Available online at https://rsgmodelling.co.uk/
Commun. Nonlinear Anal. 1 (2023), 1–16
Research Article

Communications in Nonlinear Analysis
Publisher
Research & Scinece Group Ltd.

An Application of Fixed Point Theory to A Nonlinear Integral Equation in Banach Spaces

Austine Efut Ofem*

Department of Mathematics, University of Uyo, Uyo, Nigeria

Abstract

In this paper we propose a new iterative scheme, called the AF iteration process, for approximating the unique solution of a mixed type Volterra-Fredholm functional nonlinear integral equation. We prove in the sense of Berinde [8] that our new iterative scheme converges at a rate faster than some of the leading iterative schemes in the literature which have been employed recently to approximate the unique solution of a mixed type Volterra Fredholm functional nonlinear integral equation. We also prove that our new iterative method converges strongly to the unique solution of a mixed type Volterra Fredholm functional nonlinear integral equation. In addition, we give data dependence result for the solution of the nonlinear integral equation which we are considering with the help of our new iterative scheme. Our results improve and unify some well known results in the existing literature.

Keywords: Fixed point, Banach space, strong convergence, contraction mapping, nonlinear integral equation, data dependence
2010 MSC: 46A19, 47L07

1. Introduction

Through out this paper, let $\mathbb{N}$ denote the set of all positive integers and let $\Re$ denote the set of real numbers. Let Υ be a nonempty closed convex subset of a real Banach space E. A mapping $T : \Upsilon \to \Upsilon$ is called contraction if there exists a constant $\vartheta \in (0, 1)$ such that $\|Tx - Ty\| \leq \vartheta\|x - y\|$, $\forall\, x, y \in \Upsilon$.

It well known that integral equations cover many mathematical models of various phenomena in physics, economics, biology, engineering and even mathematics and other related fields of applied science. The demonstrative examples of such models can be found in the literature (see for example, [3] [7], [9], [12] [20], [22], [25], [34], [33] and the references there in). Many problems of applied science and engineering are often reduced to Volterra–Fredholm integral equations (see for example, [2], [19], [26] and the references there in).

*Corresponding author
Email address: ofemaustine@gmail.com, austineofem080@gmail.com (Austine Efut Ofem)

Received 2022-11-04

Analytical solutions of integral equations either do not exist or are difficult to compute (see [9], [26]). Eventually, an exact solution is computable, but the required calculation may be tedious, or the resulting solution may be difficult to interpret. Due to this, it is required to obtain an efficient numerical or iterative solution [6]. There are numerous results in the literature regarding the numerical solution of integral equations (see for example, [5], [13], [17], [23], [28], [30] [37]).

On the other hand, fixed point theory has become the most interesting branch of nonlinear analysis. It is well known that several mathematical and real-world problems are naturally formulated as a fixed point problem, that is, a problem for finding a point x in a domain of an appropriate mapping T such

$$Tx = x. \tag{1.1}$$

A point x satisfying the equation (1.1) is called a fixed point of the mapping T. Furthermore, fixed point theory has been effectively applied in several areas including differential equations, integral equations, matrix equations, convex minimization, as well as for finding the zeros of contraction mappings. Fixed point theory has continually been studied by many authors (see for example [1], [27], [36] and the references there in). It is well known that the contraction-type conditions are very indispensable in the study of fixed point theory.

Overtime, especially recently, there have been many papers devoted to the study of nonlinear integral equations such as Volterra–Fredholm integral equations and their properties. Procedures for approximating their solutions numerically have been developed via collocation methods, CAS wavelets, Taylor expansion methods, block-pulse functions, linear programming, etc, (see for example, [10], [11], [12], [14], [11], [21], [29] and the references therein).

As part of the beauty of fixed point theory, many researchers in nonlinear analysis have come up with several iteration schemes for solving functional nonlinear integral equations. Many iterative processes have been constructed since Picard iteration scheme failed to converge to the fixed point of nonexpansive mappings.

Recently, many authors have employed different iterative schemes to solving the following mixed type Volterra-Fredholm functional nonlinear integral equation which was considered by Crăciun and Şerban [12]:

$$x(t) = F\left(t, x(t), \int_{u_1}^{q_1} \cdots \int_{u_m}^{q_m} K(t,s,x(s))ds, \int_{u_1}^{v_1} \cdots \int_{u_m}^{v_m} H(t,s,x(s))ds\right), \tag{1.2}$$

where $[u_1; v_1] \times \cdots \times [u_m; v_m]$ is an interval in $\Re^m$, $K, H : [u_1; v_1] \times \cdots \times [u_m; v_m] \times [u_1; v_1] \times \cdots \times [u_m; v_m] \times \Re \to \Re$ continuous functions and $F : [u_1; v_1] \times \cdots \times [u_m; v_m] \times \Re^3 \to \Re$. The following iterative schemes which are known as normal S-iterative scheme [35], M iterative scheme [39], Gordian and Uddin iterative scheme [15] respectively, have been used by Gursoy [18], Okeke and Abbas [32], Gordian and Uddin [15] respectively, to approximate the unique solution of the Volterra-Fredholm functional nonlinear integral equation (1.2):

$$\begin{cases} a_0 \in \Upsilon, \\ b_n = (1 - r_n)a_n + r_n T a_n, & \forall n \geq 1; \\ a_{n+1} = T b_n, \end{cases} \tag{1.3}$$

$$\begin{cases} m_0 \in \Upsilon, \\ c_n = (1 - r_n)m_n + r_n T m_n, & \\ \delta_n = T c_n, & \forall n \geq 1; \\ m_{n+1} = T \delta_n, \end{cases} \tag{1.4}$$

$$\begin{cases} d_0 \in \Upsilon, \\ u_n = T d_n, & \\ v_n = (1 - r_n)u_n + r_n T u_n, & \forall n \geq 1. \\ d_{n+1} = T v_n, \end{cases} \tag{1.5}$$

It is no more a surprise that multi-steps iteration processes perform better than single step and two steps iteration processes respectively. Glowinski and Le-Tallec [16] used a multi step iterative process to solve elasto-viscoplasticity, liquid crystal and eigenvalue problems. They established that three-step iterative scheme performs better than one-step (Mann) and two-step (Ishikawa) iterative schemes. Haubruge et al. [24] studied the convergence analysis of the three-step iterative processes of Glowinski and Le-Tallec [16] and used the three-step iteration to obtain some new splitting type algorithms for solving variational inequalities, separable convex programming and minimization of a sum of convex functions. They also proved that three-steps iteration processes also lead to highly parallelized algorithms under certain conditions.

Many researchers have recently been active in constructing multi-steps iteration schemes to obtain faster rate of convergence (see [18], [39] and the references there in). Hence, we see that multi-steps iteration processes play pivotal role in nonlinear analysis and gives faster convergence rate.

