Available online at http://cna-journal.com
Commun. Nonlinear Anal. 2 (2020), 28–43
Research Article

Communications in Nonlinear Analysis
Journal Homepage: www.cna-journal.com

New Iteration Algorithms for Finite Family of Two Quasi-nonexpansive Mappings Satisfying Jointly Demiclosedness Principle in Banach Spaces

Imo Kalu Agwu[a,*]

[a]Department of Mathematics, Micheal Okpara University of Agriculture, Umudike, Umuahia, Abia State, Nigeria

Abstract

In this paper, we propose and study two iteration schemes (modified Halpern's type and HS-iteration schemes). Furthermore, it is proved that if two finite families of quasi-nonexpansive mappings satisfy jointly demiclosedness principle, then under appropriate conditions on the iteration parameters, the schemes so introduced strongly converge to the common fixed points of the mappings. Our main results improve and generalize the results in the literature and many other existing results currently in literature.

Keywords: Quasi-Nonexpansive Mappings, Modified Halpern's Type, HS-Iteration Schemes, Jointly Demiclosedness Principle, Uniformly Convex Banach Space.
2010 MSC: 47H10, 54H25

1. Introduction

Let E be a Banach space and D a nonempty closed convex subset of E. Throughout the paper, N, $\rightarrow$ and $\rightharpoonup$ will denote the set of positive integers,strong convergence and weak convergence respectively.

Let $Q : D \longrightarrow D$ be a mapping of D into itself and $S_i, T_i : D \longrightarrow CB(D)$, for $i = 1, 2, \cdots, N$, be finite families of two quasi-nonexpansive mappings of D into itself. The set of fixed points of Q will be denoted by $F(Q) = \{x \in D : Qx = x\}$. A point x is a common fixed point of S_i and T_i, for $i = 1, 2, \cdots, N$, if $x \in \mathcal{F} = \cap_{i=1}^{N} F(S) \cap \cap_{i=1}^{N} F(T)$.

In 1965, Browder [13] established the first fixed point theorem for single-valued nonexpansive self mappings. More precisely, he proved that if C is a bounded closed convex subset of a Hilbert space H and T is a nonexpansive mapping of C into itself, then T has a fixed point in C. Soon after that, both Browder [14] and Gohde [15] simultaneously proved that the same is true if E is a uniformly convex Banach space. Kirk [16] also proved the following theorem:

*Corresponding author
Email address: agwuimo@gmail.com (Imo Kalu Agwu)

Received 2020-03-11

Theorem 1.1. ($\star$) *Let E be a reflexive Banach space and let C be a nonempty bounded closed convex subset of E which has normal structure. Let T be a nonexpansive mapping of C into itself. Then $F(T)$ is nonempty.*

After Kirk's theorem, many fixed point theorems concerning single-valued mappings have been proved in a Hilbert space or a Banach space (see, e.g., [2, 4, 6, 8, 8, 9, 10] and the references contained in them). In particular, Baillon and Schoneberg [17] introduced the concept of asymptotic normal structure and generalized Kirk's fixed point theorem as follows:

Theorem 1.2. ($\star\star$) *Let E be a reflexive Banach space and let C be a nonempty bounded closed convex subset of E which has asymptotic normal structure. Let T be a nonexpansive mapping of C into itself. Then $F(T)$ is nonempty.*

ln the sequel, the following definitions will be needed:

Definition 1.3. Recall that a single-valued mapping $Q : D \longrightarrow D$ is called nonexpansive [1] if

$$\|Qx - Qy\| \leq \|x - y\|, \forall x, y \in D. \tag{1.1}$$

It is important to note that if $F(Q) \neq \emptyset$ in (1.1), then we obtain a class of mapping called quasi-nonexpansive mapping; that is, a mapping such that for all $x \in D$ and $q \in F(Q)$,

$$\|Qx - q\| \leq \|x - q\|.$$

A subset D of E is said to be a retract of E if there exists a continuous mapping $Q : E \longrightarrow D$ (called retraction) such that $Qx = x$ for all $x \in E$. A retraction Q from D onto E is called sunny if the following property holds: $Q(Qx + t(x + Qx)) = Qx$ for all $x \in D$ and $t \geq 0$ with $Qx + t(x + Qx) \in D$. A retract of a Hausdorff space must be a closed subset. It is well known that every closed convex subset of a uniformly convex Banach space E is a retract of E.

It has been established [3, Theorem 13.1] that in a smooth Banach space E, a retraction Q from D onto E is both sunny and nonexpansive if and only if

$$\langle x - Qx, J(y - Qx) \leq 0, \forall x \in D \quad \text{and} \quad y \in E. \tag{1.2}$$

Hence, there is at most one sunny nonexpansive retraction from D onto E. For example, if E is a nonempty, closed and convex subset of a Hilbert space H, then the nearest point projection P_E from E onto E is the unique sunny nonexpansive retraction of E onto E. This is not true in general for all Banach spaces, since outside Hilbert space, nearest point projections, although sunny are no longer nonexpansive. Conversely, sunny nonexpansive retraction do sometimes play a similar role in Banach space as the nearest point projections in Hilbert space. Thus, it becomes necessary to ask the following question:

Question 1.4. Which subsets of a Banach space does a sunny nonexpansive retraction exists? If it does exists, how can one finds it?

