k}{\alpha+k}\right)^{1-\frac{1}{q}}\left(\frac{|f'(x)|^{\frac{x-a}{b-a}}}{|f'(a)|^{-\frac{b-x}{b-a}}}\left(\frac{x-a}{b-a}\right)^{\frac{\alpha}{k}+1}\left(\sum_{i=1}^{\infty}\frac{\left(-\frac{x-a}{b-a}\ln\frac{|f'(x)|^q}{|f'(a)|^q}\right)^{i-1}}{\left(\frac{\alpha}{k}+1\right)_i}\right)^{\frac{1}{q}}\right. \\ \left.+|f'(b)|\left(\frac{b-x}{b-a}\right)^{\frac{\alpha}{k}+1}\left(\frac{k}{\alpha+k}\right)^{\frac{1}{q}}\right) \\ \quad \textit{if } |f'(a)|\neq|f'(x)|=|f'(b)|, \\ (b-a)\left(\frac{k}{\alpha+k}\right)^{1-\frac{1}{q}}\left(\frac{|f'(x)|^{\frac{x-a}{b-a}}}{|f'(a)|^{-\frac{b-x}{b-a}}}\left(\frac{x-a}{b-a}\right)^{\frac{\alpha}{k}+1}\left(\sum_{i=1}^{\infty}\frac{\left(-\frac{x-a}{b-a}\ln\frac{|f'(x)|^q}{|f'(a)|^q}\right)^{i-1}}{\left(\frac{\alpha}{k}+1\right)_i}\right)^{\frac{1}{q}}\right. \\ \left.+\frac{|f'(x)|^{\frac{b-x}{b-a}}}{|f'(b)|^{-\frac{x-a}{b-a}}}\left(\frac{b-x}{b-a}\right)^{\frac{\alpha}{k}+1}\left(\sum_{i=1}^{\infty}\frac{\left(-\frac{b-x}{b-a}\ln\frac{|f'(x)|^q}{|f'(b)|^q}\right)^{i-1}}{\left(\frac{\alpha}{k}+1\right)_i}\right)^{\frac{1}{q}}\right) \\ \quad \textit{if } |f'(a)|\neq|f'(x)|\neq|f'(b)|. \end{cases}$$

Corollary 3.10. *In Theorem 3.8, if we taking $k=1$, we obtain*

$$\left|\left(\left(\frac{x-a}{\eta(b,a)}\right)^{\alpha}+\left(\frac{a+\eta(b,a)-x}{\eta(b,a)}\right)^{\alpha}\right)f(x)-\frac{\Gamma(\alpha+1)}{(\eta(b,a))^{\alpha}}\left(J_{x-}^{\alpha}f\left(a\right)+J_{x+}^{\alpha}f\left(a+\eta\left(b,a\right)\right)\right)\right|$$

