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Some Inequalities for the Polar Derivative of a Polynomial Having S-Fold Zeros at the Origin

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Abstract

Let $P(z)$ be a polynomial of degree n having all its zeros in $|z| \leq 1$ then for all $(\alpha_i)_{i=1}^{t} \in \mathbb{C}$ with $|\alpha_i| \geq 1, 1 \leq i \leq t < n$, it was proved by Jain[V. K. Jain, Generalization of an inequality involving maximum moduli of a polynomial and its polar derivative, Bull Math Soc Sci Math Roum Tome. 98, 6774 (2007)] that

$$\max_{|z|=1} |D_{\alpha_t}...D_{\alpha_2}D_{\alpha_1}P(z)| \geq \frac{n_t}{2^t}\left[A_{\alpha_t}\max_{|z|=1}|P(z)| + \left(2^t\prod_{i=1}^{t}|\alpha_i| - A_{\alpha_t}\right)\min_{|z|=1}|P(z)|\right].$$

where $n_t = n(n-1)...(n-t+1)$ and $A_{\alpha_t} = (|\alpha_1|-1)(|\alpha_2|-1)...(|\alpha_t|-1)$.

In this paper, we generalize this and some other results.

Keywords: s-fold zeros, Polar Derivative, Inequalities, maximum modulus.
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1. Introduction

Let $\mathcal{P}_n$ denote the space of all complex polynomials $P(z) = \sum_{j=0}^{n} a_j z^j$ of degree at most n. A famous result known as Bernstein's inequality [5] states if $P \in \mathcal{P}_n$, then

$$\max_{|z|=1} |P'(z)| \leq n \max_{|z|=1} |P(z)|. \tag{1.1}$$

This result is best possible and equality holds for the polynomial having all zeros at the origin. If $P(z)$ has all zeros in $|z| \leq 1$ then it was proved by P. Turan [15] that

$$\max_{|z|=1} |P'(z)| \geq \frac{n}{2} \max_{|z|=1} |P(z)|. \tag{1.2}$$

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Inequality (1.2) is best possible and equality holds for polynomials which have all zeros on $|z| = 1$. As a refinement of (1.2) Aziz and Dawood [2] proved that if $P(z)$ has all its zeros in $|z| \leq 1$, then

$$\max_{|z|=1} |P'(z)| \geq \frac{n}{2} \left\{ \max_{|z|=1} |P(z)| + \min_{|z|=1} |P(z)| \right\} \tag{1.3}$$

The equality in (1.3) holds for $P(z) = \alpha z^n + \beta$ where $|\beta| \leq |\alpha|$.

Inequality(1.2) was generalised by Malik [12] who proved that if $P(z)$ is a polynomial of degree n having all its zeros in $|z| \leq k, k \leq 1$, then

$$\max_{|z|=1} |P'(z)| \geq \frac{n}{1+k} \max_{|z|=1} |P(z)|. \tag{1.4}$$

The result is sharp and equality holds for $P(z) = (z + k)^n$.

Inequality (1.4) was generalized by Aziz and Shah [4] by proving that if $P(z)$ is a polynomial of degree n having all its zeros in $|z| \leq k, k \leq 1$ with s-fold zeros at the origin, then

$$\max_{|z|=1} |P'(z)| \geq \frac{n+ks}{1+k} \max_{|z|=1} |P(z)|. \tag{1.5}$$

The result is sharp and the extremal polynomial is $P(z) = z^s(z + k)^{n-s}, 0 \leq s \leq n$.