It has been proved [3, Theorem 13.2] that if C is a closed convex subset of a uniformly smooth Banach space and $T : C \longrightarrow C$ is nonexpansive, then the fixed point set is a sunny retraction of C. Bruck [2, Theorem 2] also proved that if C is a closed convex subset of a reflexive Banach space, every bounded, closed and convex subset of which has the fixed point property for nonexpansive mappings and $T : C \longrightarrow C$ is nonexpansive, then its fixed point set is a nonexpansive retraction of C.

Due its connection with so many contractive-type mappings, several authors have studied iterative methods for approximating fixed points of nonexpansive and quasi-nonexpansive mappings (see, e.g., [1, 3, 6, 8, 10]), etc. and the references contained therein).

Definition 1.5. Let E be a smooth, strictly convex and reflexive Banach space and J be the normalized duality mapping of E. Let C be a nonempty, closed and convex subset of E. A mapping $T : C \longrightarrow C$ is said to be nonspreading (see [19, 20]) if

$$\phi(Tx, Ty) + \phi(Ty, Tx) \leq \phi(Tx, y) + \phi(Tx, y), \forall x, y \in C,$$

where $\phi(x, y) = \|x\|^2 - 2\langle x, Jy\rangle + |y\|^2, \forall x, y \in E$. Note that if E is a real Hilbert space, then J is the identity mapping and $\phi(x, y) = \|x\|^2 - 2\langle x, Jy\rangle + |y\|^2 = \|x - y\|^2$. They mapping $T : C \longrightarrow C$ called nonspreading with respect to Hilbert if the inequality below holds

$$2\|Tx - Ty\|^2 \leq \|Tx - y\|^2 + \|x - Ty\|^2, \forall x, y \in C.$$

ln 1953, Mann [22] introduced the following iteration scheme, which is referred to as Mann iteration method:

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)Tx_n, \tag{1.3}$$

where an initial guess $x_0 \in C$ is arbitrary and $\{\alpha_n\}_{n\in N} \subseteq [0, 1]$ such that $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$. The Mann iteration scheme has been extensively investigated (see, e.g., [23]). ln an infinitely dimensional Hilbert spaces, the Mann iteration sequence can only guarantee weak convergence (see [24]). To achieve strong convergence, different authors have modified the Mann iteration method (see [18]) in many ways.

In 1967, Halpern [18] studied the following iteration scheme :

$$x_{n+1} = \alpha_n u + (1 - \alpha_n)Tx_n, \tag{1.4}$$

where $\{\alpha_n\}_{n\in N}$ is a real sequence in $[0, 1]$ satisfying some appropriate conditions. He proved strong convergence of $\{x_n\}_{n\in N}$ to the fixed point of T, where $\alpha_n = n^a$, for $a \in (0, 1)$, in the setting of Hilbert space. Recently, many researchers have used (1.4) in its original form, and the modified version, in approximating the fixed points of nonexpansive mappings and other classes of nonlinear mappings in different spaces (see [1, 25, 26]) and the references therein).

For approximating the fixed points of Lipschitz pseudocontractive mapping T, lshikawa [27] introduced the following algorithm, which is called lshikawa iteration algorithm:

$$\begin{cases} x_1 = x \in C; \\ x_{n+1} = \alpha_n x_n + (1 - \alpha_n)Ty_n; \\ y_n = (1 - \beta_n)x_n + \beta_n Tx_n, n \geq 0 \end{cases} \tag{1.5}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences in $(0, 1)$ satisfying the following conditions:

1. $0 \leq \alpha_n \leq \beta_n \leq 1$;
2. $\lim_{n\to\infty} \beta_n = 0$;
3. $\sum_{n=1}^{\infty} \alpha_n\beta_n = \infty$

He showed that the sequence defined by (1.5) converges strongly to a fixed point of the mapping T provided C is a compact convex subset of a Hilbert space H.

Recently, Agarwal. O'Regan and Sahu [28] introduced the S-iteration algorithm in Banach space as follows:

$$\begin{cases} x_1 = \in K; \\ x_{n+1} = (1 - \alpha_n)Tx_n + \alpha_n Ty_n; \\ y_n = (1 - \beta_n)x_n + \beta_n Tx_n, n \geq 1, \end{cases} \tag{1.6}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences in $(0, 1)$. They showed that the algorithm is independent of (1.3) and (1.5) and converges faster than both (1.3) and (1.5) .

Most recently, Naraghirad [1] introduced (1.7) as a generalization of (1.4) as follows:

$$\begin{cases} x_1 = x \in D; \\ x_{n+1} = \alpha_n u + (1-\alpha_n)y_n; \\ y_n = (1-\beta_n)S_n x_n + \beta_n T x_n, \end{cases} \tag{1.7}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences in $(0, 1)$. They proved strong convergence of (1.7) to $Q_{\mathcal{F}}u$, where $Q_{\mathcal{F}}$ is the sunny nonexpansive retraction from E onto $\mathcal{F}$.