Motivated by the above results, we introduce the following four steps iterative scheme, called the AF iterative scheme, for approximating the solution of the mixed type Volterra-Fredholm functional nonlinear integral equation (1.2):

$$\begin{cases} x_0 \in \Upsilon, \\ z_n = Tx_n, \\ w_n = Tz_n, & \forall n \geq 1. \\ y_n = Tw_n, \\ x_{n+1} = (1-r_n)y_n + r_n Ty_n, \end{cases} \tag{1.6}$$

It is our purpose in this paper to show that AF iterative scheme (1.6) converges faster than iterative schemes (1.3)-(1.5) in the sense of Berinde [8]. Furthermore, we prove that AF iterative scheme (1.6) converges strongly to the unique solution of the mixed type Volterra-Fredholm functional nonlinear integral equation (1.2). In addition, we give the data dependence result for the solution of the equation (1.2) via AF iterative scheme (1.6). Our result improve and unify the corresponding results in [12, 15, 18, 32], and several others in the existing literature.

2. Preliminaries

The following definitions, lemmas and theorem will be useful in proving our main results.

Definition 2.1 (see Berinde [8]). Let $\{\mu_n\}_{n=0}^{\infty}$ and $\{\eta_n\}_{n=0}^{\infty}$ be two sequences of real numbers converging to μ and η respectively. Then we say that $\{\mu_n\}_{n=0}^{\infty}$ converges faster than $\{\eta_n\}_{n=0}^{\infty}$ if

$$\lim_{n\to\infty} \frac{\|\mu_n - \mu\|}{\|\eta_n - \eta\|} = 0. \tag{2.1}$$

Definition 2.2 (see Berinde [8]). Let $\{\xi_n\}_{n=0}^{\infty}$ and $\{\zeta_n\}_{n=0}^{\infty}$ be two fixed point iteration procedure sequences that converge to the same point p. If $\|\xi_n - p\| \leq \mu_n$ and $\|\zeta_n - p\| \leq \eta_n$ for all $n \in \mathbb{N}$, where $\{\mu_n\}_{n=0}^{\infty}$ and $\{\eta_n\}_{n=0}^{\infty}$ are two sequences of positive numbers (converging to zero). Then we say that $\{\xi_n\}_{n=0}^{\infty}$ converges faster than $\{\zeta_n\}_{n=0}^{\infty}$ to p if $\{\mu_n\}_{n=0}^{\infty}$ converges faster than $\{\eta_n\}_{n=0}^{\infty}$.

Lemma 2.3. *Let $\{\rho_n\}$ and ψ_n be two nonnegative real sequences satisfying the following inequalities:*

$$\rho_{n+1} \leq (1-\tau_n)\rho_n + \tau_n\psi_n, \tag{2.2}$$

where $\tau_n \in (0,1)$ for all $n \in \mathbb{N}$, $\sum_{n=0}^{\infty} \tau_n = \infty$ and $\psi_n \geq 0$ for all $n \in \mathbb{N}$, then

$$0 \leq \limsup_{n\to\infty} \rho_n \leq \limsup_{n\to\infty} \psi_n. \tag{2.3}$$

Theorem 2.4 (see [12]). *We assume that the following conditions are satisfied:*

(B_1) $K, H \in C([u_1; v_1] \times \cdots \times [u_m; v_m] \times [u_1; v_1] \times \cdots \times [u_m; v_m] \times \Re)$;

(B_2) $F \in ([u_1; v_1] \times \cdots \times [u_m; v_m] \times \Re^3)$;

(B_3) *there exists nonnegative constants* α, β, γ *such that*

$$|F(t, f_1, g_1, h_1) - F(t, f_2, g_2, h_2)| \leq \alpha|f_1 - f_2| + \beta|g_1 - g_2| + \gamma|h_1 - h_2|,$$

for all $t \in [u_1; v_1] \times \cdots \times [u_m; v_m]$, $f_1, g_1, h_1, f_2, g_2, h_2 \in \Re$;

(B_4) *there exist nonnegative constants* L_K *and* L_H *such that*

$$|K(t, s, f) - K(t, s, g)| \leq L_K|f - g|,$$
$$|H(t, s, f) - H(t, s, g)| \leq L_H|f - g|,$$

for all $t, s \in [u_1; v_1] \times \cdots \times [u_m; v_m]$, $f, g \in \Re$;

(B_5) $\alpha + (\beta L_K + \gamma L_H)(v_1 - u_1) \cdots (v_m - u_m) < 1$.

Then, the nonlinear integral equation (1.2) has a unique solution $p \in C([u_1; v_1] \times \cdots \times [u_m; v_m])$.

3. Rate of Convergence

In this section, we prove that the AF iterative process (1.6) converges at a rate faster than all of normal S-iterative process (1.3), M iterative process (1.4) and Garodia and Uddin iterative process (1.5) in the sense of Berinde [8].

Theorem 3.1. *Let* Υ *be a nonempty closed convex subset of a Banach space* E *and* $T : \Upsilon \to \Upsilon$ *be a contraction mapping with contraction constant* $\vartheta \in (0, 1)$ *such that* $F(T) \neq \emptyset$*. If* $\{x_n\}$ *is the sequence defined by (1.6), then* $\{x_n\}$ *converges faster than all the other three processes.*

Proof. For any $p \in F(T)$, from (1.6), we have

$$\begin{aligned} \|z_n - p\| &= \|Tx_n - p\| \\ &\leq \vartheta\|x_n - p\| \end{aligned}$$

and

$$\begin{aligned} \|w_n - p\| &= \|Tz_n - p\| \\ &\leq \vartheta\|z_n - p\| \\ &\leq \vartheta^2\|x_n - p\| \end{aligned}$$

also,

$$\begin{aligned} \|y_n - p\| &= \|Tw_n - p\| \\ &\leq \vartheta\|w_n - p\| \\ &\leq \vartheta^3\|x_n - p\|. \end{aligned}$$

Since $\{r_n\}$ is a sequence in (0,1), then we can always find a constant $r \in \Re$ such that $r_n \leq r < 1$ for all $n \in \mathbb{N}$. So,

$$\begin{aligned} \|x_{n+1} - p\| &= \|(1 - r_n)y_n + r_nTy_n - p\| \\ &\leq (1 - r_n)\|y_n - p\| + r_n\|Ty_n - p\| \\ &\leq (1 - r_n)\|y_n - p\| + r_n\vartheta\|y_n - p\| \\ &= (1 - (1 - \vartheta)r_n)\|y_n - p\| \\ &\leq \vartheta^3(1 - (1 - \vartheta)r_n)\|x_n - p\| \\ &\vdots \\ &\leq \vartheta^{3n}(1 - (1 - \vartheta)r)^n\|x_1 - p\|. \end{aligned}$$

Let

$$h_n = \vartheta^{3n}(1-(1-\vartheta)r)^n\|x_1 - p\|. \tag{3.1}$$

Now, from (1.3), we have

$$\begin{aligned}
\|b_n - p\| &= \|(1-r_n)a_n + r_nTa_n - p\| \\
&\leq (1-r_n)\|a_n - p\| + r_n\|Ta_n - p\| \\
&\leq (1-r_n)\|a_n - p\| + r_n\vartheta\|a_n - p\| \\
&= (1-(1-\vartheta)r_n)\|a_n - p\|.
\end{aligned}$$