Let $D_\alpha P(z)$ be an operator that carries n^{th} degree polynomial $P(z)$ to the polynomial

$$D_\alpha P(z) = nP(z) + (\alpha - z)P'(z), \quad \alpha \in \mathbb{C}$$

of degree at most $(n - 1)$. $D_\alpha P(z)$ generalizes the ordinary derivative $P'(z)$ in the sense that

$$\lim_{\alpha \to \infty} \frac{D_\alpha P(z)}{\alpha} = P'(z)$$

Now corresponding to a given n^{th} degree polynomial $P(z)$, we construct a sequence of polar derivatives

$$D_\alpha P(z) = nP(z) + (\alpha - z)P'(z)$$

$$\begin{aligned} D_{\alpha_k} D_{\alpha_{k-1}} ... D_{\alpha_1} P(z) =& (n - k + 1) D_{\alpha_{k-1}} ... D_{\alpha_1} P(z) \\ &+ (\alpha_k - z)(D_{\alpha_{k-1}} ... D_{\alpha_1} P(z))' \quad \text{for} \quad k = 2, 3, ..., n. \end{aligned}$$

The points $\alpha_1, \alpha_2, ..., \alpha_k, k = 1, 2, ..., n$, may be equal or unequal. Like the k^{th} ordinary derivative $P^{(k)}(z)$ of $P(z)$, the k^{th} polar derivative $D_{\alpha_k} D_{\alpha_{k-1}} ... D_{\alpha_1} P(z)$ of $P(z)$ is a polynomial of degree at most $n - k$.

As an extension of (1.1) for the polar derivative Aziz and Shah [3] used polar derivative and established that if $P(z)$ is a polynomial of degree n, then for every real or complex number α with $|\alpha| > 1$ and for $|z| \geq 1$,

$$|D_\alpha P(z)| \leq n|\alpha z^{n-1}| \max_{|z|=1} |P(z)| \tag{1.6}$$

Aziz [1] extended (1.6) to the j^{th} polar derivative and proved that if $P(z)$ is a polynomial of degree n then for all $(\alpha_i)_{i=1}^t \in \mathbb{C}$ with $|\alpha_i| \geq 1$ for all $i = 1, 2, ..., t (t < n)$ then for $|z| \geq 1$,

$$\max_{|z|=1} |D_{\alpha_t} ... D_{\alpha_2} D_{\alpha_1} P(z)| \leq n(n-1)...(n-t+1)|\alpha_1 \alpha_2 ... \alpha_t||z|^{n-t} \max_{|z|=1} |P(z)|.$$

W. M. Shah [14] extended (1.2) to the polar derivative and proved that if $P \in \mathcal{P}_n$ and has all zeros in $|z| \leq 1$, then for $|\alpha| \geq 1$

$$\max_{|z|=1} |D_\alpha P(z)| \geq \frac{n(|\alpha| - 1)}{2} \max_{|z|=1} |P(z)|. \tag{1.7}$$

As an extension of (1.7) to the j^{th} polar derivative, Jain [10] proved that if $P(z)$ has all its zeros in $|z| \leq 1$, then for all $(\alpha_i)_{i=1}^{t} \in \mathbb{C}$ with $|\alpha_i| \geq 1, 1 \leq i \leq t < n$,

$$\max_{|z|=1} |D_{\alpha_t}...D_{\alpha_2}D_{\alpha_1}P(z)| \geq \frac{n_t}{2^t}\left[A_{\alpha_t}\max_{|z|=1}|P(z)| + \left(2^t\prod_{i=1}^{t}|\alpha_i| - A_{\alpha_t}\right)\min_{|z|=1}|P(z)|\right]. \tag{1.8}$$

where

$$n_t = n(n-1)...(n-t+1) \quad \text{and} \quad A_{\alpha_t} = (|\alpha_1|-1)(|\alpha_2|-1)...(|\alpha_t|-1). \tag{1.9}$$

This result is best possible and extremal polynomial is $P(z) = (z-1)^n$ with $\alpha_i \geq 1, 1 \leq i \leq t < n$.

2. Preliminaries

For the proof of these Theorems, we need the following Lemmas. The first Lemma is due to Laguerre [11].

Lemma 2.1. *If all the zeros of an n^{th} degree polynomial $P(z)$ lie in a circular region C and if none of the points $(\alpha_i)_{i=1}^{t}$ lie in the region C then each of the polar derivatives $(D_{\alpha_i})_{i=1}^{t}$, $t < n$ has all its zeros in region C.*

Lemma 2.2. *Let A and B be any two complex numbers, then*

(i) If $|A| \geq |B|$ and $B \neq 0$, then $A = vB$ for all complex numbers v with $|v| < 1$.