The demiclosedness principle, which was first studied by Opial [6], is one of the indispensable tools in proving weak and strong convergence theorems for both single-valued and multivalued nonlinear mappings. Notably, the theory of fixed points with the associated mappings satisfying demiclosedness principle due to Opial [6] has been deeply investigated for the past forty (40) years or so, and much more intensively recently (see, e.g, [1] and the references therein). Although some interesting results have been obtained, it is worth mentioning that, in some cases, the mapping T of the class of nonexpansive mappings defined in the setting of a Hilbert space H does not necessarily satisfies the demiclosedness principle due to Opial [6] (see [1], Example 2.1 for details). Consequently, it is natural to ask:

Question 1.6. Is there any way one can obtain strong convergence theorems of Halpern's type for such mappings that fail to satisfy the original demiclosedness principle due to Opial in the setting of Banach spaces?

Naraghrad [1] gave an affirmative answer to the above question using the idea of jointly demiclosedness principle (Recall that if C is a nonempty subset of of a Banach space E, then a pair $S, T : C \longrightarrow C$ satisfies jointly demiclosedness principle if $x_n \subset C$ converges weakly to a point $z \in C$ and $lim_{n\to\infty}\|Sx_n - Tx_n\| = 0$, then $Sz = z$ and $Tz = z$; that is, $S - T$ is jointly demiclosed at zero) which they introduced. More precisely, they prove the following theorem:

Theorem 1.7. *(EN) Let E be a Banach space and C a nonempty, closed and convex subset of E and $v \in C$. Let $S, T : C \longrightarrow C$ be two quasi-nonexpansive self mappings such that $\mathcal{F} = F(S) \cap F(T) \neq \emptyset$ is closed and convex. Let S, T satisfies jointly demiclosedness principle on C and $\{x_n\}_{n\geq 1}$ be the sequence defined by*

$$\begin{cases} x_1 = x \in D; \\ x_{n+1} = \alpha_n u + (1-\alpha_n)y_n; \\ y_n = (1-\beta_n)Sx_n + \beta_n T x_n, \forall n \in N, \end{cases} \tag{1.8}$$

where $\{\alpha_n\}_{n\in N}$ and $\{\beta_n\}_{n\in N}$ are real sequences in $(0, 1)$. If the following conditions hold:

1. $limn_{n\to\infty}\alpha_n = 0$;
2. $\sum_{n=1}^{\infty}\alpha_n = \infty$;
3. $0 < \liminf_{n\to\infty}\beta_n(1-\beta_n) \leq \limsup_{n\to\infty}\beta_n(1-\beta_n) < 1$.

Then, the sequence defined by (1.8) converges strongly to $Q_{\mathcal{F}}u$, where $Q_{\mathcal{F}}$ is the sunny nonexpansive retraction from E onto $\mathcal{F}$.

Remark 1.8. It is remarked in [1] that if $S = I$, where I is the identity mapping on E, then $I - T$ is demiclosed at zero. Again, if S and T satisfy the demiclosedness principle due to Opial [6], then (S, T) satisfies the jointly demiclosedness . Regrettably, the converse is not in general true as could be seen in [1, Example 2.1].

Example 1.9. (see [1]) Let $E = \ell^2(N)$, where

$$\ell^2(N) = \left\{\sigma = (\sigma_1, \sigma_2, \cdots, \sigma_n \cdots) : \sum_{n=1}^{\infty}\|\sigma_n\|^2 < \infty\right\}$$

$$\|\sigma\| = \left(\sum_{n=1}^{\infty} \|\sigma_n\|^2\right)^{\frac{1}{2}}, \forall \sigma \in \ell^2(N),$$

$$(\sigma, \eta) = \sum_{n=1}^{\infty} \sigma_n \eta_n, \forall \sigma = (\sigma_1, \sigma_2, \cdots, \sigma_n \cdots), \eta = (\eta_1, \eta_2, \cdots, \eta_n \cdots) \in \ell^2(N).$$

Let $\{y_n\}_{n \in N \cup \{0\}} \subset E$ be a sequence defined by

$$\begin{array}{lcl} y_0 & = & (1, 0, 0, 0, 0 \cdots) \\ y_1 & = & (1, 1, 0, 0, 0, 0, \cdots) \\ y_2 & = & (1, 0, 1, 0, 0, 0, \cdots) \\ y_2 & = & (1, 0, 0, 1, 0, 0, 0, \cdots) \\ & \cdots\cdots\cdots & \\ y_n & = & (\sigma_{n,1}, \sigma_{n,2} \cdots, \sigma_{n,k}, \cdots) \\ & \cdots\cdots\cdots & \end{array}$$

where

$$\sigma_{n,k} = \begin{cases} 1, & \text{if} \quad k = 1, n+1 \\ 0, & \text{if} \quad k \neq 1, k \neq n+1 \end{cases}$$

Define two mappings $S, T : E \longrightarrow E$ by

$$S(y) = \begin{cases} \dfrac{-n}{n+1} y, & \text{if} \quad y = y_0 \\ -y, & \text{if} \quad y \neq y_0, \forall n \geq 0 \end{cases}$$

and

$$T(y) = \begin{cases} \dfrac{n}{n+1} y, & \text{if} \quad y = y_0 \\ y, & \text{if} \quad y \neq y_0 \end{cases}$$

Then, T does not satisfy the original demiclosedness principle but (S, T) does satisfies the jointly demiclosedness principle (see, e.g., [1] for details).