So,

$$\begin{aligned}
\|a_{n+1} - p\| &= \|Tb_n - p\| \\
&\leq \vartheta\|b_n - p\| \\
&\leq \vartheta(1-(1-\vartheta)r_n)\|a_n - p\| \\
&\vdots \\
&\leq \vartheta^n(1-(1-\vartheta)r)^n\|a_1 - p\|.
\end{aligned}$$

Let

$$\lambda_n = \vartheta^n(1-(1-\vartheta)r)^n\|a_1 - p\| \tag{3.2}$$

Again, from (1.4), we get

$$\begin{aligned}
\|c_n - p\| &= \|(1-r_n)m_n + r_nTm_n - p\| \\
&\leq (1-r_n)\|m_n - p\| + r_n\|Tm_n - p\| \\
&\leq (1-r_n)\|m_n - p\| + r_n\vartheta\|m_n - p\| \\
&= (1-(1-\vartheta)r_n)\|m_n - p\|
\end{aligned}$$

and

$$\begin{aligned}
\|\delta_n - p\| &= \|Tc_n - p\| \\
&\leq \vartheta\|c_n - p\| \\
&\leq \vartheta(1-(1-\vartheta)r_n)\|m_n - p\|
\end{aligned}$$

So,

$$\begin{aligned}
\|m_{n+1} - p\| &= \|T\delta_n - p\| \\
&\leq \vartheta\|\delta_n - p\| \\
&\leq \vartheta^2(1-(1-\vartheta)r_n)\|m_n - p\| \\
&\vdots \\
&\leq \vartheta^{2n}(1-(1-\vartheta)r)^n\|m_1 - p\|.
\end{aligned}$$

Set

$$t_n = \vartheta^{2n}(1-(1-\vartheta)r)^n\|m_1 - p\| \tag{3.3}$$

Now, using (1.5), we get

$$\begin{aligned}
\|u_n - p\| &= \|Td_n - p\| \\
&\leq \vartheta\|d_n - p\|
\end{aligned}$$

and

$$\begin{aligned}\|v_n - p\| &= \|(1-r_n)u_n + r_n T u_n - p\| \\ &\le (1-r_n)\|u_n - p\| + r_n\|Tu_n - p\| \\ &\le (1-r_n)\|u_n - p\| + r_n\vartheta\|u_n - p\| \\ &= (1-(1-\vartheta)r_n)\|u_n - p\| \\ &\le \vartheta(1-(1-\vartheta)r_n)\|d_n - p\|.\end{aligned}$$

So,

$$\begin{aligned}\|d_{n+1} - p\| &= \|Tv_n - p\| \\ &\le \vartheta\|v_n - p\| \\ &\le \vartheta^2(1-(1-\vartheta)r_n)\|d_n - p\| \\ &\vdots \\ &\le \vartheta^{2n}(1-(1-\vartheta)r)^n\|d_1 - p\|.\end{aligned}$$

Set

$$\varpi_n = \vartheta^{2n}(1-(1-\vartheta)r)^n\|d_1 - p\|. \tag{3.4}$$

Now we compute the rate of convergence of our iterative scheme (1.6) as follows:

(i) Observe that

$$\frac{h_n}{\lambda_n} = \frac{\vartheta^{3n}(1-(1-\vartheta)r)^n\|x_1 - p\|}{\vartheta^{n}(1-(1-\vartheta)r)^n\|a_1 - p\|} = \vartheta^{2n}\frac{\|x_1 - p\|}{\|a_1 - p\|} \to 0 \text{ as } n \to \infty. \tag{3.5}$$

Thus, $\{x_n\}$ converges faster to p than $\{a_n\}$. This implies that, the AF iterative process (1.6) converges faster to p than the normal S-iterative process (1.3).

(ii) Also,

$$\frac{h_n}{t_n} = \frac{\vartheta^{3n}(1-(1-\vartheta)r)^n\|x_1 - p\|}{\vartheta^{2n}(1-(1-\vartheta)r)^n\|m_1 - p\|} = \vartheta^{n}\frac{\|x_1 - p\|}{\|m_1 - p\|} \to 0 \text{ as } n \to \infty. \tag{3.6}$$

Thus, $\{x_n\}$ converges faster to p than $\{m_n\}$. This implies that, the AF iterative process (1.6) converges faster to p than the normal M iterative process (1.4).

(iii) Finally, we see that

$$\frac{h_n}{\varpi_n} = \frac{\vartheta^{3n}(1-(1-\vartheta)r)^n\|x_n - p\|}{\vartheta^{2n}(1-(1-\vartheta)r)^n\|d_1 - p\|} = \vartheta^{n}\frac{\|x_1 - p\|}{\|d_1 - p\|} \to 0 \text{ as } n \to \infty. \tag{3.7}$$

Thus, $\{x_n\}$ converges faster to p than $\{d_n\}$. This implies that, the AF iterative process (1.6) converges faster to p than all of Garodia and Uddin iterative process (1.5). This completes the prove.

□

4. Convergence result

In this section, we prove strong convergence theorem of a sequence generated by AF iteration process (1.6) for the mixed type Volterra-Fredholm functional nonlinear integral equation defined by (1.2) in a real Banach space. And also, we give data dependence result for the solution of the mixed type Volterra-Fredholm functional nonlinear integral equation (1.2) with the help of our new iterative scheme (1.6).

Theorem 4.1. *Assume that all the conditions* $(B_1) - (B_5)$ *in Theorem 2.4 are satisfied. Let* $\{x_n\}$ *be defined by AF iteration process* (1.6) *with real sequence* $r_n \in [0,1]$*, satisfying* $\sum_{n=1}^{\infty} r_n = \infty$*. Then* (1.2) *has a unique solution and the AF iteration process* (1.6) *converges strongly to the unique solution of the mixed type Volterra–Fredholm functional nonlinear integral equation* (1.2)*, say* $p \in C([u_1; v_1] \times \cdots \times [u_m; v_m])$*.*

Proof. We now consider the Banach space $E = C([u_1; v_1] \times \cdots \times [u_m; v_m], \|\cdot\|_C)$, where $\|\cdot\|_C$ is the Chebyshev's norm. Let $\{x_n\}$ be the iterative sequence generated by AF iterative scheme (1.6) for the operator $A : E \to E$ define by

$$A(x)(t) = F\left(t, x(t), \int_{u_1}^{q_1} \cdots \int_{u_m}^{q_m} K(t,s,x(s))ds, \int_{u_1}^{v_1} \cdots \int_{u_m}^{v_m} H(t,s,x(s))ds\right). \tag{4.1}$$