$$\leq \begin{cases} \eta(b,a)\left(\frac{1}{\alpha+1}\right)|f'(a)|\left(\left(\frac{x-a}{\eta(b,a)}\right)^{\alpha+1}+\left(1-\frac{x-a}{\eta(b,a)}\right)^{\alpha+1}\right) \\ \text{if } |f'(a)|=|f'(x)|=|f'(b)|, \\ \eta(b,a)\left(\frac{1}{\alpha+1}\right)^{1-\frac{1}{q}}\left(|f'(a)|\left(\frac{x-a}{\eta(b,a)}\right)^{\alpha+1}\left(\frac{1}{\alpha+1}\right)^{\frac{1}{q}}\right. \\ \left.+\frac{|f'(a)|^{1-\frac{x-a}{\eta(b,a)}}}{|f'(b)|^{-\frac{x-a}{\eta(b,a)}}}\left(1-\frac{x-a}{\eta(b,a)}\right)^{\alpha+1}\left(\sum_{i=1}^{\infty}\frac{\left(-\left(1-\frac{x-a}{\eta(b,a)}\right)\ln\frac{|f'(a)|^q}{|f'(b)|^q}\right)^{i-1}}{(\alpha+1)_i}\right)^{\frac{1}{q}}\right) \\ \text{if } |f'(a)|=|f'(x)|\neq|f'(b)|, \\ \eta(b,a)\left(\frac{1}{\alpha+1}\right)^{1-\frac{1}{q}}\left(\frac{|f'(x)|^{\frac{x-a}{\eta(b,a)}}}{|f'(a)|^{\frac{x-a}{\eta(b,a)}-1}}\left(\frac{x-a}{\eta(b,a)}\right)^{\alpha+1}\left(\sum_{i=1}^{\infty}\frac{\left(-\frac{x-a}{\eta(b,a)}\ln\frac{|f'(x)|^q}{|f'(a)|^q}\right)^{i-1}}{(\alpha+1)_i}\right)^{\frac{1}{q}}\right. \\ \left.+|f'(b)|\left(1-\frac{x-a}{\eta(b,a)}\right)^{\alpha+1}\left(\frac{1}{\alpha+1}\right)^{\frac{1}{q}}\right) \\ \text{if } |f'(a)|\neq|f'(x)|=|f'(b)|, \\ \eta(b,a)\left(\frac{1}{\alpha+1}\right)^{1-\frac{1}{q}}\left(\frac{|f'(x)|^{\frac{x-a}{\eta(b,a)}}}{|f'(a)|^{\frac{x-a}{\eta(b,a)}-1}}\left(\frac{x-a}{\eta(b,a)}\right)^{\alpha+1}\left(\sum_{i=1}^{\infty}\frac{\left(-\frac{x-a}{\eta(b,a)}\ln\frac{|f'(x)|^q}{|f'(a)|^q}\right)^{i-1}}{(\alpha+1)_i}\right)^{\frac{1}{q}}\right. \\ \left.+\frac{|f'(x)|^{1-\frac{x-a}{\eta(b,a)}}}{|f'(b)|^{-\frac{x-a}{\eta(b,a)}}}\left(1-\frac{x-a}{\eta(b,a)}\right)^{\alpha+1}\left(\sum_{i=1}^{\infty}\frac{\left(-\left(1-\frac{x-a}{\eta(b,a)}\right)\ln\frac{|f'(x)|^q}{|f'(b)|^q}\right)^{i-1}}{(\alpha+1)_i}\right)^{\frac{1}{q}}\right) \\ \text{if } |f'(a)|\neq|f'(x)|\neq|f'(b)|. \end{cases}$$

Moreover if we take $\eta(b,a)=b-a$, *we get*

$$\left|\left(\left(\frac{x-a}{b-a}\right)^{\alpha}+\left(\frac{b-x}{b-a}\right)^{\alpha}\right)f(x)-\frac{\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left(J_{x-}^{\alpha}f(a)+J_{x+}^{\alpha}f(b)\right)\right|$$

$$\leq \begin{cases} (b-a)\left(\frac{1}{\alpha+1}\right)|f'(a)|\left(\left(\frac{x-a}{b-a}\right)^{\alpha+1}+\left(\frac{b-x}{b-a}\right)^{\alpha+1}\right) \\ \quad \text{if } |f'(a)|=|f'(x)|=|f'(b)|, \\ (b-a)\left(\frac{1}{\alpha+1}\right)^{1-\frac{1}{q}}\left(|f'(a)|\left(\frac{x-a}{b-a}\right)^{\alpha+1}\left(\frac{1}{\alpha+1}\right)^{\frac{1}{q}}\right. \\ \left.+\frac{|f'(a)|^{\frac{b-x}{b-a}}}{|f'(b)|^{-\frac{x-a}{b-a}}}\left(\frac{b-x}{b-a}\right)^{\alpha+1}\left(\sum\limits_{i=1}^{\infty}\frac{\left(-\frac{b-x}{b-a}\ln\frac{|f'(a)|^q}{|f'(b)|^q}\right)^{i-1}}{(\alpha+1)_i}\right)^{\frac{1}{q}}\right) \\ \quad \text{if } |f'(a)|=|f'(x)|\neq|f'(b)|, \\ (b-a)\left(\frac{1}{\alpha+1}\right)^{1-\frac{1}{q}}\left(\frac{|f'(x)|^{\frac{x-a}{b-a}}}{|f'(a)|^{-\frac{b-x}{b-a}}}\left(\frac{x-a}{b-a}\right)^{\alpha+1}\left(\sum\limits_{i=1}^{\infty}\frac{\left(-\frac{x-a}{b-a}\ln\frac{|f'(x)|^q}{|f'(a)|^q}\right)^{i-1}}{(\alpha+1)_i}\right)^{\frac{1}{q}}\right. \\ \left.+|f'(b)|\left(\frac{b-x}{b-a}\right)^{\alpha+1}\left(\frac{1}{\alpha+1}\right)^{\frac{1}{q}}\right) \\ \quad \text{if } |f'(a)|\neq|f'(x)|=|f'(b)|, \\ (b-a)\left(\frac{1}{\alpha+1}\right)^{1-\frac{1}{q}}\left(\frac{|f'(x)|^{\frac{x-a}{b-a}}}{|f'(a)|^{-\frac{b-x}{b-a}}}\left(\frac{x-a}{b-a}\right)^{\alpha+1}\left(\sum\limits_{i=1}^{\infty}\frac{\left(-\frac{x-a}{b-a}\ln\frac{|f'(x)|^q}{|f'(a)|^q}\right)^{i-1}}{(\alpha+1)_i}\right)^{\frac{1}{q}}\right. \\ \left.+\frac{|f'(x)|^{\frac{b-x}{b-a}}}{|f'(b)|^{-\frac{x-a}{b-a}}}\left(\frac{b-x}{b-a}\right)^{\alpha+1}\left(\sum\limits_{i=1}^{\infty}\frac{\left(-\frac{b-x}{b-a}\ln\frac{|f'(x)|^q}{|f'(b)|^q}\right)^{i-1}}{(\alpha+1)_i}\right)^{\frac{1}{q}}\right) \\ \quad \text{if } |f'(a)|\neq|f'(x)|\neq|f'(b)|. \end{cases}$$