Moltivated by these facts, we introduce and study the following iterative algorithms for finite families of nonlinear mappings :
Let D be a nonempty subset of a real Banach space E and $S_i, T_i : D \longrightarrow D$, for $i = 1, 2, \cdots, N$, be finite family of two nonlinear mappings with $\mathcal{F}(T) \neq \emptyset$. Compute the sequence $\{x_n\}_{n \in N}$ by the iterative schemes

$$\begin{cases} x_1 = x \in D; \\ x_{n+1} = \sum_{i=1}^N \alpha_{n,i} u + (1 - \sum_{i=1}^N \alpha_{n,i}) T_i y_n; \\ y_n = (1 - \sum_{i=1}^N \beta_{n,i}) S_i x_n + \sum_{i=1}^N \beta_n T_i x_{n,i}, \forall n \in N \& \end{cases} \tag{1.9}$$

and

$$\begin{cases} x_1 = x \in D; \\ x_{n+1} = \sum_{i=1}^N \alpha_{n,i} u + (1 - \sum_{i=1}^N \alpha_{n,i}) y_n; \\ y_n = (1 - \sum_{i=1}^N \beta_{n,i}) S_i x_n + \sum_{i=1}^N \beta_n T_i x_{n,i}, \forall n \in N \& \end{cases} \tag{1.10}$$

where $\{\alpha_n\}_{n \in N}$ and $\{\beta_n\}_{n \in N}$ are real sequences in $(0, 1)$ with $\sum_{i=0}^N \alpha_{n,i} = 1 = \sum_{i=0}^N \beta_{n,i}$. It is important to note that if $i = 1$, then (1.9) is independent of (1.7). We also note that (1.9) and (1.10) are independent. The purpose of this paper is to establish strong convergence theorems of the iterative algorithm (1.9) and (1.10) for finite families of two quasi-nonexpansive mappings in uniformly convex Banach space.

2. Preliminary

For the sake of convenience, we restate the following concepts and results:
Let E be a Banach space with its dimension greater than or equal to 2. The modulus of convexity of E is a function $\delta_E(\varepsilon) : (0, 2] \longrightarrow (0, 2]$ defined by

$$\delta_E(\varepsilon) = \inf\{1 - \|\frac{1}{2}(x + y)\| : \|x\| = 1, \|y\| = 1, \varepsilon = \|x - y\|\}.$$

A Banach space E is uniformly convex if and if $\delta_E(\varepsilon) > 0,$ for all $\varepsilon \in (0, 2]$.
We recall the following:

Definition 2.1. A multivalued mapping $T : D \longrightarrow CB(D)$ is said to satisfy condition E_μ, where $\mu \geq 0$, if for each $x, y \in D$,

$$d(x, Ty) \leq \mu d(x, Tx) + \|x - y\|.$$

We say that T satisfy condition (E) whenever T satisfies (E_μ) for some $\mu \geq 0$.

Definition 2.2. The space E has Opial condition [6] if for any sequence $\{x_n\}$ in E, x_n converges to x weakly,it follows that $\limsup_{n\to\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - y\|$ for all $y \in E$ with $x \neq y$.

Examples of Banach spaces satisfying Opial conditions are Hilbert spaces and all spaces $l^p(1 < p < \infty)$. On the other hand, $L^p[0, \pi]$ with $1 < p \neq 2$ fails to satify Opial condition.

Lemma 2.3. *[1] Let $\{s_n\}_{n\in N}$ be a sequence of nonnegative real numbers satisfyingthe inequality:*

$$s_{n+1} \leq (1 - \gamma_n)s_n + \gamma_n\delta_n, \forall n \geq 1,$$

where $\{\gamma_n\}_{n\in N}$ and $\{\delta_n\}_{n\in N}$ satisfy the following conditions:

(i) $\{\gamma_n\}_{n\in N} \subset [0, 1]$ and $\sum_{n=1}^{\infty} \gamma_n = \infty$, or equivalently, $\prod_{n=1}^{\infty}(1 - \gamma_n) = 0$;
(ii) $\limsup_{n\to\infty} \delta_n \leq 0$ or $(ii)'$ $\sum_{n1}^{\infty} \gamma_n\delta_n < \infty$.

Then,$\lim_{n\to\infty} s_n = 0$.

Lemma 2.4. *(see [1]) Let X be a uniformly convex Banach space and let $r > 0$. Then, there exists a strictly increasing continous function $g : [0, \infty) \longrightarrow [0, \infty)$ with $g(0) = 0$ such that $\|\lambda x + (1 - \lambda)y\|^2 \leq \lambda\|x\|^2 + (1 - \lambda)\|y\|^2 - \lambda(1 - \lambda)g(\|x - y\|)$ for all $x, y \in B_r = \{z \in X : \|z\| \leq r, \lambda \in [0, 1]$.*

Lemma 2.5. *(see [1]) Let E be a real Banach space and $J : E \longrightarrow 2^{E^\star}$ be the normalized duality mapping of E. Then,*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, J(x + y)\rangle, \forall x, y \in E. \tag{2.1}$$

Lemma 2.6. *(see [1]) Let $\{a_n\}_{n\in N}$ be a sequence of real numbers such that there exists a subsequence $\{n_i\}_{n\in N}$ of $\{n\}_{n\in N}$ such that $a_{n_i} < a_{n_i+1}$ for all $i \in N$. Then, there exists a subsequence $\{m_k\}_{k\in N}$ such that $m_k \to \infty$ and the following properties are satisfied by all (sufficiently large) $k \in N$:*

$$a_{m_k} \leq a_{m_k+1} \quad and \quad a_k \leq a_{m_k+1} \tag{2.2}$$

In fact, $m_k = maxj \leq k : a_j < a_j + 1$.