Our intention is to prove that $x_n \to p$ as $n \to \infty$. Now, by using (1.6), (1.2), (4.1) and the assumptions (B_1)–(B_5), we have that

$$\begin{aligned}
\|z_n - p\| &= |A(x_n)(t) - A(p)(t)| \\
&= |F\left(t, x_n(t), \int_{u_1}^{q_1} \cdots \int_{u_m}^{q_m} K(t,s,x_n(s))ds, \int_{u_1}^{v_1} \cdots \int_{u_m}^{v_m} H(t,s,x_n(s))ds\right) \\
&\quad -F\left(t, p(t), \int_{u_1}^{q_1} \cdots \int_{u_m}^{q_m} K(t,s,p(s))ds, \int_{u_1}^{v_1} \cdots \int_{u_m}^{v_m} H(t,s,p(s))ds\right)| \\
&\leq \alpha|x_n(t) - p(t)| + \beta|\int_{u_1}^{q_1} \cdots \\
&\quad \int_{u_m}^{q_m} K(t,s,x_n(s))ds - \int_{u_1}^{q_1} \cdots \int_{u_m}^{q_m} K(t,s,p(s))ds| \\
&\quad +\gamma|\int_{u_1}^{v_1} \cdots \int_{u_m}^{v_m} H(t,s,x_n(s))ds - \int_{u_1}^{v_1} \cdots \int_{u_m}^{v_m} H(t,s,p(s))ds| \\
&\leq \alpha|x_n(t) - p(t)| + \beta \int_{u_1}^{q_1} \cdots \int_{u_m}^{q_m} |K(t,s,x_n(s)) - K(t,s,p(s))|ds \\
&\quad +\gamma \int_{u_1}^{v_1} \cdots \int_{u_m}^{v_m} |H(t,s,x_n(s)) - H(t,s,p(s))|ds \\
&\leq \alpha|x_n(t) - p(t)| + \beta \int_{u_1}^{q_1} \cdots \int_{u_m}^{q_m} L_K|x_n(s) - p(s)|ds \\
&\quad +\gamma \int_{u_1}^{v_1} \cdots \int_{u_m}^{v_m} L_H|x_n(s) - p(s)|ds \\
&\leq \alpha\|x_n - p\| + \beta\Pi_{i=1}^m(v_i - u_i)L_K\|x_n - p\| \\
&\quad +\gamma\Pi_{i=1}^m(v_i - u_i)L_H\|x_n - p\| \\
&= [\alpha + (\beta L_K + \gamma L_H)\Pi_{i=1}^m(v_i - u_i)]\|x_n - p\|.
\end{aligned} \tag{4.2}$$

$$
\begin{aligned}
\|w_n - p\| &= |A(z_n)(t) - A(p)(t)| \\
&= |F\left(t, z_n(t), \int_{u_1}^{q_1}\cdots\int_{u_m}^{q_m} K(t,s,z_n(s))ds, \int_{u_1}^{v_1}\cdots\int_{u_m}^{v_m} H(t,s,z_n(s))ds\right) \\
&\quad -F\left(t, p(t), \int_{u_1}^{q_1}\cdots\int_{u_m}^{q_m} K(t,s,p(s))ds, \int_{u_1}^{v_1}\cdots\int_{u_m}^{v_m} H(t,s,p(s))ds\right)| \\
&\leq \alpha|z_n(t) - p(t)| + \beta|\int_{u_1}^{q_1}\cdots \\
&\quad \int_{u_m}^{q_m} K(t,s,z_n(s))ds - \int_{u_1}^{q_1}\cdots\int_{u_m}^{q_m} K(t,s,p(s))ds| \\
&\quad +\gamma|\int_{u_1}^{v_1}\cdots\int_{u_m}^{v_m} H(t,s,z_n(s))ds - \int_{u_1}^{v_1}\cdots\int_{u_m}^{v_m} H(t,s,p(s))ds|
\end{aligned}
$$

$$
\begin{aligned}
&\leq \alpha|z_n(t) - p(t)| + \beta\int_{u_1}^{q_1}\cdots\int_{u_m}^{q_m} |K(t,s,z_n(s)) - K(t,s,p(s))|ds \\
&\quad +\gamma\int_{u_1}^{v_1}\cdots\int_{u_m}^{v_m} |H(t,s,z_n(s)) - H(t,s,p(s))|ds \\
&\leq \alpha|z_n(t) - p(t)| + \beta\int_{u_1}^{q_1}\cdots\int_{u_m}^{q_m} L_K|z_n(s) - p(s)|ds \\
&\quad +\gamma\int_{u_1}^{v_1}\cdots\int_{u_m}^{v_m} L_H|z_n(s) - p(s)|ds \\
&\leq \alpha\|z_n - p\| + \beta\Pi_{i=1}^m(v_i - u_i)L_K\|z_n - p\| \\
&\quad +\gamma\Pi_{i=1}^m(v_i - u_i)L_H\|z_n - p\| \\
&= [\alpha + (\beta L_K + \gamma L_H)\Pi_{i=1}^m(v_i - u_i)]\|z_n - p\|. \qquad (4.3)
\end{aligned}
$$

Substituting (4.2) into (4.3) we obtain

$$
\|w_n - p\| \leq ([\alpha + (\beta L_K + \gamma L_H)\Pi_{i=1}^m(v_i - u_i)])^2\|x_n - p\|. \qquad (4.4)
$$

Also,

$$\begin{aligned}
\|y_n - p\| &= |A(w_n)(t) - A(p)(t)| \\
&= |F\left(t, w_n(t), \int_{u_1}^{q_1}\cdots\int_{u_m}^{q_m} K(t,s,w_n(s))ds, \int_{u_1}^{v_1}\cdots\int_{u_m}^{v_m} H(t,s,w_n(s))ds\right) \\
&\quad -F\left(t, p(t), \int_{u_1}^{q_1}\cdots\int_{u_m}^{q_m} K(t,s,p(s))ds, \int_{u_1}^{v_1}\cdots\int_{u_m}^{v_m} H(t,s,p(s))ds\right)| \\
&\leq \alpha|w_n(t) - p(t)| + \beta|\int_{u_1}^{q_1}\cdots \\
&\quad \int_{u_m}^{q_m} K(t,s,w_n(s))ds - \int_{u_1}^{q_1}\cdots\int_{u_m}^{q_m} K(t,s,p(s))ds| \\
&\quad +\gamma|\int_{u_1}^{v_1}\cdots\int_{u_m}^{v_m} H(t,s,w_n(s))ds - \int_{u_1}^{v_1}\cdots\int_{u_m}^{v_m} H(t,s,p(s))ds| \\
&\leq \alpha|z_n(t) - p(t)| + \beta\int_{u_1}^{q_1}\cdots\int_{u_m}^{q_m} |K(t,s,w_n(s)) - K(t,s,p(s))|ds \\
&\quad +\gamma\int_{u_1}^{v_1}\cdots\int_{u_m}^{v_m} |H(t,s,w_n(s)) - H(t,s,p(s))|ds \\
&\leq \alpha|w_n(t) - p(t)| + \beta\int_{u_1}^{q_1}\cdots\int_{u_m}^{q_m} L_K|w_n(s) - p(s)|ds \\
&\quad +\gamma\int_{u_1}^{v_1}\cdots\int_{u_m}^{v_m} L_H|w_n(s) - p(s)|ds \\
&\leq \alpha\|w_n - p\| + \beta\Pi_{i=1}^m(v_i - u_i)L_K\|w_n - p\| \\
&\quad +\gamma\Pi_{i=1}^m(v_i - u_i)L_H\|w_n - p\| \\
&= [\alpha + (\beta L_K + \gamma L_H)\Pi_{i=1}^m(v_i - u_i)]\|w_n - p\|. \qquad (4.5)
\end{aligned}$$