References

[1] M, Alomari, M. Darus, S. S. Dragomir and P. Cerone, Ostrowski type inequalities for functions whose derivatives are s-convex in the second sense. Appl. Math. Lett. 23 (2010), no. 9, 1071–1076. 1

[2] M. Alomari and M. Darus, Some Ostrowski type inequalities for quasi-convex functions with applications to special means, RGMIA, 2010, 13 (2) Article no 3. 1

[3] R. Diaz and E. Pariguan, On hypergeometric functions and Pochhammer k-symbol. Divulg. Mat. 15 (2007), no. 2, 179–192. 2, 2.7, 2.8

[4] İ. İşcan, Ostrowski type inequalities for functions whose derivatives are preinvex. Bull. Iranian Math. Soc. 40 (2014), no. 2, 373–386. 1

[5] A. Kashuri, R. Liko and T. Du, Ostrowski type fractional integral operators for generalized beta (r, g)-preinvex functions. Khayyam J. Math. 4 (2018), no. 1, 39–58. 1

[6] A. Kashuri and R. Liko, Ostrowski type fractional integral operators for generalized $(r; g, s, m, \phi)$-preinvex functions. Stud. Univ. Babeş-Bolyai Math. 63 (2018), no. 2, 155–173. 1

[7] A. Kashuri and R. Liko, Some new Ostrowski type fractional integral inequalities for generalized relative semi-(r, m, h)-preinvex mappings via Caputo k-fractional derivatives. Proyecciones 38 (2019), no. 2, 363–394. 1

[8] W. Liu, Ostrowski type fractional integral inequalities for MT-convex functions. Miskolc Math. Notes 16 (2015), no. 1, 249–256. 1

[9] B. Meftah, Ostrowski inequalities for functions whose first derivatives are logarithmically preinvex. Chin. J. Math. (N.Y.) 2016, Art. ID 5292603, 10 ptype. 1

[10] B. Meftah, Some new Ostrwoski's inequalities for functions whose n^{th} derivatives are r-convex. International Journal of Analysis, 2016, 7 pages. 1

[11] B. Meftah, Some new Ostrowski's inequalities for n-times differentiable mappings which are φ-convex, Revista Colombiana de Matemáticas. 51(2017), no.1, 57-69. 1

[12] B. Meftah, Ostrowski inequality for functions whose first derivatives are s-preinvex in the second sense. Khayyam J. Math. 3 (2017), no. 1, 61–80. 1

[13] B. Meftah, Fractional Ostrowski type inequalities for functions whose first derivatives are φ-preinvex. J. Adv. Math. Stud. 10 (2017), no. 3, 335-347. 1

[14] B. Meftah, Some new Ostrowski's inequalities for n-times differentiable mappings which are quasi-convex. Facta Universitatis (NIŠ) Ser. Math. Inform. 32 (2017), no. 3, 319–327. 1

[15] B. Meftah Fractional Ostrowski type inequalities for functions whose first derivatives are s-preinvex in the second sense. International Journal of Analysis and Applications 15 (2017), no. 2, 146–154. 1