Lemma 2.7. *(see [1]) Let C and D be nonempty subsets of a Banach space E with $D \subset C$ and let $Q_D : C \longrightarrow D$ be retraction from C into D. then, Q_D is sunny and nonexpansive if and only if*

$$\langle z - Q_D(z), J(y - Q_D(z))\rangle \leq 0, \forall z \in C \quad and \quad \forall y \in D, \tag{2.3}$$

where J is the normalized duality mapping of E.

Lemma 2.8. *(see [6]) Let X be the Banach space which satisfies the Opial property and $\{x_n\}$ be a sequence in X. Let $u, v \in X$ be such that $\|x_n - u\|$ and $\|x_n - v\|$ exists. If $\{x_{n_i}\}$ and $\{x_{n_j}\}$ are subsequences of $\{x_n\}$ which converges to u and v respectively, then $u = v$.*

3. Main Results

Theorem 3.1. *Let E be a Banach space and D a nonempty, closed and convex subset of E and $v \in D$. Let $\{S_i\}_{i=1}^N, \{T_i\}_{i=1}^N : D \longrightarrow D$ be finite families of two quasi-nonexpansive self mappings such that $\mathcal{F} = \cap_{i=1}^N F(S_i) \cap \cap_{i=1}^N F(T) \neq \emptyset$ is closed and convex.. Let (S_i, T_i), for $i = 1, 2, \cdots, N$, satifies jointly demiclosedness principle on D and $\{x_n\}$ be the sequence defined by*

$$x_1 = x \in D; \tag{3.1}$$

$$x_{n+1} = \sum_{i=1}^N \alpha_{n,i} u + (1 - \sum_{i=1}^N \alpha_{n,i}) T_i y_n; \tag{3.2}$$

$$y_n = (1 - \sum_{i=1}^N \beta_{n,i}) S_i x_n + \sum_{i=1}^N \beta_{n,i} T_i x_n, \forall n \in N \tag{3.3}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences in $(0,1)$ such that $\sum_{i=0}^N \alpha_{n,i} = 1 = \sum_{i=0}^N \beta_{n,i}$. If the following conditions hold:

i. $\lim_{n\to\infty} \alpha_n = 0$

ii. $\sum_{n=1}^\infty \alpha_n = \infty$;

iii. $0 < \liminf_{n\to\infty} \beta_n(1-\beta_n) \leq \limsup_{n\to\infty} \beta_n(1-\beta_n) < 1$.

Then, the sequence defined in (3.1) converges strongly to $Q_{\mathcal{F}} v$, where $Q_{\mathcal{F}}$ is the sunny nonexpansive retaction from E onto $\mathcal{F}$.

Proof. Since S and T are quasi-nonexpansive mappings,it follows that $\mathcal{F}$ is closed and convex. Set

$$z = Q_{\mathcal{F}} v.$$

Let $q \in \mathcal{F}$ be fixed. From Lemma 2.2, it follows that there exists a strictly increasing function $\phi : [0,\infty) \longrightarrow [0,\infty)$ with $\phi(0) = 0$ such that the following estimates remain valid:

$$\begin{aligned}
\|y_n - q\|^2 &= \|(1 - \sum_{i=1}^N \beta_{n,i})(S_i x_n - q) + \sum_{i=1}^N \beta_{n,i}(T_i x_n - q)\|^2 \\
&\leq (1 - \sum_{i=1}^N \beta_{n,i})\|S_i x_n - q\|^2 + \sum_{i=1}^N \beta_{n,i}\|T_i x_n - q\|^2 - \sum_{i=1}^N \beta_{n,i}(1-\beta_{n,i})\phi(\|S_i x_n - T_i x_n\|) \\
&\leq (1 - \sum_{i=1}^N \beta_{n,i})\|x_n - q\|^2 + \sum_{i=1}^N \beta_{n,i}\|x_n - q\|^2 - \sum_{i=1}^N \beta_{n,i}(1-\beta_{n,i})\phi(\|S_i x_n - T_i x_n\|) \\
&= \|x_n - q\|^2 - \sum_{i=1}^N \beta_{n,i}(1-\beta_{n,i})\phi(\|S_i x_n - T_i x_n\|) \qquad (3.4) \\
&\leq \|x_n - q\|^2 \qquad (3.5)
\end{aligned}$$

Again, from (3.1) and (3.4), we obtain

$$\begin{aligned}
\|x_{n+1}-q\|^2 &= \|\sum_{i=1}^{N}\alpha_{n,i}(u-q)+(1-\sum_{i=1}^{N}\alpha_{n,i})(T_iy_n-q)\|^2\\
&\le \sum_{i=1}^{N}\alpha_{n,i}\|u-q\|^2+(1-\sum_{i=1}^{N}\alpha_{n,i})\|T_iy_n-q\|^2\\
&\le \sum_{i=1}^{N}\alpha_{n,i}\|u-q\|^2+(1-\sum_{i=1}^{N}\alpha_{n,i})\|y_n-q\|^2\\
&\le \sum_{i=1}^{N}\alpha_{n,i}\|u-q\|^2+(1-\sum_{i=1}^{N}\alpha_{n,i})[\|x_n-q\|^2\\
&\quad -\sum_{i=1}^{N}\beta_{n,i}(1-\beta_{n,i})\phi(\|S_ix_n-T_ix_n\|)] \qquad (3.6)\\
&\le \sum_{i=1}^{N}\alpha_{n,i}\|u-q\|^2+(1-\sum_{i=1}^{N}\alpha_{n,i})\|x_n-q\|^2\\
&\le max\{\|u-q\|^2,\|x_n-q\|^2\}
\end{aligned}$$