Substituting (4.4) into (4.5) we obtain

$$\|y_n - p\| \leq ([\alpha + (\beta L_K + \gamma L_H)\Pi_{i=1}^m(v_i - u_i)])^3\|x_n - p\|. \qquad (4.6)$$

Finally,

$$\begin{aligned}
\|x_{n+1} - p\| &\leq (1 - r_n)|y_n(t) - p(t)| + r_n|A(y_n)(t) - A(p)(t)| \\
&= (1 - r_n)|y_n(t) - p(t)| \\
&\quad +r_n|F\left(t, y_n(t), \int_{u_1}^{q_1}\cdots\int_{u_m}^{q_m} K(t,s,y_n(s))ds, \int_{u_1}^{v_1}\cdots\int_{u_m}^{v_m} H(t,s,y_n(s))ds\right) \\
&\quad -F\left(t, p(t), \int_{u_1}^{q_1}\cdots\int_{u_m}^{q_m} K(t,s,p(s))ds, \int_{u_1}^{v_1}\cdots\int_{u_m}^{v_m} H(t,s,p(s))ds\right)| \\
&\leq (1 - r_n)|y_n(t) - p(t)| + r_n\alpha|y_n(t) - p(t)| + r_n\beta\int_{u_1}^{q_1}\cdots\int_{u_m}^{q_m} L_K|y_n(s) - p(s)|ds \\
&\quad +r_n\gamma\int_{u_1}^{v_1}\cdots\int_{u_m}^{v_m} L_H|y_n(s) - p(s)|ds \\
&\leq \{1 - r_n(1 - [\alpha + (\beta L_K + \gamma L_H)\Pi_{i=1}^m(v_i - u_i)])\}\|y_n - p\|. \qquad (4.7)
\end{aligned}$$

substituting (4.6) into (4.7) we obtain

$$\begin{aligned}
\|x_{n+1} - p\| &\leq ([\alpha + (\beta L_K + \gamma L_H)\Pi_{i=1}^m(v_i - u_i)])^3 \\
&\quad \times\{1 - r_n(1 - [\alpha + (\beta L_K + \gamma L_H)\Pi_{i=1}^m(v_i - u_i)])\}\|x_n - p\|.
\end{aligned} \qquad (4.8)$$

Since from condition (B_5) we have $[\alpha + (\beta L_K + \gamma L_H)\Pi_{i=1}^{m}(v_i - u_i)] < 1$, it follows that $([\alpha + (\beta L_K + \gamma L_H)\Pi_{i=1}^{m}(v_i - u_i)])^3 < 1$. Thus, (4.8) reduces to

$$\|x_{n+1} - p\| \leq \{1 - r_n(1 - [\alpha + (\beta L_K + \gamma L_H)\Pi_{i=1}^{m}(v_i - u_i)])\}\|x_n - p\|. \quad (4.9)$$

From (4.9), we have the following inequalities:

$$\begin{aligned} \|x_{n+1} - p\| &\leq \{1 - r_n(1 - [\alpha + (\beta L_K + \gamma L_H)\Pi_{i=1}^{m}(v_i - u_i)])\}\|x_n - p\| \\ \|x_n - p\| &\leq \{1 - r_{n-1}(1 - [\alpha + (\beta L_K + \gamma L_H)\Pi_{i=1}^{m}(v_i - u_i)])\}\|x_{n-1} - p\| \\ &\vdots \\ \|x_1 - p\| &\leq \{1 - r_0(1 - [\alpha + (\beta L_K + \gamma L_H)\Pi_{i=1}^{m}(v_i - u_i)])\}\|x_0 - p\|. \end{aligned} \quad (4.10)$$

From (4.10), we have

$$\|x_{n+1} - p\| \leq \|x_0 - p\|\Pi_{k=0}^{n}\left\{1 - r_k\left(1 - [\alpha + (\beta L_K + \gamma L_H)\Pi_{i=1}^{m}(v_i - u_i)]\right)\right\}. \quad (4.11)$$

Since $r_k \in [0, 1]$ for all $k \in \mathbb{N}$ and recalling from assumption (B_5) that $[\alpha + (\beta L_K + \gamma L_H)\Pi_{i=1}^{m}(v_i - u_i)] < 1$, then we have

$$1 - r_k(1 - [\alpha + (\beta L_K + \gamma L_H)\Pi_{i=1}^{m}(v_i - u_i) < 1. \quad (4.12)$$

We recall the inequality $1 - x \leq e^{-x}$ for all $x \in [0, 1]$, thus from (4.11), we have

$$\|x_{n+1} - p\| \leq \|x_0 - p\|e^{-(1-[\alpha+(\beta L_K+\gamma L_H)\Pi_{i=1}^{m}(v_i-u_i)])\sum_{k=0}^{n} r_k}. \quad (4.13)$$

Taking the limit of both sides of the above inequalities, we have $\lim_{n\to\infty} \|x_n - p\| = 0$. Hence, (1.6) converges strongly to the unique solution of the mixed type Volterra-Fredholm functional nonlinear integral equation (1.2). □

We now turn our attention to proving the data dependence of the solution for the integral equation (1.2) with help of AF iteration process (1.6).

Let E be as in the proof of Theorem 4.1 and $T, \tilde{T} : E \to E$ be two operators defined by:

$$T(x)(t) = F\left(t, x(t), \int_{u_1}^{q_1} \cdots \int_{u_m}^{q_m} K(t, s, x(s))ds, \int_{u_1}^{v_1} \cdots \int_{u_m}^{v_m} H(t, s, x(s))ds\right), \quad (4.14)$$

$$\tilde{T}(x)(t) = F\left(t, x(t), \int_{u_1}^{q_1} \cdots \int_{u_m}^{q_m} \tilde{K}(t, s, x(s))ds, \int_{u_1}^{v_1} \cdots \int_{u_m}^{v_m} \tilde{H}(t, s, x(s))ds\right), \quad (4.15)$$

where $K, \tilde{K}, H$ and $\tilde{H} \in C([u_1; v_1] \times \cdots \times [u_m; v_m] \times [u_1; v_1] \times \cdots \times [u_m; v_m] \times \Re)$.