(ii) Conversely, if $A \neq vB$ for all complex number v with $|v| < 1$, then $|A| \geq |B|$.

Lemma(2.2) is due to Xin Li [16]

Lemma 2.3. *If $P(z) = a_0 + a_1 z + \sum_{j=2}^{n} a_j z^j$ is a polynomial of degree n, having no zeros in $|z| < k$, $\quad k \geq 1$, then*

$$\frac{k|a_1|}{|a_0|} \leq n.$$

This Lemma is due to Gardner et al. [7]

Lemma 2.4. *If $P(z) = \sum_{j=0}^{n} a_j z^j$ is a polynomial of degree n, having all its zeros in $|z| \leq k$, $\quad k \leq 1$, then*

$$\frac{|a_{n-1}|}{|a_n|} \leq nk.$$

Proof. Since $P(z)$ has all zeros in $|z| \leq k$, $\quad k \leq 1$, therefore $q(z) = z^n\overline{P\left(\frac{1}{\bar{z}}\right)} = \overline{a_n} + \overline{a_{n-1}}z + ... + \overline{a_1}z^{n-1} + \overline{a_0}z^n$, is a polynomial of degree at most n, which does not vanish in $|z| < \frac{1}{k}$, $\quad \frac{1}{k} \geq 1$. Apply Lemma 2.3 to $q(z)$, we get the desired result. □

Lemma 2.5. *If $P(z)$ is a polynomial of degree n having all its zeros in $|z| \leq k \leq 1$ with s-fold zeros at the origin then for every $\alpha \in \mathbb{C}$ with $|\alpha| \geq k$,*

$$|D_\alpha P(z)| \geq \frac{(|\alpha|-k)(n+ks)}{1+k}|P(z)|$$

where $0 \leq s \leq n$.

The above Lemma is due to Dewan et al. [6]

3. Main Result

The main aim of this paper is to obtain inequalities similar to (1.8) for the polynomial having s-fold zeros at the origin.

Theorem 3.1. *If $P(z)$ is a polynomial of degree n having all zeros in $|z| \leq k, k \leq 1$ with s-fold zeros at origin then for all $(\alpha_i)_{i=1}^{t} \in \mathbb{C}$ with $|\alpha_i| \geq k, 1 \leq i \leq t < n$,*

$$\max_{|z|=1} |D_{\alpha_t}...D_{\alpha_2} D_{\alpha_1} P(z)| \geq \frac{A^k_{\alpha_t}}{(1+k)^t} \prod_{i=0}^{t-1} [(n-i) + k(s-i)] \max_{|z|=1} |P(z)|. \tag{3.1}$$

where $A^k_{\alpha_t} = (|\alpha_1| - k)(|\alpha_2| - k)...(|\alpha_t| - k)$ and $0 \leq s \leq n$.

Proof. If $|\alpha_j| = k$ for at least one $j, 1 \leq j \leq t$, then result is trivial. Therefore, we assume that $|\alpha_j| > k$ for all $j; 1 \leq j \leq t$. We will prove the result by mathematical induction. The result is true for $t = 1$ by Lemma 2.5 that means if $|\alpha_1| > k$, then

$$|D_{\alpha_1} P(z)| \geq \frac{(|\alpha_1| - k)(n + ks)}{1 + k} |P(z)| \tag{3.2}$$

Now for $t = 2$, since $D_{\alpha_1} P(z) = (na_n\alpha_1 + a_{n-1})z^{n-1} + ... + (na_0 + \alpha_1 a_1)$, and $|\alpha_1| > k$, then $D_{\alpha_1} P(z)$ will be a polynomial of degree $(n-1)$. If it is not true, then the coefficient of z^{n-1} must be equal to zero, which implies

$$na_n\alpha_1 + a_{n-1} = 0,$$

i.e,

$$|\alpha_1| = \frac{|a_{n-1}|}{n|a_n|}.$$

Applying Lemma 2.4, we get

$$|\alpha_1| = \frac{|a_{n-1}|}{n|a_n|} \leq k.$$

But this contradicts the fact that $|\alpha_1| > k$. Hence, the polynomial $D_{\alpha_1} P(z)$ must be of degree $(n-1)$.