By induction, we obtain from the last inequality that

$$\|x_{n+1}-q\|^2 \le max\{\|u-q\|^2,\|x_1-q\|^2\}, \forall n\in N.$$

It is easy to see that the sequence $\{\|x_n-q\|\}_{n\in N}$ is bounded and so is $\{x_n\}_{n\in N}$. Consquently, using (3.1), the following sequences $\{y_n\}_{n\in N}$, $\{Ty_n\}_{n\in N}$, $\{Tx_n\}_{n\in N}$, $\{Sx_n\}_{n\in N}$ are bounded. Let

$$M=\sup\left\{\|u-q\|^2-\|x_n-q\|^2+\beta_{n,i}(1-\beta_{n,i})\phi(\|S_ix_n-T_ix_n\|):n\in N\right\}\ge 0 \qquad (3.7)$$

Then, it follows from (3.6) and (3.7) that

$$\sum_{i=1}^{N}\beta_{n,i}(1-\beta_{n,i})\phi(\|S_ix_n-T_ix_n\|)\le\|x_n-q\|^2-\|x_{n+1}-q\|^2+\sum_{i=1}^{N}\alpha_{n,i}M \qquad (3.8)$$

Next, we show that

$$\|x_{n+1}-q\|^2\le(1-\alpha_{n,i})\|x_n-q\|^2+2\sum_{i=1}^{N}\alpha_{n,i}\langle u-q,J(x_{n+1}-q)\rangle, i\in N. \qquad (3.9)$$

From Lemma 2.3, (3.1) and (3.5), we get

$$\begin{aligned}
\|x_{n+1}-q\|^2 &= \|\sum_{i=1}^{N}\alpha_{n,i}(u-q)+(1-\sum_{i=1}^{N}\alpha_{n,i})(T_iy_n-q)\|^2\\
&\leq (1-\sum_{i=1}^{N}\alpha_{n,i})^2\|T_iy_n-q\|^2+2\sum_{i=1}^{N}\alpha_{n,i}\langle u-q,J(x_{n+1}-q)\rangle\\
&\leq (1-\sum_{i=1}^{N}\alpha_{n,i})^2\|y_n-q\|^2+2\sum_{i=1}^{N}\alpha_{n,i}\langle u-q,J(x_{n+1}-q)\rangle\\
&\leq (1-\sum_{i=1}^{N}\alpha_{n,i})^2\|x_n-q\|^2+2\sum_{i=1}^{N}\alpha_{n,i}\langle u-q,J(x_{n+1}-q)\rangle\\
&\leq (1-\alpha_{n,i})^2\|x_n-q\|^2+2\sum_{i=1}^{N}\alpha_{n,i}\langle u-q,J(x_{n+1}-q)\rangle\\
&\leq (1-\alpha_{n,i})\|x_n-q\|^2+2\sum_{i=1}^{N}\alpha_{n,i}\langle u-q,J(x_{n+1}-q)\rangle, i\in N. \qquad (3.10)
\end{aligned}$$

Now, we show that $x_n \to q$ as $n\to\infty$. To do this, we consider two possible cases. Case A. Suppose $\{\|x_n-q\|\}_{n\in N}$ is a monotonically decreasing sequence, then there exists $n_0\in N$ such that $\|x_n-q\|^2-\|x_{n+1}-q\|^2\to 0$ as $n\to\infty$. Thus, from (3.8) and conditions [(i) and (iii)], we have

$$\lim_{n\to\infty}\sum_{i=1}^{N}\beta_{n,i}(1-\beta_{n,i})\phi(\|S_ix_n-T_ix_n\|)=0 \qquad (3.11)$$

and from the properties of ϕ, we get

$$\lim_{n\to\infty}\|S_ix_n-T_ix_n\|=0, \qquad (3.12)$$

for each $i\in N$. Since $\{x_n\}_{n\in N}$ is bounded, there exists a subsequence $\{x_{n_k}\}_{k\in N}$ of $\{x_n\}_{n\in N}$ such that $x_{n_k}\rightharpoonup p\in D$ as $k\to\infty$. Furthermore, from (3.12), we have

$$\lim_{k\to\infty}\|S_ix_{n_k}-T_ix_{n_k}\|=0, i\in N. \qquad (3.13)$$

Again, since for each $i\in N$ (S_i,T_i) is assumed to satisfy jointly demiclosedness principle, it follows from (3.12) that $p\in\mathcal{F}$. Now, assume that there exists another subsequence $\{x_{n_i}\}_{k\in N}$ of $\{x_n\}_{n\in N}$ such that $x_{n_i}\rightharpoonup q\in D$ as $i\to\infty$ with $p\neq q$, where $q\in\mathcal{F}$. Then, (3.10) implies

$$\|p-q\|^2\leq(1-\alpha_n)\|p-q\|^2+2\sum_{i=1}^{N}\alpha_{n,i}\langle u-q,J(p-q)\rangle \qquad (3.14)$$

Similarly,

$$\|q-p\|^2\leq(1-\alpha_n)\|q-p\|^2+2\sum_{i=1}^{N}\alpha_{n,i}\langle u-q,J(q-p)\rangle \qquad (3.15)$$

Adding (3.14) and (3.15), we obtain

$$\|p-q\|\leq\|q-p\|,$$

which is a contradiction. Hence, $p = q$.

