Available online at http://cna-journal.com
Commun. Nonlinear Anal. 2 (2020), 44–55
Research Article

Communications in Nonlinear Analysis

Journal Homepage: www.cna-journal.com

Study a semi-linear pseudo-parabolic problem with Neumann and integral conditions by using Galerkin mixed finite element method

Sihem Benamira[a,*], Taki-Eddine Oussaeif[b], Abdalfeteh Bouziani[b]

[a] *Department of Mathematics University Mentouri Constantine 1, Algeria.*
[b] *Department of Mathematics and Computer science, Oum El Bouaghi University, Algeria*

Abstract

In this paper, we establish sufficient conditions for the existence, uniqueness, and continuous dependence of generalized solution of a semi-linear pseudo-parabolic problem with Neumann condition and integral boundary condition of first type. The results are by the application of the method based on a priori estimate "energy inequality" and the finite element method based on the Faedo-Galerkin technique.

Keywords: Nonlocal conditions, Integral condition, Finite element method, A priori estimates, Non-Classical Space,, pseudo-parabolic problem.
2010 MSC: 35A05, 35A07, 35K50, 35Q80.

1. Introduction

In the recent years, a new attention has been given to non-linear partial differential equations problem which involve an integral over the spatial domain of a function of the desired solution on the boundary conditions ; see [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22].

The purpose of this paper is to prove the existence and uniqueness of a solution for the following pseudo-parabolic problem with Neumann condition and integral boundary condition of first type. The plan of this paper is as follows. In section 2 we give some notations used through out the paper. Section 3 is devoted to statement of the problem . In section 4 we construct an approximate solution using finite element method. in section 5 we give some a priori estimates. Finally in the section 6, we prove the convergence and we give the existence result where we prove the uniqueness and the continuous dependence of solution.

*Corresponding author
Email addresses: s.benamira@yahoo.fr (Sihem Benamira), takieddine.oussaeif@gmail.com, taki_maths@live.fr (Taki-Eddine Oussaeif), aefbouziani@yahoo.fr (Abdalfeteh Bouziani)

Received 2020-04-25

2. Notation

Let $L^2(\Omega)$ be the usual space of square integrable functions ; its scalar product is denoted by $(.,.)$ and its associated norm by $\|.\|$. We denote by $C_0(\Omega)$ the space of continuous functions with compact support in Ω.

Definition 2.1. We denote by $B_2^m(\Omega)$ called the Bouziani space, the Hilbert space defined of $C_0(\Omega)$ for the scalar product

$$(z, w)_{B_2^m(\Omega)} = \int_\Omega \Im_x^m z.\Im_x^m w dx, \tag{2.1}$$

where

$$\Im_x^m z = \int_\Omega \frac{(x-\xi)^{m-1}}{(m-1)!} z(\xi)\, d\xi,$$

by the norm of the function z from $B_2^m(\Omega)$, the nonnegative number

$$\|z\|_{B_2^m(\Omega)} = \left(\int_\Omega (\Im_x^m z)^2\, dx\right)^{\frac{1}{2}} < \infty, \tag{2.2}$$

then the inequality

$$\|z\|^2_{B_2^m(\Omega)} \leq \frac{(\beta-\alpha)^2}{2} \|z\|^2_{B_2^{m-1}(\Omega)}, \; m \geq 1, \tag{2.3}$$

holds for every $z \in B_2^{m-1}(\Omega)$, and the embedding

$$B_2^{m-1}(\Omega) \hookrightarrow B_2^m(\Omega), \tag{2.4}$$

is continuous .

Remark 2.2. If $m = 0$, the space $B_2^0(\Omega)$ coincides with $L^2(\Omega)$.

Definition 2.3. We denote by $L_0^2(\Omega)$ the space consisting of elements $z(x)$ of the space $L^2(\Omega)$ verifying

$$\int_\Omega x^k z(x)\, dx = 0\, (k = 0, 1).$$

Let X be a space with a norm denoted by $\|.\|_X$

Definition 2.4. (i) Denote by $L^2(I, X)$ the set of all measurable abstract functions $u(., t)$ from I into X such that

$$\|u\|_{L^2(I,X)} = \left(\int_I \|u(., t)\|_x^2\, dt\right)^{\frac{1}{2}} < \infty. \tag{2.5}$$

(ii)Let $C\left(\bar{I}; X\right)$ be the set of all continuous functions $u(., t) : \bar{I} \longrightarrow X$ with

$$\|u\|_{C(\bar{I};X)} = \max \|u(., t)\|_X < \infty.$$

Lemma 2.5. *Let be $v : [0, T] \to H$ be a Bochner integrable function and let $A \subset [0, T]$, any measurable subset, so:*

$i)$ the function $\|v(.)\|_H : [0, T] \to H$ is Lebesgue integrable and we have,

$$\left\|\int_A v(t)\, dt\right\|_H \leq \int_A \|v(t)\|_H\, dt, \tag{2.6}$$

$ii)$ for each $\varphi \in H$, the function $(v(.), \varphi)_H : [0, T] \to \mathbb{R}$ is Lebesgue integrable and we have,

$$\left(\int_A v(t)\, dt, \varphi\right)_H = \int_A (v(t), \varphi)_H\, dt. \tag{2.7}$$