[16] B. Meftah, Some new Ostrowski's inequalities for functions whose n^{th} derivatives are logarithmically convex. Ann. Math. Sil. 32 (2017), no. 1, 275–284. 1

[17] B. Meftah, Some Ostrowski's inequalities for functions whose nth derivatives are s-convex. An Univ Oradea Fasc. Mat. 25 (2018), no. 2, 185–212. 1

[18] B. Meftah, Ostrowski's inequalities for functions whose first derivatives are s-logarithmically preinvex in the second sense. Math. Morav. 20 (2018), no. 2, 11–28. 1

[19] B. Meftah, M. Merad and A. Souahi, Fractional Ostrowski type inequalities for functions whose mixed derivatives are prequasiinvex and α-prequasiinvex functions. J. Indones. Math. Soc. 25 (2019), no. 02, pp. 92-107. 1

[20] B. Meftah, Fractional Ostrowski type inequalities for functions whose certain power of modulus of the first derivatives are prequasiinvex via power mean inequality, J. Appl. Anal. 25 (2019) no 1, 83-90. 1

[21] B. Meftah and A. Azaizia, Fractional Ostrowski type inequalities for functions whose first derivatives are MT-preinvex. Revista De Matemáticas De la Universidad del Atlántico Páginas. 6 (2019), no. 1, 33–43. 1

[22] B. Meftah, Fractional Ostrowski type inequalities for functions whose modulus of the first derivatives are prequasiinvex, J. Appl. Anal. 25 (2019), no 2, 165-171. 1

[23] B. Meftah, M. Merad, and A. Souahi, Fractional Ostrowski type inequalities for functions whose modulus of the first derivatives are s-preinvex. Extracta Mathematicae Journal, 34 (2019), no 2, 285-301. 1

[24] B. Meftah, M. Merad, New Ostrowski type inequalities for differentiable harmonically convex functions via fractional integral, Indian Journal of Mathematics. 61 (2019), no. 3, 343-357. 1

[25] K.S. Miller and B. Ross, An introduction to the fractional calculus and fractional differential equations. A Wiley-Interscience Publication. John Wiley & Sons, Inc., New York, 1993. 2.6

[26] S. Mubeen and G. M. Habibullah, k-fractional integrals and application. Int. J. Contemp. Math. Sci. 7 (2012), no. 1-4, 89–94. 2.9

[27] M. A. Noor, Variational-like inequalities. Optimization 30 (1994), no. 4, 323-330. 2.5

[28] M. A. Noor, K. I. Noor and M. U. Awan, Fractional Ostrowski inequalities for (s, m)-Godunova-Levin functions. Facta Univ. Ser. Math. Inform. 30 (2015), no. 4, 489–499. 1

[29] A. M. Ostrowski, Über die Absolutabweichung einer differentiierbaren Funktion von ihrem Integralmittelwert. (German) Comment. Math. Helv. 10 (1937), no. 1, 226–227. 1.1

[30] J. E. Pečarić, F. Proschan and Y. L. Tong, Convex functions, partial orderings, and statistical applications. Mathematics in Science and Engineering, 187. Academic Press, Inc., Boston, MA, 1992. 2.1, 2.2

[31] M. Z. Sarikaya and H. Filiz, Note on the Ostrowski type inequalities for fractional integrals. Vietnam J. Math. 42 (2014), no. 2, 187–190. 1

[32] M. Z. Sarikaya and H. Budak, Generalized Ostrowski type inequalities for local fractional integrals. Proc. Amer. Math. Soc. 145 (2017), no. 4, 1527–1538. 1

[33] E. Set, New inequalities of Ostrowski type for mappings whose derivatives are s-convex in the second sense via fractional integrals. Comput. Math. Appl. 63 (2012), no. 7, 1147–1154. 1

[34] E. Set, M. E. Özdemir, M. Z. Sarıkaya and A. O. Akdemir, Ostrowski-type inequalities for strongly convex functions. Georgian Math. J. 25 (2018), no. 1, 109–115. 1

[35] J. Wang, J. Deng and M. Fěckan, Hermite-Hadamard type inequalities for r-convex functions via Riemann-Liouville fractional integrals. Ukrainian Math. J., 65(2013), 193-211. 2.10

[36] T. Weir and B. Mond, Pre-invex functions in multiple objective optimization. J. Math. Anal. Appl. 136 (1988), no. 1, 29–38.

2.3, 2.4