Using the above information and Lemma 2.5, we get

$$\begin{aligned}\limsup_{n\to\infty}\langle u-q, J(x_{n+1}-q)\rangle &= \lim_{k\to\infty}\langle u-q, J(x_{n_k+1}-q)\rangle \\ &= \langle u-q, J(p-q)\rangle \\ &\leq 0 \end{aligned} \tag{3.16}$$

Hence, putting $\delta_n = \langle u-q, J(x_{n+1}-q)\rangle, \gamma_n = \sum_{i=1}^{N}\alpha_{n,i}, s_n = \|x_n - q\|^2$, then it follows from (3.10), (3.16) and Lemma 2.1 that $\lim_{n\to}\|x_n - q\| = 0$, which is the desired result.

Case B. If $\{\|x_n - q\|\}_{n\in N}$ is not monotonically decreasing sequence, then there exists a nondecreasing sequence $\{n_i\}_{i\in N}$ of $\{n\}_{n\in N}$ such that

$$\|x_{n_i} - q\| < \|x_{n_i+1} - q\|. \tag{3.17}$$

Thus, by Lemma 2.4, there exists a nondecreasing sequence $\{x_{n_j}\}_{j\in N}$ such that

$$m_j \to \infty, \|x_{n_j} - q\| \leq \|x_{m_j+1} - q\| \quad \text{and} \quad \|x_j - q\| \leq \|x_{m_j+1} - q\| \tag{3.18}$$

By substituting m_j for n in (3.8), using the first part of the last inequality, we get

$$\begin{aligned}\sum_{i=1}^{N}\beta_{m_j,i}(1-\beta_{m_j,i})\phi(\|Sx_{m_j} - Tx_{m_j}\|) &\leq \|x_{m_j} - q\|^2 - \|x_{m_j+1} - q\|^2 + \sum_{i=1}^{N}\alpha_{m_j,i}M \\ &\leq \sum_{i=1}^{N}\alpha_{m_j,i}M, \forall j \in N. \end{aligned} \tag{3.19}$$

Thus, by conditions [(i) and (iii)]and the properties of ϕ, we get

$$\lim_{j\to\infty}\|S_i x_{m_j} - T_i x_{m_j}\| = 0, i \in N. \tag{3.20}$$

Using similar argument as in Case A, it is easy to show that

$$\begin{aligned}\limsup_{n\to\infty}\langle u-q, J(x_{n+1}-q)\rangle &= \lim_{j\to\infty}\langle u-q, J(x_{m_j+1}-q)\rangle \\ &\leq 0 \end{aligned} \tag{3.21}$$

Again, substituting m_j for n in (3.10), we have

$$\|x_{m_j+1} - q\|^2 \leq (1-\alpha_{m_j,i})\|x_{m_j} - q\|^2 + 2\sum_{i=1}^{N}\alpha_{m_j,i}\langle u-q, J(x_{m_j+1}-q)\rangle. \tag{3.22}$$

Using the last inequality with $\alpha_{m_j,i} \in (0,1)$, we obtain

$$0 \leq \|x_{m_j} - q\|^2 - |x_{m_j} - q\|^2 \leq 2\sum_{i=1}^{N}\alpha_{m_j,i}[\langle u-q, J(x_{m_j+1}-q)\rangle - \|x_{m_j} - q\|].$$

Hence, from (3.21), we have

$$\lim_{j\to\infty}\|x_{m_j} - q\| = 0. \tag{3.23}$$

Also, from (3.22) and (3.23), we have

$$\lim_{j\to\infty}\|x_{m_j+1} - q\| = 0. \tag{3.24}$$

Finally, from (3.24) and the second part of the inequalities in (3.18), for all $j \in N$, we have $x_j \to q$ as $j \to \infty$.Thus, we have $x_n \to q$ as $n \to \infty$ as desired. This completes the proof. This completes the proof. □

Theorem 3.2. *Let E be a Banach space and D a nonempty, closed and convex subset of E and $v \in D$. Let $\{S_i\}_{i=1}^N, \{T_i\}_{i=1}^N : D \longrightarrow D$ be finite families of two quasi-nonexpansive self mappings such that $\mathcal{F} = \cap_{i=1}^N F(S_i) \cap \cap_{i=1}^N F(T) \neq \emptyset$ is closed and convex.. Let (S_i, T_i), for $i = 1, 2, \cdots, N$, satifies jointly demiclosedness principle on D and $\{x_n\}$ be the sequence defined by*

$$x_1 = x \in D; \tag{3.25}$$

$$x_{n+1} = \sum_{i=1}^N \alpha_{n,i} u + (1 - \sum_{i=1}^N \alpha_{n,i}) y_n; \tag{3.26}$$

$$y_n = (1 - \sum_{i=1}^N \beta_{n,i}) S_i x_n + \sum_{i=1}^N \beta_{n,i} T_i x_n, \forall n \in N \tag{3.27}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences in $(0,1)$ such that $\sum_{i=0}^N \alpha_{n,i} = 1 = \sum_{i=0}^N \beta_{n,i}$. If the following conditions hold:

i. $\lim_{n\to\infty} \alpha_n = 0$

ii. $\sum_{n=1}^\infty \alpha_n = \infty$;

iii. $0 < \liminf_{n\to\infty} \beta_n(1 - \beta_n) \leq \limsup_{n\to\infty} \beta_n(1 - \beta_n) < 1$.