Also $P(z)$ has all zeros in $|z| \leq k, k \leq 1$ with s-fold zeros at origin, so $P(z) = z^s h(z)$ where $h(z)$ is a polynomial of degree $n - s$ having all zeros in $|z| \leq, k \leq 1$. Now $D_{\alpha_1} P(z) = z^s D_{\alpha_1} h(z) + t\alpha_1 z^{s-1} h(z)$. Hence $D_{\alpha_1} P(z)$ is a polynomial of degree $n-1$ having all zeros in $|z| \leq k$ with $(s-1)$ fold zeros at origin. By Lemma 2.5 we have for $|\alpha_2| > k$

$$|D_{\alpha_2}(D_{\alpha_1} P(z))| \geq \frac{[(n-1) + k(s-1)]}{1+k} (|\alpha_2| - k)|D_{\alpha_1} P(z)|. \tag{3.3}$$

Using (3.2) we have

$$|D_{\alpha_2} D_{\alpha_1} P(z)| \geq \frac{(n+ks)\,[(n-1) + k(s-1)]}{(1+k)^2} (|\alpha_1| - k)(|\alpha_2| - k)|P(z)|. \tag{3.4}$$

This implies result is true for $t = 2$. Assume that the result is true for $t = q < n$; so for $|z| = 1$, we have

$$\begin{aligned} |D_{\alpha_q}...D_{\alpha_2} D_{\alpha_1} P(z)| \geq & \frac{(n+ks)\,[(n-1) + k(s-1)] \ldots [(n-q+1) + k(s-q+1)]}{(1+k)^q} \times \\ & (|\alpha_1| - k)(|\alpha_2| - k)...(|\alpha_q| - k)|P(z)| \end{aligned} \tag{3.5}$$

and we will prove that the result is true for $t = q+1 < n$. According to above procedure, one can conclude that $D_{\alpha_q}...D_{\alpha_2}D_{\alpha_1}P(z)$ will be a polynomial of degree $(n-q)$ for all $(\alpha_i)_{i=1}^t \in \mathbb{C}$ with $|\alpha_i| \geq k; 1 \leq i \leq q < n$ and has all zeros in $|z| \leq k, k \leq 1$ with $(s-q)$ fold zeros at origin. Therefore, for $|\alpha_{q+1}| > k$, by applying Lemma 2.5 to $D_{\alpha_q}...D_{\alpha_2}D_{\alpha_1}P(z)$, we get

$$|D_{\alpha_{q+1}}\left\{D_{\alpha_q}...D_{\alpha_2}D_{\alpha_1}P(z)\right\}| \geq \frac{[(n-q)+k(s-q)]}{1+k}(|\alpha_{q+1}|-k)|D_{\alpha_q}...D_{\alpha_2}D_{\alpha_1}P(z)| \tag{3.6}$$

Using (3.5) in (3.6) we have for $|z| = 1$

$$\begin{aligned}&|D_{\alpha_{q+1}}D_{\alpha_q}...D_{\alpha_2}D_{\alpha_1}P(z)| \geq \\ &\frac{(n+ks)\left[(n-1)+k(s-1)\right]...\left[(n-q+1)+k(s-q+1)\right]\left[(n-q)+k(s-q)\right]}{(1+k)^{q+1}}\times \\ &(|\alpha_{q+1}|-k)(|\alpha_q|-k)...(|\alpha_2|-k)(|\alpha_1|-k)|P(z)|.\end{aligned} \tag{3.7}$$

This implies result is true for $t = q+1$.

□

If $\alpha_1 = \alpha_2 = ... = \alpha_t = \alpha$, then by dividing both sides of (3.1) by $|\alpha|^t$ and letting $|\alpha| \to \infty$, we get the following result.