Theorem 4.2. *Let F, K and H be as defined in Theorem 4.1. Let $\{x_n\}$ be an iterative sequence generated by AF iteration process (1.6) associated with T. Let $\{\tilde{x_n}\}$ be the an iterative sequence generated by*

$$\begin{cases} \tilde{x_0} \in E, \\ \tilde{z}_n = \tilde{T}\tilde{x_n}, \\ \tilde{w}_n = \tilde{T}\tilde{z}_n, \\ \tilde{y}_n = \tilde{T}\tilde{w}_n, \\ \tilde{x}_{n+1} = (1 - r_n)\tilde{y_n} + r_n\tilde{T}\tilde{y_n}, \end{cases} \quad \forall n \geq 1. \quad (4.16)$$

where E is defined as in the proof of Theorem 4.1 and $r_n \in [0, 1]$ is a real sequence satisfying

(D_1) $\frac{1}{2} \leq r_n$*, for all* $n \geq 1$*;*

(D_2) $\sum\limits_{n=1}^{\infty} r_n = \infty$*. In addition, suppose that;*

(D_3) there exist nonnegative constants φ_1 *and* φ_2 *such that* $|K(t,s,f) - \tilde{K}(t,s,f)| \leq \varphi_1$ *and* $|H(t,s,f) - \tilde{H}(t,s,f)| \leq \varphi_2$*, for all* $f \in \Re$ *and* $t, s \in [u_1; v_1] \times \cdots \times [u_m; v_m]$*.*

If p *is the solution of* (4.14) *and also* $\tilde{p}$ *the solution of* (4.15)*, then we have*

$$\|p - \tilde{p}\| \leq \frac{7(\beta\varphi_1 + \gamma\varphi_2)\Pi_{i=1}^m (v_i - u_i)}{1 - [\alpha + (\beta L_K + \gamma L_H)\Pi_{i=1}^m (v_i - u_i)}. \tag{4.17}$$

Proof. Using (1.6), (4.14), (4.15), (4.16), conditions $(D_1) - (D_3)$ and assumptions $(B_1) - (B_5)$, we obtain

$$\begin{aligned}
\|z_n - \tilde{z}_n\| &= \|Tx_n - \tilde{T}\tilde{x_n}\| \\
&= |F\left(t, x_n(t), \int_{u_1}^{q_1} \cdots \int_{u_m}^{q_m} K(t,s,x_n(s))ds, \int_{u_1}^{v_1} \cdots \int_{u_m}^{v_m} H(t,s,x_n(s))ds\right) \\
&\quad -F\left(t, \tilde{x_n}(t), \int_{u_1}^{q_1} \cdots \int_{u_m}^{q_m} \tilde{K}(t,s,\tilde{x_n}(s))ds, \int_{u_1}^{v_1} \cdots \int_{u_m}^{v_m} \tilde{H}(t,s,\tilde{x_n}(s))ds\right)| \\
&\leq \alpha|x_n(t) - \tilde{x_n}(t)| + \beta|\int_{u_1}^{q_1} \cdots \\
&\quad \int_{u_m}^{q_m} K(t,s,x_n(s))ds - \int_{u_1}^{q_1} \cdots \int_{u_m}^{q_m} \tilde{K}(t,s,\tilde{x}(s))ds| \\
&\quad +\gamma|\int_{u_1}^{v_1} \cdots \int_{u_m}^{v_m} H(t,s,x_n(s))ds - \int_{u_1}^{v_1} \cdots \int_{u_m}^{v_m} \tilde{H}(t,s,\tilde{x_n}(s))ds| \\
&\leq \alpha|x_n(t) - \tilde{x_n}(t)| + \beta \int_{u_1}^{q_1} \cdots \int_{u_m}^{q_m} (|K(t,s,x_n(s)) - K(t,s,\tilde{x_n}(s))| \\
&\quad +|K(t,s,\tilde{x}_n(s)) - \tilde{K}(t,s,\tilde{x_n}(s))|)ds \\
&\quad +\gamma \int_{u_1}^{v_1} \cdots \int_{u_m}^{v_m} (|H(t,s,x_n(s)) - H(t,s,\tilde{x_n}(s))| + \\
&\quad +|H(t,s,\tilde{x}_n(s)) - \tilde{H}(t,s,\tilde{x_n}(s))|)ds \\
&\leq \alpha|x_n(t) - \tilde{x_n}(t)| + \beta \int_{u_1}^{q_1} \cdots \int_{u_m}^{q_m} (L_K|x_n(s) - \tilde{x_n}(s)| + \varphi_1)ds \\
&\quad +\gamma \int_{u_1}^{v_1} \cdots \int_{u_m}^{v_m} (L_H|x_n(s) - \tilde{x_n}(s)| + \varphi_2)ds \\
&\leq \alpha\|x_n - \tilde{x_n}\| + \beta(L_K\|x_n - \tilde{x_n}\| + \varphi_1)\Pi_{i=1}^m (v_i - u_i) \\
&\quad +\gamma(L_H\|x_n - \tilde{x_n}\| + \varphi_2)\Pi_{i=1}^m (v_i - u_i) \\
&= [\alpha + (\beta L_K + \gamma L_H)\Pi_{i=1}^m (v_i - u_i)]\|x_n - \tilde{x_n}\| \\
&\quad +(\beta\varphi_1 + \gamma\varphi_2)\Pi_{i=1}^m (v_i - u_i).
\end{aligned} \tag{4.18}$$

$$\begin{aligned}
\|w_n-\tilde{w}_n\| &= \|Tz_n-\tilde{T}\tilde{z_n}\| \\
&= |F\left(t, z_n(t), \int_{u_1}^{q_1}\cdots\int_{u_m}^{q_m} K(t,s,z_n(s))ds, \int_{u_1}^{v_1}\cdots\int_{u_m}^{v_m} H(t,s,z_n(s))ds\right) \\
&\quad -F\left(t, \tilde{z_n}(t), \int_{u_1}^{q_1}\cdots\int_{u_m}^{q_m} \tilde{K}(t,s,\tilde{z_n}(s))ds, \int_{u_1}^{v_1}\cdots\int_{u_m}^{v_m} \tilde{H}(t,s,\tilde{z_n}(s))ds\right)| \\
&\leq \alpha|z_n(t)-\tilde{z_n}(t)| + \beta|\int_{u_1}^{q_1}\cdots \\
&\quad \int_{u_m}^{q_m} K(t,s,z_n(s))ds - \int_{u_1}^{q_1}\cdots\int_{u_m}^{q_m}\tilde{K}(t,s,\tilde{z}_n(s))ds| \\
&\quad +\gamma|\int_{u_1}^{v_1}\cdots\int_{u_m}^{v_m} H(t,s,z_n(s))ds - \int_{u_1}^{v_1}\cdots\int_{u_m}^{v_m}\tilde{H}(t,s,\tilde{z_n}(s))ds|
\end{aligned}$$