Lemma 2.6. *Let M be a linear closed subspace from a Hilbert space H. So for every $h \in H$, there exists a unique $u \in M$ such that:*

$$\|h-u\|_H = \min_{v\in M} \|h-v\|_H , \tag{2.8}$$

the element u is called the orthogonal projection of h on M relatively to the inner product $(.,.)$ and we note $u = P_M h$. Furthermore, we have the following Pythagorean relation

$$\|h\|_H^2 = \|P_M h\|_H^2 + \|h - P_M h\|_H^2 . \tag{2.9}$$

Theorem 2.7 (Cauchy- Schwarz inequality)**.** *Let be f and g two functions of $L^2(\Omega)$; so*

$$f.g \in L^1(\Omega),$$

and

$$\int_\Omega |f.g| \le \|f\|_{L^2} . \|g\|_{L^2} . \tag{2.10}$$

Theorem 2.8 (The Cauchy inequality)**.** *Let be $a, b \in \mathbb{R}$, and every $\varepsilon > 0$, we have*

$$|ab| \le \frac{\varepsilon}{2}a^2 + \frac{1}{2\varepsilon}b^2.$$

Lemma 2.9 (Gronwall lemma)**.** *Let $h(t)$ and $y(t)$ be two real integrable functions on the interval I, $h(\tau)$ nondeceasing, and c a positive constant if*

$$y(t) \le h(t) + c\int_0^t y(\tau)\, d\tau \qquad \forall t \in I,$$

then

$$y(t) \le h(t)\, e^{ct} \qquad \forall t \in I.$$

Definition 2.10. We call a nonlinear differential system the system of the form

$$\dot{X}(t) = F[X(t)] \tag{2.11}$$

t is a real

$$X(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ . \\ . \\ . \\ x_n(t) \end{pmatrix}, \qquad F(t) = \begin{pmatrix} f_1(t) \\ f_2(t) \\ . \\ . \\ . \\ f_n(t) \end{pmatrix},$$

where f_i are continuous functions.

Definition 2.11. Let be

$$X(t) : \begin{array}{lll} I \subset \mathbb{R} & \longrightarrow & \mathbb{R}^n \\ x & \longrightarrow & x(t) \end{array} , \tag{2.12}$$

X is the solution of the system (2.11), if X is derivable and continuous function, for every each $t \in I$, $X(t) \in I$ and $\dot{X}(t) = F(X(t))$.

Theorem 2.12 (The unicity of solution)**.** *We suppose that F is derivable continuous function on $E \subset \mathbb{R}^n$. So for every each initial condition for $t_0 \in I$ and $X_0 \in E$ the solution of the system (2.11) if it exists it is unique.*

Theorem 2.13 (Local existence of solution)**.** *Let be* $t_0 \in \mathbb{R}$ *and* $X_0 \in \mathbb{R}^n$ *. If* F *is derivable continuous on* X_0*, it exists* $h > 0$ *such that the solution of the system* (2.11) *verifying* $X(t_0) = X_0$ *exists on the interval* $[t_0, t_0 + h]$.

Theorem 2.14 (Global existence of solution)**.** *If* F *is derivable continuous function on* $\mathbb{R}^n$ *and if the solution of the system* (2.11) *verifying* $X(0) = X_0$ *is bounded on the interval which it exists so the solution exists on* $I = [0, +\infty]$.

See artical [22].

3. Statement of the problem

Let be the problem

$$\frac{\partial u(x,t)}{\partial t} - \alpha \frac{\partial^2 u(x,t)}{\partial x^2} - \beta \frac{\partial}{\partial t}\left(\frac{\partial^2 u(x,t)}{\partial x^2}\right) - (u(x,t))^p = f(x,t), \tag{3.1}$$

with the initial condition

$$u(x,0) = u^0, \tag{3.2}$$

and the boundary conditions

$$\begin{cases} \dfrac{\partial u}{\partial x}(1,t) = 0 \\ \int_0^1 u(x,t)\, dx = 0 \end{cases}, \tag{3.3}$$

with $t \in [0,T]$, $T < \infty$, $\alpha \in \mathbb{R}_+^*$, $p \in \mathbb{N}^*$, $x \in [0,1]$.

Through the paper, we will make the following assumptions:

(H_1) : $f \in L^2\left(0,T; B_2^1(0,1)\right)$,

(H_2) : $u^0 \in V$ where V is defined in the following way

$$V = \left\{ v \in L^2(0,1) : \int_0^1 v(x,t)\, dx = \frac{\partial v}{\partial x}(1,t) = 0 \right\}. \tag{3.4}$$

Consequently V is a Hilbert space for $(.,.)$. Moreover for a given function $w(x,t)$, the notation $w(t)$ is used for the same function considered as an abstract function of the variable t.

(H_3) : $f(t,w) \in L^2(0,1)$ for each $(t,w) \in I \times L^2(0,1)$ and the following Lipschitz condition

$$\left\| f(t,w) - f(t',w') \right\|_{B_2^1(0,1)} \leq M\left[\left|t - t'\right| \left(1 + \|w\|_{B_2^1(0,1)} + \left\|w'\right\|_{B_2^1(0,1)}\right) + \left\|w - w'\right\|_{B_2^1(0,1)} \right].$$

Definition 3.1. A weak solution of problem (3.1) − (3.3) means a function

$$u : [0,T] \longrightarrow L^2(0,1)$$

such that

(i) $u \in L^2\left(0,T; B_2^1(0,1)\right)$,

(ii) u has a strong derivative $\dfrac{du}{dt} \in L^2\left(0,T; B_2^1(0,1)\right)$,

(iii) $u(0) = u^0$,

(iv) The identity :

$$\left(\frac{du(t)}{\partial t}, v\right)_{B_2^1(0,1)} + \alpha(u(t), v) + \beta\left(\frac{\partial u}{\partial t}, v\right) - (u^p(x,t), v)_{B_2^1(0,1)} = (f(x,t), v)_{B_2^1(0,1)}.$$