Corollary 3.2. *If $P(z)$ is a polynomial of degree n having all zeros in $|z| \leq k, k \leq 1$ with s-fold zeros at origin, then*

$$\max_{|z|=1}|P^{(t)}(z)| \geq \frac{\prod_{i=0}^{t-1}[(n-i)+k(s-i)]}{(1+k)^t}\max_{|z|=1}|P(z)|.$$

where $0 \leq s \leq n$.

For $k = 1$, Theorem 3.1 reduces to the following result which is generalization of (1.8).

Corollary 3.3. *If $P(z)$ is a polynomial of degree n having all zeros in $|z| \leq 1$ with s-fold zeros at origin then for all $(\alpha_i)_{i=1}^t \in \mathbb{C}$ with $|\alpha_i| \geq 1, 1 \leq i \leq t < n$,*

$$\max_{|z|=1}|D_{\alpha_t}...D_{\alpha_2}D_{\alpha_1}P(z)| \geq \frac{A_{\alpha_t}}{2^t}\prod_{i=0}^{t-1}[(n-i)+(s-i)]\max_{|z|=1}|P(z)|$$

where A_{α_t} is defined in (1.9) and $0 \leq s \leq n$.

Theorem 3.4. *If $P(z)$ is a polynomial of degree n having all zeros in $|z| \leq k, k \leq 1$ with s-fold zeros at origin then for all $(\alpha_i)_{i=1}^t \in \mathbb{C}$ with $|\alpha_i| \geq k, 1 \leq i \leq t < n$,*

$$\begin{aligned}\max_{|z|=1}|D_{\alpha_t}...D_{\alpha_2}D_{\alpha_1}P(z)| \geq{}& \frac{A^k_{\alpha_t}}{(1+k)^t}\prod_{i=0}^{t-1}[(n-i)+k(s-i)]\max_{|z|=1}|P(z)| \\ &+ k^{-n}\left\{n_t\prod_{i=1}^{t}|\alpha_i| - \frac{A^k_{\alpha_t}}{(1+k)^t}\prod_{i=0}^{t-1}[(n-i)+k(s-i)]\right\}m\end{aligned}$$

where $m = \min_{|z|=k}|P(z)|$, $0 \leq s \leq n$, n_t is defined in (1.9) and $A^k_{\alpha_t}$ is defined in Theorem 3.1.

Proof. Let $m = \min_{|z|=k} |P(z)|$. If $P(z)$ has a zero on $|z| = k$,then $m = 0$ and the result follows from Theorem 3.1. So we suppose that all the zeros of $P(z)$ lie in $|z| < k$, with s-fold zeros at origin, so that $m > 0$. Now $m \leq |P(z)|$ for $|z| = k$. Since all zeros of $P(z)$ lie in $|z| < k$ with s-fold zeros at origin, by *Rouche's Theorem* all zeros of the polynomial $T(z) = P(z) - \lambda m \left(\frac{z}{k}\right)^n$ lie in $|z| < k$ with s-fold zeros at origin with $|\lambda| < 1$. Applying Theorem 3.1 to the polynomial $T(z)$, we get for all $(\alpha_i)_{i=1}^t \in \mathbb{C}$ with $|\alpha_i| \geq k, 1 \leq i \leq t < n$ on $|z| = 1$,

$$|D_{\alpha_t}...D_{\alpha_2}D_{\alpha_1}T(z)| \geq \frac{A^k_{\alpha_t}}{(1+k)^t}\prod_{i=0}^{t-1}[(n-i)+k(s-i)]|T(z)|.$$

Equivalently,

$$\left|D_{\alpha_t}...D_{\alpha_2}D_{\alpha_1}\left(P(z) - \lambda m\left(\frac{z}{k}\right)^n\right)\right| \geq \frac{A^k_{\alpha_t}}{(1+k)^t}\prod_{i=0}^{t-1}[(n-i)+k(s-i)] \left|P(z) - \lambda m\left(\frac{z}{k}\right)^n\right|.$$

Or

$$\left|D_{\alpha_t}...D_{\alpha_2}D_{\alpha_1}P(z) - \lambda m n_t \alpha_1\alpha_2...\alpha_t\frac{z^{n-t}}{k^n}\right| \geq \frac{A^k_{\alpha_t}}{(1+k)^t}\prod_{i=0}^{t-1}[(n-i)+k(s-i)] \left|P(z) - \lambda m\left(\frac{z}{k}\right)^n\right|. \tag{3.8}$$