$$\begin{aligned}
&\leq \alpha|z_n(t)-\tilde{z_n}(t)| + \beta\int_{u_1}^{q_1}\cdots\int_{u_m}^{q_m}(|K(t,s,z_n(s)) - K(t,s,\tilde{z_n}(s))| \\
&\quad +|K(t,s,\tilde{z}_n(s)) - \tilde{K}(t,s,\tilde{z_n}(s))|)ds \\
&\quad +\gamma\int_{u_1}^{v_1}\cdots\int_{u_m}^{v_m}(|H(t,s,z_n(s)) - H(t,s,\tilde{z_n}(s))| + \\
&\quad +|H(t,s,\tilde{z}_n(s)) - \tilde{H}(t,s,\tilde{z_n}(s))|)ds \\
&\leq \alpha|z_n(t)-\tilde{z_n}(t)| + \beta\int_{u_1}^{q_1}\cdots\int_{u_m}^{q_m}(L_K|z_n(s)-\tilde{z_n}(s)|+\varphi_1)ds \\
&\quad +\gamma\int_{u_1}^{v_1}\cdots\int_{u_m}^{v_m}(L_H|z_n(s)-\tilde{z_n}(s)|+\varphi_2)ds \\
&\leq \alpha\|z_n-\tilde{z_n}\| + \beta(L_K\|z_n-\tilde{z_n}\|+\varphi_1)\Pi_{i=1}^m(v_i-u_i) \\
&\quad +\gamma(L_H\|z_n-\tilde{z_n}\|+\varphi_2)\Pi_{i=1}^m(v_i-u_i) \\
&= [\alpha+(\beta L_K+\gamma L_H)\Pi_{i=1}^m(v_i-u_i)]\|z_n-\tilde{z_n}\| \\
&\quad +(\beta\varphi_1+\gamma\varphi_2)\Pi_{i=1}^m(v_i-u_i). \qquad (4.19)
\end{aligned}$$

Putting (4.18) into (4.19) we obtain

$$\begin{aligned}
\|w_n-\tilde{w}_n\| &\leq ([\alpha+(\beta L_K+\gamma L_H)\Pi_{i=1}^m(v_i-u_i)])^2\|x_n-\tilde{x}_n\| \\
&\quad +[\alpha+(\beta L_K+\gamma L_H)\Pi_{i=1}^m(v_i-u_i)](\beta\varphi_1+\gamma\varphi_2)\Pi_{i=1}^m(v_i-u_i) \\
&\quad +(\beta\varphi_1+\gamma\varphi_2)\Pi_{i=1}^m(v_i-u_i). \qquad (4.20)
\end{aligned}$$

From (4.20) and assumption (B_5) we obtain

$$\begin{aligned}
\|w_n-\tilde{w}_n\| &\leq ([\alpha+(\beta L_K+\gamma L_H)\Pi_{i=1}^m(v_i-u_i)])^2\|x_n-\tilde{x}_n\| \\
&\quad +2(\beta\varphi_1+\gamma\varphi_2)\Pi_{i=1}^m(v_i-u_i). \qquad (4.21)
\end{aligned}$$

$$\begin{aligned}
\|y_n - \tilde{y}_n\| &= \|Tw_n - \tilde{T}\tilde{w_n}\| \\
&= |F\left(t, w_n(t), \int_{u_1}^{q_1}\cdots\int_{u_m}^{q_m} K(t,s,w_n(s))ds, \int_{u_1}^{v_1}\cdots\int_{u_m}^{v_m} H(t,s,w_n(s))ds\right) \\
&\quad -F\left(t, \tilde{w_n}(t), \int_{u_1}^{q_1}\cdots\int_{u_m}^{q_m} \tilde{K}(t,s,\tilde{w_n}(s))ds, \int_{u_1}^{v_1}\cdots\int_{u_m}^{v_m} \tilde{H}(t,s,\tilde{w_n}(s))ds\right)| \\
&\leq \alpha|w_n(t) - \tilde{w_n}(t)| + \beta|\int_{u_1}^{q_1}\cdots \\
&\quad \int_{u_m}^{q_m} K(t,s,w_n(s))ds - \int_{u_1}^{q_1}\cdots\int_{u_m}^{q_m} \tilde{K}(t,s,\tilde{w_n}(s))ds| \\
&\quad +\gamma|\int_{u_1}^{v_1}\cdots\int_{u_m}^{v_m} H(t,s,w_n(s))ds - \int_{u_1}^{v_1}\cdots\int_{u_m}^{v_m} \tilde{H}(t,s,\tilde{w_n}(s))ds|
\end{aligned}$$

$$\begin{aligned}
&\leq \alpha|w_n(t) - \tilde{w_n}(t)| + \beta\int_{u_1}^{q_1}\cdots\int_{u_m}^{q_m}(|K(t,s,w_n(s)) - K(t,s,\tilde{w_n}(s))| \\
&\quad +|K(t,s,\tilde{w_n}(s)) - \tilde{K}(t,s,\tilde{w_n}(s))|)ds \\
&\quad +\gamma\int_{u_1}^{v_1}\cdots\int_{u_m}^{v_m}(|H(t,s,w_n(s)) - H(t,s,\tilde{w_n}(s))| + \\
&\quad +|H(t,s,\tilde{w_n}(s)) - \tilde{H}(t,s,\tilde{w_n}(s))|)ds \\
&\leq \alpha|w_n(t) - \tilde{w_n}(t)| + \beta\int_{u_1}^{q_1}\cdots\int_{u_m}^{q_m}(L_K|w_n(s) - \tilde{w_n}(s)| + \varphi_1)ds \\
&\quad +\gamma\int_{u_1}^{v_1}\cdots\int_{u_m}^{v_m}(L_H|w_n(s) - \tilde{w_n}(s)| + \varphi_2)ds \\
&\leq \alpha\|w_n - \tilde{w_n}\| + \beta(L_K\|w_n - \tilde{w_n}\| + \varphi_1)\Pi_{i=1}^m(v_i - u_i) \\
&\quad +\gamma(L_H\|w_n - \tilde{w_n}\| + \varphi_2)\Pi_{i=1}^m(v_i - u_i) \\
&= [\alpha + (\beta L_K + \gamma L_H)\Pi_{i=1}^m(v_i - u_i)]\|w_n - \tilde{w_n}\| \\
&\quad +(\beta\varphi_1 + \gamma\varphi_2)\Pi_{i=1}^m(v_i - u_i). \qquad (4.22)
\end{aligned}$$

Putting (4.21) into (4.22)

$$\begin{aligned}
\|y_n - \tilde{y}_n\| &\leq ([\alpha + (\beta L_K + \gamma L_H)\Pi_{i=1}^m(v_i - u_i)])^3\|x_n - \tilde{x}_n\| \\
&\quad +[\alpha + (\beta L_K + \gamma L_H)\Pi_{i=1}^m(v_i - u_i)](\beta\varphi_1 + \gamma\varphi_2)\Pi_{i=1}^m(v_i - u_i) \\
&\quad +2(\beta\varphi_1 + \gamma\varphi_2)\Pi_{i=1}^m(v_i - u_i). \qquad (4.23)
\end{aligned}$$

From (4.23) and assumption (B_5) we obtain

$$\begin{aligned}
\|y_n - \tilde{y}_n\| &\leq ([\alpha + (\beta L_K + \gamma L_H)\Pi_{i=1}^m(v_i - u_i)])^3\|x_n - \tilde{x}_n\| \\
&\quad +3(\beta\varphi_1 + \gamma\varphi_2)\Pi_{i=1}^m(v_i - u_i). \qquad (4.24)
\end{aligned}$$