4. Construction of an approximate solution

Let $\varphi_1, \varphi_2, ..., \varphi_N, ...$ be a Hilbertian basis of V, such that we devise $[\alpha, \beta]$ on $N+1$ parts ($N \in \mathbb{N}^*$) and we pose

$$h = \frac{1}{N+1} \quad , \quad t_i = ih \quad , \quad i = 0, 1, 2, ..., N+1.$$

We define functions (φ_i) by

$$\varphi_i(x) = \begin{cases} \dfrac{x - x_{i-1}}{x_i - x_{i-1}}, & x_{i-1} \le x \le x_i, \\ \dfrac{x - x_i}{x_{i+1} - x_i}, & x_i \le x \le x_{i+1}, \\ 0, & ailleurs. \end{cases}$$

For every each functions (φ_i) are of degree 1 with $\varphi_i(x_j) = \delta_{ij}$.

Let (V_n) the subspace from V generated by the first n elements of the basis.

We have to find for each $n \in \mathbb{N}^*$, the approximate solution which has the following form

$$u_n(x,t) = \sum_{i=1}^{n} g_{in}(t)\, \varphi_i(x), \qquad (x,t) \in (0,1) \times [0,T], \tag{4.1}$$

where $g_{in} \in H^1(0,T)$ are unknown functions for the moment.

As we have that $u^0 \in V$ and V_n is a closed subspace from V, we can define in a unique way u_n^0 by

$$u_n^0 = P_{V_n} u^0, \tag{4.2}$$

where P_{V_n} is define in lemma (2.1). By the virtue of the density of $\cup V_n$ in V it follows that

$$u_n^0 \longrightarrow u^0 \text{ in } V \text{ if } n \longrightarrow \infty. \tag{4.3}$$

We note by $\left(g_{in}^0\right)$ the coordinates of u_n^0 in the basis $(\varphi_i)_{i=1}^n$ of V_n that is

$$u_n^0 = \sum_{i=1}^{n} g_{in}^0 \varphi_i, \tag{4.4}$$

so, we have to find

$$u_n \in H^1(0,T;V_n) \tag{4.5}$$

solution of the differential system

$$\left(\frac{du_n}{dt}, \varphi_j\right)_{B_2^1(0,1)} + \alpha(u_n, \varphi_j) - \beta\left(\frac{du_n}{dt}, \varphi_j\right) - (u_n^p, \varphi_j)_{B_2^1(0,1)} = (f(x,t), \varphi_j)_{B_2^1(0,1)}, \tag{4.6}$$

$$u_n(0) = u_n^0, \tag{4.7}$$

By replacing (u_n) by (4.1) and by using the following notations

$$\begin{array}{lll} \alpha_{ij} = (\varphi_i, \varphi_j)_{B_2^1(\Omega)} & , & A = (\alpha_{ij})_{1 \le i,j \le n}, \\ B_{ij} = (\varphi_i, \varphi_j) & , & B = (B_{ij})_{1 \le i,j \le n}, \\ C_j = (u_n^p, \varphi_j)_{B_2^1(0,1)} & , & C = (C_j)_{1 \le j \le n,} \\ F_j(t) = (f, \varphi_j)_{B_2^1(0,1)} & , & \overrightarrow{F(t)} = (F_j(t))_{j=1}^n, \end{array}$$

and

$$\overrightarrow{g_n(t)} = (g_{in}(t))_{i=1}^n \quad , \quad \overrightarrow{g_n^0} = \left(g_{in}^0\right)_{i=1}^n.$$

The system (4.6) can be written as follows

$$(A-\beta B)\,\frac{\overrightarrow{dg_n}}{dt}+\alpha B\overrightarrow{g_n}+C\overrightarrow{g_n^p}=\overrightarrow{F\,(t)}, \tag{4.8}$$

which is a nonlinear differential system.

We easily prove that $(A-\beta B)$ is regular matrix, and by virtue Definition (2.10), (2.11) and Theorems (2.12), (2.13) and (2.14), so the system (4.8) has a unique solution $\overrightarrow{g_n}\in\left[H^1\left(0,T\right)\right]^n$.

Lemma 4.1. *For every $n\geq 1$, problem (4.5) − (4.8) has a unique solution $u_n\in H^1\left(0,T;V_n\right)$ which has the form (4.1).*

5. A-priori estimates for approximations

Lemma 5.1. *For every $n\in\mathbb{N}^*$ functions $u_n\in H^1\left(0,T;V_n\right)$ solutions of (4.6) verify*

$$\int_0^t \left\Vert u_n\right\Vert^2 d\tau\leq KT, \tag{5.1}$$

and

$$\int_0^t \left\Vert \frac{du_n}{dt}\right\Vert^2_{B_2^1(0,1)} d\tau\leq \frac{L}{\beta}, \tag{5.2}$$

where K and L are two positive constants.