By Lemma 2.1, the polynomial $R(z) = D_{\alpha_t}...D_{\alpha_2}D_{\alpha_1}T(z) \neq 0$ for $|z| > k$ that is for every λ with $|\lambda| < 1$ and $|z| > k$, the polynomial $R(z) = D_{\alpha_t}...D_{\alpha_2}D_{\alpha_1}P(z) - \lambda m n_t\alpha_1\alpha_2...\alpha_t\frac{z^{n-t}}{k^n} \neq 0$. Thus by (ii) of Lemma 2.2, we have for $|z| > k$

$$|D_{\alpha_t}...D_{\alpha_2}D_{\alpha_1}P(z)| \geq \frac{m}{k^n}n_t\prod_{i=1}^{t}|\alpha_i||z|^{n-t} \tag{3.9}$$

Taking a relevant choice of argument of λ in (3.8) which is possible by (3.9) we get

$$|D_{\alpha_t}...D_{\alpha_2}D_{\alpha_1}P(z)| - |\lambda|\frac{m}{k^n}n_t\prod_{i=1}^{t}|\alpha_i||z|^{n-t} \geq \frac{A^k_{\alpha_t}}{(1+k)^t}\prod_{i=0}^{t-1}[(n-i)+k(s-i)]|P(z)| - |\lambda|\frac{A^k_{\alpha_t}}{(1+k)^t}\prod_{i=0}^{t-1}[(n-i)+k(s-i)]m\frac{|z|^n}{k^n}.$$

Which on simplification gives for $|z| = 1$,

$$|D_{\alpha_t}...D_{\alpha_2}D_{\alpha_1}P(z)| \geq \frac{A^k_{\alpha_t}}{(1+k)^t}\prod_{i=0}^{t-1}[(n-i)+k(s-i)]\max_{|z|=1}|P(z)| + |\lambda|k^{-n}\left\{n_t\prod_{i=1}^{t}|\alpha_i| - \frac{A^k_{\alpha_t}}{(1+k)^t}\prod_{i=0}^{t-1}[(n-i)+k(s-i)]\right\}m \tag{3.10}$$

Making $|\lambda| \to 1$, the desired result follows. □

If $\alpha_1 = \alpha_2 = ... = \alpha_t = \alpha$, then by dividing both sides of (3.1) by $|\alpha|^t$ and letting $|\alpha| \to \infty$, we get the following result.

Corollary 3.5. *If $P(z)$ is a polynomial of degree n having all zeros in $|z| \leq k, k \leq 1$ with s-fold zeros at origin, then*

$$\max_{|z|=1} |P^{(t)}(z)| \geq \frac{\prod_{i=0}^{t-1}[(n-i)+k(s-i)]}{(1+k)^t} \max_{|z|=1} |P(z)| + k^{-n}\left\{ n_t - \frac{\prod_{i=0}^{t-1}[(n-i)+k(s-i)]}{(1+k)^t} \right\} m$$

where $m = \min_{|z|=k} |P(z)|$, $0 \leq s \leq n$ and n_t is defined in (1.9).

For $t = 1$ Theorem 3.4 reduces to the following result which is refinement of a result of Dewan et al [6].

Corollary 3.6. *If $P(z)$ is a polynomial of degree n having all zeros in $|z| \leq k, k \leq 1$ with s-fold zeros at origin then for all $\alpha \in \mathbb{C}$ with $|\alpha| \geq k$, we have for $|z| = 1$*

$$\max_{|z|=1} |D_\alpha P(z)| \geq \frac{(|\alpha| - k)}{(1+k)}(n+ks) \max_{|z|=1} |P(z)| + k^{-n}\left\{ n|\alpha| - \frac{|\alpha| - k}{(1+k)}(n+ks) \right\} m$$

where $m = \min_{|z|=k} |P(z)|$ and $0 \leq s \leq n$.

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