Finally

$$\begin{aligned}
\|x_{n+1}-\tilde{x}_{n+1}\| &\leq (1-r_n)|y_n(t)-\tilde{y_n}(t)|+r_n|T(y_n)(t)-\tilde{T}(\tilde{y_n})(t)| \\
&= (1-r_n)|y_n(t)-\tilde{y_n}(t)| \\
&\quad +r_n|F\left(t,y_n(t),\int_{u_1}^{q_1}\cdots\int_{u_m}^{q_m}K(t,s,y_n(s))ds,\int_{u_1}^{v_1}\cdots\int_{u_m}^{v_m}H(t,s,y_n(s))ds\right) \\
&\quad -F\left(t,\tilde{y_n}(t),\int_{u_1}^{q_1}\cdots\int_{u_m}^{q_m}K(t,s,\tilde{y_n}(s))ds,\int_{u_1}^{v_1}\cdots\int_{u_m}^{v_m}H(t,s,\tilde{y_n}(s))ds\right)| \\
&\leq (1-r_n)|y_n(t)-\tilde{y_n}(t)| \\
&\quad +r_n\alpha|y_n(t)-\tilde{y_n}(t)|+r_n\beta\int_{u_1}^{q_1}\cdots\int_{u_m}^{q_m}(L_K|y_n(s)-\tilde{y_n}(s)|+\varphi_1)ds \\
&\quad +r_n\gamma\int_{u_1}^{v_1}\cdots\int_{u_m}^{v_m}(L_H|y_n(s)-\tilde{y_n}(s)|+\varphi_2)ds \\
&\leq \{1-r_n(1-[\alpha+(\beta L_K+\gamma L_H)\Pi_{i=1}^m(v_i-u_i)])\}\|y_n-\tilde{y_n}\| \\
&\quad +r_n(\beta\varphi_1+\gamma\varphi_2)\Pi_{i=1}^m(v_i-u_i). \qquad (4.25)
\end{aligned}$$

Putting (4.24) into (4.25) we obtain

$$\begin{aligned}
\|x_{n+1}-\tilde{x}_{n+1}\| &\leq \{1-r_n(1-[\alpha+(\beta L_K+\gamma L_H)\Pi_{i=1}^m(v_i-u_i)])\} \\
&\quad \times([\alpha+(\beta L_K+\gamma L_H)\Pi_{i=1}^m(v_i-u_i)])^3\|x_n-\tilde{x}_n\| \\
&\quad +r_n(\beta\varphi_1+\gamma\varphi_2)\Pi_{i=1}^m(v_i-u_i)+3(\beta\varphi_1+\gamma\varphi_2)\Pi_{i=1}^m(v_i-u_i).
\end{aligned} \qquad (4.26)$$

Since from assumption (B_5) we have $[\alpha+(\beta L_K+\gamma L_H)\Pi_{i=1}^m(v_i-u_i)]<1$, it follows that $([\alpha+(\beta L_K+\gamma L_H)\Pi_{i=1}^m(v_i-u_i)])^3<1$, thus (4.26) becomes

$$\begin{aligned}
\|x_{n+1}-\tilde{x}_{n+1}\| &\leq \{1-r_n(1-[\alpha+(\beta L_K+\gamma L_H)\Pi_{i=1}^m(v_i-u_i)])\}\|x_n-\tilde{x}_n\| \\
&\quad +r_n(\beta\varphi_1+\gamma\varphi_2)\Pi_{i=1}^m(v_i-u_i)+3(\beta\varphi_1+\gamma\varphi_2)\Pi_{i=1}^m(v_i-u_i).
\end{aligned} \qquad (4.27)$$

From our assumption $\frac{1}{2}\leq r_n$, we have that

$$1-r_n\leq r_n\Rightarrow 1=1-r_n+r_n\leq r_n+r_n=2r_n.$$

Thus, we have from (4.27) that

$$\begin{aligned}
\|x_{n+1}-\tilde{x}_{n+1}\| &\leq \{1-r_n(1-[\alpha+(\beta L_K+\gamma L_H)\Pi_{i=1}^m(v_i-u_i)])\}\|x_n-\tilde{x}_n\| \\
&\quad +r_n(\beta\varphi_1+\gamma\varphi_2)\Pi_{i=1}^m(v_i-u_i) \\
&\quad +3(1-r_n+r_n)(\beta\varphi_1+\gamma\varphi_2)\Pi_{i=1}^m(v_i-u_i). \\
&\leq \{1-r_n(1-[\alpha+(\beta L_K+\gamma L_H)\Pi_{i=1}^m(v_i-u_i)])\}\|x_n-\tilde{x}_n\| \\
&\quad +7r_n(\beta\varphi_1+\gamma\varphi_2)\Pi_{i=1}^m(v_i-u_i) \\
&= \{1-r_n(1-[\alpha+(\beta L_K+\gamma L_H)\Pi_{i=1}^m(v_i-u_i)])\}\|x_n-\tilde{x_n}\| \\
&\quad +r_n(1-[\alpha+(\beta L_K+\gamma L_H)\Pi_{i=1}^m(v_i-u_i)]) \\
&\quad \times\left(\frac{7(\beta\varphi_1+\gamma\varphi_2)\Pi_{i=1}^m(v_i-u_i)}{1-[\alpha+(\beta L_K+\gamma L_H)\Pi_{i=1}^m(v_i-u_i)}\right). \qquad (4.28)
\end{aligned}$$

For all $n \geq 1$, from (4.28) put

$$\begin{aligned}\rho_n &= \|x_n - \tilde{x_n}\|,\\ \tau_n &= r_n(1-[\alpha+(\beta L_K+\gamma L_H)\Pi_{i=1}^m(v_i-u_i)]) \in (0,1),\\ \psi_n &= \frac{7(\beta\varphi_1+\gamma\varphi_2)\Pi_{i=1}^m(v_i-u_i)}{1-[\alpha+(\beta L_K+\gamma L_H)\Pi_{i=1}^m(v_i-u_i)} \geq 0.\end{aligned}$$

Therefore, all the conditions of Lemma 2.3 are satisfied. Hence, we obtain that

$$\|x_n - \tilde{x}_n\| \leq \frac{7(\beta\varphi_1+\gamma\varphi_2)\Pi_{i=1}^m(v_i-u_i)}{1-[\alpha+(\beta L_K+\gamma L_H)\Pi_{i=1}^m(v_i-u_i)}. \tag{4.29}$$

□

Remark 4.3. Since our new iteration process converges at a rate faster than some of the iterative methods in the existing literature which have been used to obtain the solution of the mixed type Volterra-Fredholm functional nonlinear integral equation (1.2). Hence, our new iterative scheme is an efficient method for solving (1.2). Hence, our results improves and unify the corresponding results in [12, 15, 18, 32] and several others in the existing literature.

Acknowledgement.

The author is grateful to Associate Professor Edet P. Akapn (Departmental of Mathematics, University of Uyo, Uyo, Nigeria) for making me to see the beauty of pure mathematics and for constantly believing in my ability to solve any form of integral problem. Also to the reviewers who carefully read through this paper for their useful contributions which helped to improve this paper.

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