Proof. Multiplying the integral identity (4.6) by $g_{jn}\left(t\right)$ and summing up for $j=1,...,n$ and integrating the resulting over $(0,t)$, we obtain

$$\begin{aligned}&\frac{1}{2}\left\Vert u_n\right\Vert^2_{B_2^1(0,1)}+\alpha\textstyle\int_0^t\left\Vert u_n\right\Vert^2 d\tau+\frac{\beta}{2}\left\Vert u_n\right\Vert^2\\&=\textstyle\int_0^t\left(f,u_n\right)_{B_2^1(0,1)}d\tau+\int_0^t\left(u_n^p,u_n\right)_{B_2^1(0,1)}d\tau+\frac{1}{2}\left\Vert u_n^0\right\Vert^2_{B_2^1(0,1)}+\frac{\beta}{2}\left\Vert u_n^0\right\Vert^2.\end{aligned} \tag{5.3}$$

We have

$$\left\Vert u_n^0\right\Vert^2_{B_2^1(0,1)}\leq\left\Vert u^0\right\Vert^2_{B_2^1(0,1)}\leq\frac{1}{2}\left\Vert u^0\right\Vert^2, \tag{5.4}$$

so

$$\begin{aligned}&\left\Vert u_n\right\Vert^2_{B_2^1(0,1)}+2\alpha\textstyle\int_0^t\left\Vert u_n\right\Vert^2 d\tau+\beta\left\Vert u_n\right\Vert^2\\&=2\textstyle\int_0^t\left(f,u_n\right)_{B_2^1(0,1)}d\tau+2\int_0^t\left(u_n^p,u_n\right)_{B_2^1(0,1)}d\tau+\left(\frac{1}{2}+\beta\right)\left\Vert u^0\right\Vert^2,\end{aligned} \tag{5.5}$$

hence, thanks to the Cauchy inequality (5.5)

$$\begin{aligned}&\left\Vert u_n\right\Vert^2_{B_2^1(0,1)}+2\alpha\textstyle\int_0^t\left\Vert u_n\right\Vert^2 d\tau+\beta\left\Vert u_n\right\Vert^2\\&\leq\textstyle\int_0^t\left\Vert f\right\Vert^2_{B_2^1(0,1)}d\tau+\int_0^t\left\Vert u_n\right\Vert^2_{B_2^1(0,1)}d\tau+\int_0^t\left\Vert u_n^p\right\Vert^2_{B_2^1(0,1)}d\tau\\&+\textstyle\int_0^t\left\Vert u_n\right\Vert^2_{B_2^1(0,1)}d\tau+\left(\frac{1}{2}+\beta\right)\left\Vert u^0\right\Vert^2,\end{aligned} \tag{5.6}$$

but we have

$$\left\Vert u_n\right\Vert^2_{B_2^1(0,1)}\leq\frac{1}{2}\left\Vert u_n\right\Vert^2,$$

we get

$$\begin{aligned}&\left\Vert u_n\right\Vert^2_{B_2^1(0,1)}+(2\alpha-1)\textstyle\int_0^t\left\Vert u_n\right\Vert^2 d\tau+\beta\left\Vert u_n\right\Vert^2\\&\leq\textstyle\int_0^t\left\Vert f\right\Vert^2_{B_2^1(0,1)}d\tau+\left(\frac{1}{2}+\beta\right)\left\Vert u^0\right\Vert^2+\int_0^t\left\Vert u_n^p\right\Vert^2_{B_2^1(0,1)}d\tau,\end{aligned} \tag{5.7}$$

we have that

$$\begin{aligned} \int_0^t \|u_n^p\|^2_{B_2^1(0,1)} d\tau &= \int_0^t \left\|u_n^{p-1} \cdot u_n\right\|^2_{B_2^1(0,1)} d\tau \\ &\le \frac{1}{2}\int_0^t \left\|u_n^{p-1}\right\|^2_{B_2^1(0,1)} d\tau + \frac{1}{2}\int_0^t \|u_n\|^2_{B_2^1(0,1)} d\tau \\ &\le \frac{1}{2}\int_0^t \left\|u_n^{p-1}\right\|^2_{B_2^1(0,1)} d\tau + \frac{1}{4}\int_0^t \|u_n\|^2 d\tau, \end{aligned} \tag{5.8}$$

substituting (5.8) in (5.7) we have

$$\begin{aligned} &\|u_n\|^2_{B_2^1(0,1)} + \left(2\alpha - \frac{5}{4}\right)\int_0^t \|u_n\|^2 d\tau \\ &\le \int_0^t \|f\|^2_{B_2^1(0,1)} d\tau + \left(\frac{1}{2}+\beta\right)\left\|u^0\right\|^2 + \int_0^t \left\|u_n^{p-1}\right\|^2_{B_2^1(0,1)} d\tau. \end{aligned} \tag{5.9}$$

But

$$\begin{aligned} \int_0^t \left\|u_n^{p-1}\right\|^2_{B_2^1(0,1)} d\tau &= \int_0^t \left\|u_n^{p-2} \cdot u_n\right\|^2_{B_2^1(0,1)} d\tau \\ &\le \frac{1}{2}\int_0^t \left\|u_n^{p-2}\right\|^2_{B_2^1(0,1)} d\tau + \frac{1}{2}\int_0^t \|u_n\|^2_{B_2^1(0,1)} d\tau \\ &\le \frac{1}{2}\int_0^t \left\|u_n^{p-2}\right\|^2_{B_2^1(0,1)} d\tau + \frac{1}{4}\int_0^t \|u_n\|^2 d\tau. \end{aligned} \tag{5.10}$$

Since (5.10) so (5.9) can be written

$$\begin{aligned} &\|u_n\|^2_{B_2^1(0,1)} + \left(2\alpha - 1 - \frac{1}{2} - \frac{1}{2}\right)\int_0^t \|u_n\|^2 d\tau \\ &\le \int_0^t \|f\|^2_{B_2^1(0,1)} d\tau + \left(\frac{1}{2}+\beta\right)\left\|u^0\right\|^2 + \int_0^t \left\|u_n^{p-2}\right\|^2_{B_2^1(0,1)} d\tau, \end{aligned} \tag{5.11}$$

after p iteration we get

$$\begin{aligned} &\|u_n\|^2_{B_2^1(0,1)} + \left(2\alpha - 1 - \frac{p}{2}\right)\int_0^t \|u_n\|^2 d\tau \\ &\le \int_0^t \|f\|^2_{B_2^1(0,1)} d\tau + \left(\frac{1}{2}+\beta\right)\left\|u^0\right\|^2 + \int_0^t \left\|(u_n)^0\right\|^2_{B_2^1(0,1)} d\tau, \end{aligned} \tag{5.12}$$

so

$$\begin{aligned} &\|u_n\|^2_{B_2^1(0,1)} + \left(2\alpha - 1 - \frac{p}{2}\right)\int_0^t \|u_n\|^2 d\tau + \beta\|u_n\|^2 \\ &\le \int_0^t \|f\|^2_{B_2^1(0,1)} d\tau + \left(\frac{1}{2}+\beta\right)\left\|u^0\right\|^2 + \frac{T}{2}. \end{aligned} \tag{5.13}$$

Let be

$$K = \int_0^t \|f\|^2_{B_2^1(0,1)} d\tau + \left(\frac{1}{2}+\beta\right)\left\|u^0\right\|^2 + \frac{T}{2}, \tag{5.14}$$

we get

$$\|u_n\|^2_{B_2^1(0,1)} \le K, \tag{5.15}$$

so,

$$\int_0^t \|u_n\|^2_{B_2^1(0,1)} \le KT$$

and

$$\int_0^t \|u_n\|^2 d\tau \le \frac{K}{2\alpha - 1 - \frac{p}{2}}, \tag{5.16}$$

$$\|u_n\|^2 \leq \frac{K}{\beta}$$

on the other hand multiplying (4.6) by $\dfrac{dg_{jn}}{dt}$ and sum up for $j = 1, ..., n$ we obtain

$$\begin{aligned}&\left\|\frac{du_n}{dt}\right\|^2_{B_2^1(0,1)} + \frac{\alpha}{2}\frac{d}{dt}\|u_n\|^2 + \beta\left\|\frac{du_n}{dt}\right\|^2 \\ &= \left(f, \frac{du_n}{dt}\right)_{B_2^1(0,1)} + \left(u_n^p, \frac{du_n}{dt}\right)_{B_2^1(0,1)},\end{aligned} \tag{5.17}$$

integrating (5.17) over $(0, t)$

$$\begin{aligned}&2\int_0^t \left\|\frac{du_n}{dt}\right\|^2_{B_2^1(0,1)} d\tau + \alpha\|u_n\|^2 + 2\beta\int_0^t \left\|\frac{du_n}{dt}\right\|^2 d\tau \\ &= 2\int_0^t \left(f, \frac{du_n}{dt}\right)_{B_2^1(0,1)} d\tau + 2\int_0^t \left(u_n^p, \frac{du_n}{dt}\right)_{B_2^1(0,1)} d\tau + \alpha\left\|u_n^0\right\|^2,\end{aligned} \tag{5.18}$$

by reference by the inequaliy (5.4) we get

$$\begin{aligned}&2\int_0^t \left\|\frac{du_n}{dt}\right\|^2_{B_2^1(0,1)} d\tau + \alpha\|u_n\|^2 + 2\beta\int_0^t \left\|\frac{du_n}{dt}\right\|^2 d\tau \\ &= 2\int_0^t \left(f, \frac{du_n}{dt}\right)_{B_2^1(0,1)} d\tau + 2\int_0^t \left(u_n^p, \frac{du_n}{dt}\right)_{B_2^1(0,1)} d\tau + \alpha\left\|u^0\right\|^2,\end{aligned} \tag{5.19}$$

applying the Cauchy inequality

$$\begin{aligned}&\alpha\|u_n\|^2 + 2\beta\int_0^t \left\|\frac{du_n}{dt}\right\|^2 d\tau \\ &\leq \int_0^t \|f\|^2_{B_2^1(0,1)} d\tau + \alpha\left\|u^0\right\|^2 + \int_0^t \|u_n^p\|^2_{B_2^1(0,1)} d\tau\end{aligned}, \tag{5.20}$$

but we have

$$\begin{aligned}\int_0^t \|u_n^p\|^2_{B_2^1(0,1)} d\tau &= \int_0^t \left\|u_n^{p-1}\cdot u_n\right\|^2_{B_2^1(0,1)} d\tau \\ &\leq \frac{1}{2}\int_0^t \left\|u_n^{p-1}\right\|^2_{B_2^1(0,1)} d\tau + \frac{1}{2}\int_0^t \|u_n\|^2_{B_2^1(0,1)} d\tau \\ &\leq \frac{1}{2}\int_0^t \left\|u_n^{p-1}\right\|^2_{B_2^1(0,1)} d\tau + \frac{1}{2}KT \qquad \text{see equation (5.15)} \\ &\leq \frac{1}{2}\int_0^t \left\|u_n^{p-2}\cdot u_n\right\|^2_{B_2^1(0,1)} d\tau + \frac{1}{2}KT \\ &\leq \frac{1}{2}\left[\frac{1}{2}\int_0^t \left\|u_n^{p-2}\right\|^2_{B_2^1(0,1)} d\tau + \frac{1}{2}\int_0^t \|u_n\|^2_{B_2^1(0,1)} d\tau\right] + \frac{1}{2}KT \\ &\leq \frac{1}{2}\cdot\frac{1}{2}\int_0^t \left\|u_n^{p-2}\right\|^2_{B_2^1(0,1)} d\tau + \frac{1}{2}\cdot\frac{1}{2}\cdot KT + \frac{1}{2}KT,\end{aligned}$$

after p iteration we get

$$\int_0^t \|u_n^p\|^2_{B_2^1(0,1)} d\tau \leq T\left(\frac{1}{2^{p+1}}\|u_0\|^2 + K\left(\frac{1}{2^p} + \frac{1}{2}\right)\right), \tag{5.21}$$

substituting (5.21) in (5.20) we get

$$\begin{aligned} &\alpha \left\|u_n\right\|^2 + 2\beta \int_0^t \left\|\frac{du_n}{dt}\right\|^2 d\tau \\ &\leq \int_0^t \|f\|^2_{B_2^1(0,1)} d\tau + \alpha \left\|u^0\right\|^2 + T\left(\frac{1}{2^{p+1}}\left\|(u)^0\right\|^2 + K\left(\frac{1}{2^p}+\frac{1}{2}\right)\right). \end{aligned} \tag{5.22}$$

Let be

$$L = \int_0^t \|f\|^2_{B_2^1(0,1)} d\tau + \alpha \left\|u^0\right\|^2 + T\left(\frac{1}{2^{p+1}} + K\left(\frac{1}{2^p}+\frac{1}{2}\right)\right), \tag{5.23}$$

so we have

$$\int_0^t \left\|\frac{du_n}{dt}\right\|^2 d\tau \leq \frac{L}{2\beta}. \tag{5.24}$$

□

6. Convergence and existence result

Theorem 6.1. *There exist a function $u \in L^2(0,T;V)$ with*

$$\frac{du}{dt} \in L^2\left(0,T;B_2^1(0,1)\right),$$

and a subsequence $(u_{n_k})_k \subseteq (u_n)_n$ such that

$$u_{n_k} \rightharpoonup u \text{ in } L^2(0,T;V), \tag{6.1}$$

and

$$\frac{du_{n_k}}{dt} \rightharpoonup \frac{du}{dt} \text{ in } L^2\left(0,T;B_2^1(0,1)\right), \tag{6.2}$$

when $n \longrightarrow \infty$.

Proof. See article [3]. □

Theorem 6.2. *The limit function u from Theorem (6.1) is the unique weak solution to problem (3.1) − (3.3) in the sense of definition (3.1).*

Proof. One: Existence. We have to show that the limit function u satisfies all conditions $(i) - (iv)$ of definition (3.1). Obviously, in light of properties of function u the first two conditions are already seen. On the other hand, from $u(t) = u^0 + \int_0^t \Psi(s)\,ds$, $t \in [0,T]$, written in the proof of Theorem (6.1), we have directly $u(0) = u^0$, so the initial condition is also fulfilled, now we have to see that integral identity obeyed by u, for this, writing (4.6) for $n = n_k$ and integrating on $[0,t]$, it comes

$$\begin{aligned} &\int_0^t \left(\frac{\partial u_{n_k}(s)}{\partial s}, \varphi_j\right)_{B_2^1(0,1)} ds + \alpha \int_0^t \left(u_{n_k}(s), \varphi_j\right) ds \\ &+\beta \int_0^t \left(\frac{\partial u_{n_k}(s)}{\partial s}, \varphi_j\right) ds - \int_0^t \left(u_{n_k}^p(s), \varphi_j\right)_{B_2^1(0,1)} ds \\ &= \int_0^t \left(f(x,s), \varphi_j\right)_{B_2^1(0,1)} ds; \qquad \forall t \in [0,T], \qquad j = 1, ..., n_k. \end{aligned} \tag{6.3}$$

By performing a limit process $k \longrightarrow \infty$ in (6.3), we get owing (6.1) and (6.2)

$$\begin{aligned} &\int_0^t \left(\frac{\partial u(s)}{\partial s}, \varphi_j\right)_{B_2^1(0,1)} ds + \alpha \int_0^t \left(u(s), \varphi_j\right) ds \\ &- \int_0^t \left(u^p(s), \varphi_j\right)_{B_2^1(0,1)} ds \\ &+\beta \int_0^t \left(\frac{\partial u(s)}{\partial s}, \varphi_j\right) ds \\ &= \int_0^t \left(f(x,s), \varphi_j\right)_{B_2^1(0,1)} ds; \qquad \forall t \in [0,T], \qquad j = 1, ..., n_k. \end{aligned} \tag{6.4}$$

Differentiating this latter with respect to t we get

$$\begin{array}{l}\left(\frac{\partial u\left(t\right)}{\partial t},\varphi_j\right)_{B_2^1(0,1)}+\alpha\left(u\left(t\right),\varphi_j\right)+\beta\left(\frac{\partial u\left(t\right)}{\partial t},\varphi_j\right)\\ -\left(u^p\left(t\right),\varphi_j\right)_{B_2^1(0,1)}\\ =\left(f\left(x,t\right),\varphi_j\right)_{B_2^1(0,1)}\qquad\forall t\in\left[0,T\right],\ j\geq 1.\end{array}\tag{6.5}$$

From where (iv) is obtained due the density of $\cup_n V_n$ in V . Thus, u weakly solves problem $(3.1)-(3.2)$. Two : Uniqueness . Writing the problem $(3.1)-(3.3)$ in the form

$$\frac{\partial u\left(x,t\right)}{\partial t}-\alpha\frac{\partial^2 u\left(x,t\right)}{\partial t^2}=f\left(x,t,u\left(x,t\right)\right),\tag{6.6}$$

which

$$f\left(x,t,u\left(x,t\right)\right)=\left(u\left(x,t\right)\right)^p+\beta\frac{\partial}{\partial x}\left(\frac{\partial^2 u\left(x,t\right)}{\partial x^2}\right)+f\left(x,t\right).\tag{6.7}$$

Let us $(\tilde{u},\mathring{u})$ two weak solutions of (6.6) we get

$$\left(\frac{d\tilde{u}\left(t\right)}{\partial t},v\right)_{B_2^1(0,1)}+\alpha\left(\tilde{u}\left(t\right),v\right)=\left(f\left(\tilde{u},x,t\right),v\right)_{B_2^1(0,1)},\tag{6.8}$$

and

$$\left(\frac{d\mathring{u}\left(t\right)}{\partial t},v\right)_{B_2^1(0,1)}+\alpha\left(\mathring{u}\left(t\right),v\right)=\left(f\left(\mathring{u},x,t\right),v\right)_{B_2^1(0,1)},\tag{6.9}$$

subtructing the identity (6.9) from (6.8) we get for $v=\mathring{u}-\tilde{u}$

$$\frac{1}{2}\frac{d}{dt}\left\|\left(\mathring{u}-\tilde{u}\right)t\right\|_{B_2^1(0,1)}+\alpha\left\|\left(\mathring{u}-\tilde{u}\right)t\right\|=f\left(t,\mathring{u}\right)_{B_2^1(0,1)}-f\left(t,\tilde{u}\right)_{B_2^1(0,1)},\tag{6.10}$$

integrating (6.10) and putting $u\left(t\right)=\mathring{u}-\tilde{u}$ we have

$$\begin{array}{rcl}\left\|u\left(t\right)\right\|_{B_2^1(0,1)}^2+2\alpha\int_0^t\left\|u\left(\tau\right)\right\|^2d\tau & = & 2\int_0^t\left(f\left(\tau,\mathring{u}\right)-f\left(\tau,\tilde{u}\right),u\right)_{B_2^1(0,1)}d\tau,\\ & \leq & 2\int_0^t\left\|f\left(\tau,\mathring{u}\right)-f\left(\tau,\tilde{u}\right)\right\|_{B_2^1(0,1)}\cdot\left\|u\left(\tau\right)\right\|_{B_2^1(0,1)}d\tau,\\ & \leq & 2M\int_0^t\left\|u\left(\tau\right)\right\|_{B_2^1(0,1)}^2d\tau.\end{array}\tag{6.11}$$

From where Gronwalls lemma yields $\left\|u\left(\tau\right)\right\|_{B_2^1(0,1)}^2=0\Longrightarrow\mathring{u}=\tilde{u}$; So, we have the uniqueness of the solution. □

Proposition 6.3. *The sequence* $\left(u_n\right)_n$ *totally converges to* u *in* $L^2\left(0,T;V\right)$.

Proof. The key point is to reason by absurdity, so we suppose that $\left(u_n\right)$ is not converging to u in $L^2\left(0,T;V\right)$ then

$$\begin{array}{c}\exists\varepsilon\geq 0,\exists v\in L^2\left(0,T;V\right),\exists\left(u_\xi\right)_\xi\subset\left(u_n\right)_n:\\ \left|\int_0^T\left(u_\xi\left(t\right)-u\left(t\right),v\left(t\right)\right)dt\right|\geq\varepsilon,\forall v,\end{array}\tag{6.12}$$

but $\left(u_\xi\right)_\xi$ is bounded in $L^2\left(0,T;V\right)$, consequently we can construct a subsequence $\left(u_{\xi_j}\right)$ which weakly converges in $L^2\left(0,T;V\right)$ towards a certain element $w\in L^2\left(0,T;V\right)$, and while reasoning exactly as for the function u from the theorem (6.1) , we prove that u is another solution for the problem $(3.1)-(3.3)$, which implies,taking into account uniqueness in the problem in question, that w is none other than u, so

$$\lim_{\xi\longrightarrow\infty}\int_0^T\left(u_\xi\left(t\right)-u\left(t\right),v\left(t\right)\right)dt=0,$$

which is in contradiction with (6.12), thus

$$u_n\rightharpoonup u\text{ in }L^2\left(0,T;V\right)$$

□

Theorem 6.4. *Let be u^0, $u^0_* \in V$, $f, f_* \in L^2\left(O,T; B^1_2(0,1)\right)$, and let u and u_* be the corresponding weak solutions satisfying assumptions $(H_1)-(H_3)$, if the following inequality*

$$\|f(t,v) - f_*(t,w)\|_{B^1_2(0,1)} \leq a(t) + b\|v-w\|_{B^1_2(0,1)}, \qquad \forall t \in I, \forall v,w \in V, \tag{6.13}$$

holds for some continuous nonnegative $a(t) \in I$ and some constant $b \geq 0$ we have the estimate

$$\|u-u_*\|^2_{B^1_2(0,1)} \leq \left(\left\|u^0-u^0_*\right\|^2_{B^1_2(0,1)} + \int_0^t a^2(\tau)\,d\tau\right) e^{(2b+1)t}. \tag{6.14}$$

Proof. We take the difference identities (6.8) − (6.9) corresponding to u, u_* and f, f_*

$$\begin{gathered}\|u-u_*\|^2_{B^1_2(0,1)} + 2\alpha\int_0^t \|u(\tau)-u_*(\tau)\|^2 d\tau \\ \leq \left\|u^0-u^0_*\right\|^2_{B^1_2(0,1)} \\ +2\int \|f(\tau,u)-f_*(\tau,u_*)\|_{B^1_2(0,1)}\cdot\|u(\tau)-u_*(\tau)\|_{B^1_2(0,1)}\,d\tau,\end{gathered} \tag{6.15}$$

applying the elementary algebraic inequality

$$2\alpha\beta \leq \alpha^2+\beta^2; \qquad \forall \alpha,\beta \in \mathbb{R},$$

to the second term in the right hand side, we derive

$$\begin{gathered}\|u-u_*\|^2_{B^1_2(0,1)} + 2\alpha\int_0^t \|u(\tau)-u_*(\tau)\|^2 d\tau \\ \leq \left\|u^0-u^0_*\right\|^2_{B^1_2(0,1)} \\ +\int_0^t a^2(\tau)\,d\tau + (2b+1)\int_0^1 \|u(\tau)-u_*(\tau)\|^2_{B^1_2(0,1)}\,d\tau\end{gathered}, \tag{6.16}$$

from which the estimate (6.14) follows by means of Gromwell's lemma. □

References

[1] A.Bouziani, On a class of nonlinear reaction-Diffusion systems with nonlocal boundary conditions, A. Analysis, 2004, 9(2004) 793-813. 1

[2] A.Bouziani, N.Merazga, S.Benamira, Galerkin method applied to a parabolic evolution problem with nonlocal boundary conditions, Nonlinear Analysis, 69(2008), 1515-1524 1

[3] N.Boudiba, Existence global pour des systems de réaction-diffusion avec controle de masse, ph.D.thesis, Université de Rennes, France, 1999. 1

[4] A. Bouziani, N.Merazga, On a time-discretisation method for a semilinear heat equation with purely integral condition in a nonclassical function, Nonlinear Analysis, 66(2007), 604–623. 1

[5] A.Bouziani, Mixed problem with boundary integral conditions for a certain parabolic equation, J. Appl. Math. Stochatic.Anal., 9(1996), 323-330. 1

[6] A.Bouziani Mixed problems with boundary integral conditions for certain partial differential equations, Ph.D. thesis, Constantine. 1

[7] A.Bouziani, on a class of parabolic equation with a non local boundary conditions , Acad. Roy.Belg. Bull. CI. .Sci., 30(1-6)(2002), 61–77. 1

[8] A.Bouziani, On the solvability of parabolic and hyperbolic problems with a boundary integral condition, Int. J. Math. Math. Sci.,31(4) (2002), 201-213. 1

[9] M.Dehghan, A finite difference method for a non-local boundary value problem for two dimensional heat equation, Appl. Math. Compute., 112(1) (2000), 133-142. 1

[10] M.Dehghan, Fully explicit finite-differenc emethod for two-dimensional diffusion with an integral condition, nonlinear Anal., 48(5) (2002), 637–650. 1

[11] M.Dehghan, Fully explicit finite-difference methods for two-dimensional diffusion with an integral condition, Nonlinear Anal., 48(5) (2002), 637-650. 1

[12] S.L.Hollis, R.H.Martin Jr, and M.pierre, Global existence and boudeness in reaction-diffusion systems, Siam J.Math.Anal., 18(3) (1987), 744-761. 1

[13] S.L.Hollis and J.Morgan, Interior estimates for a class of reaction-diffusion systems from L^1 a priori estimates in Mathematics, vol; 80,BSBB.G.Teubner Verlagsgesellschaft,Leipzig, 1985. 1

[14] R.H.Martin Jr. and M.Pierre, Nonlinear reaction-diffusion systems, Nonlinear Equations in the applied sciences (W.F.Ames, C.Rogers, and Kapell, eds) , Math, sci.Engrg, vol.185, Academic Press, Massachusetts, 1992, 363-398. 1

[15] T. E Oussaeif, A Bouziani, Solvability of nonlinear viscosity Equation with a boundary integral condition, J. Nonlinear Evol. Equ. Appl., 3 (2015), 31-45. 1

[16] T. E Oussaeif, A Bouziani; Solvability of Nonlinear Goursat Type Problem for Hyperbolic Equation with Integral Condition; Khayyam J. Math., 4(2), 2012, 198-213. 1

[17] T. E Oussaeif, A Bouziani, A priori estimates for weak solution for a time-fractional nonlinear reaction-diffusion equations with an integral condition; Chaos, Solitons & Fractals, 103 (2017), 79–89. 1

[18] S. Dhelis, A Bouziani, T. E Oussaeif, Study of Solution for a Parabolic Integrodifferential Equation with the Second Kind Integral Condition, Int. J. Anal. Appl., 16(4) (2018), 569–593. 1

[19] B.Nur-eddin and N.I.Yurchuk, A mixed problem with an integral condition for a parabolic equations with a Bessel operator, Differ. Uravn., 27(12) (1991), 2094-2098. 1

[20] C.V.Pao, Dynamics of reaction-diffusion equations with nonlocal boundary conditions, Quart Appl. Math., 53(1) (1995), 173-186. 1

[21] C.V.Pao, Asymptotic behavior of solutions of reaction-diffusion equations with nonlocal boundary conditions, J. Comput. Appl.Math., 88(1) (1998), 225-238. 1

[22] C.V.Pao, Numerical solutions of reaction-diffusion equations with nonlocal boundary conditions, J.Comput.Appl.Math., 136(1-2) (2001), 227-243. 1

[23] N.I.Yurchuk, A mixed problem with an integral condition for some parabolic equations, Differ. Urvan., 22(12) (1986), 21117-2126.

[24] Françoise Truc Systemes differentiels non lineaires October 2010.