Then, the sequence defined in (3.1) converges strongly to $Q_{\mathcal{F}} v$, where $Q_{\mathcal{F}}$ is the sunny nonexpansive retaction from E onto $\mathcal{F}$.

Proof. Since S and T are quasi-nonexpansive mappings,it follows that $\mathcal{F}$ is closed and convex. Set

$$z = Q_{\mathcal{F}} v.$$

Let $q \in \mathcal{F}$ be fixed. From Lemma 2.2, it follows that there exists a strictly increasing function $\phi : [0, \infty) \longrightarrow [0, \infty)$ with $\phi(0) = 0$ such that the following estimates remain valid:

$$\begin{aligned}
\|y_n - q\|^2 &= \|(1 - \sum_{i=1}^N \beta_{n,i})(S_i x_n - q) + \sum_{i=1}^N \beta_{n,i}(T_i x_n - q)\|^2 \\
&\leq (1 - \sum_{i=1}^N \beta_{n,i})\|S_i x_n - q\|^2 + \sum_{i=1}^N \beta_{n,i}\|T_i x_n - q\|^2 - \sum_{i=1}^N \beta_{n,i}(1 - \beta_{n,i})\phi(\|S_i x_n - T_i x_n\|) \\
&\leq (1 - \sum_{i=1}^N \beta_{n,i})\|x_n - q\|^2 + \sum_{i=1}^N \beta_{n,i}\|x_n - q\|^2 - \sum_{i=1}^N \beta_{n,i}(1 - \beta_{n,i})\phi(\|S_i x_n - T_i x_n\|) \\
&= \|x_n - q\|^2 - \sum_{i=1}^N \beta_{n,i}(1 - \beta_{n,i})\phi(\|S_i x_n - T_i x_n\|) \qquad (3.28) \\
&\leq \|x_n - q\|^2 \qquad (3.29)
\end{aligned}$$

Again, from (3.27) and (3.28), we obtain

$$\begin{aligned}
\|x_{n+1}-q\|^2 &= \|\sum_{i=1}^{N}\alpha_{n,i}(u-q)+(1-\sum_{i=1}^{N}\alpha_{n,i})(y_n-q)\|^2 \\
&\leq \sum_{i=1}^{N}\alpha_{n,i}\|u-q\|^2+(1-\sum_{i=1}^{N}\alpha_{n,i})\|y_n-q\|^2 \\
&\leq \sum_{i=1}^{N}\alpha_{n,i}\|u-q\|^2+(1-\sum_{i=1}^{N}\alpha_{n,i})[\|x_n-q\|^2 \\
&\quad -\sum_{i=1}^{N}\beta_{n,i}(1-\beta_{n,i})\phi(\|S_ix_n-T_ix_n\|)] \qquad (3.30)\\
&\leq \sum_{i=1}^{N}\alpha_{n,i}\|u-q\|^2+(1-\sum_{i=1}^{N}\alpha_{n,i})\|x_n-q\|^2 \\
&\leq max\{\|u-q\|^2,\|x_n-q\|^2\}
\end{aligned}$$

By induction, we obtain from the last inequality that

$$\|x_{n+1}-q\|^2\leq max\{\|u-q\|^2,\|x_1-q\|^2\},\forall n\in N.$$

Clearly, the sequence $\{\|x_n-q\|\}_{n\in N}$ is bounded and so is $\{x_n\}_{n\in N}$. Consquently, using (3.27), the following sequences $\{y_n\}_{n\in N}$, $\{Ty_n\}_{n\in N}$, $\{Tx_n\}_{n\in N}$, $\{Sx_n\}_{n\in N}$ are bounded. Let

$$M=\sup\left\{\|u-q\|^2-\|x_n-q\|^2+\beta_{n,i}(1-\beta_{n,i})\phi(\|S_ix_n-T_ix_n\|):n\in N\right\}\geq 0 \qquad (3.31)$$

Then, it follows from (3.30) and (3.7) that

$$\sum_{i=1}^{N}\beta_{n,i}(1-\beta_{n,i})\phi(\|S_ix_n-T_ix_n\|)\leq \|x_n-q\|^2-\|x_{n+1}-q\|^2+\sum_{i=1}^{N}\alpha_{n,i}M \qquad (3.32)$$

Next, we show that

$$\|x_{n+1}-q\|^2\leq(1-\alpha_{n,i})\|x_n-q\|^2+2\sum_{i=1}^{N}\alpha_{n,i}\langle u-q,J(x_{n+1}-q)\rangle, i\in N. \qquad (3.33)$$

From Lemma 2.3, (3.27) and (3.29), we get

$$\begin{aligned}
\|x_{n+1}-q\|^2 &= \|\sum_{i=1}^{N}\alpha_{n,i}(u-q)+(1-\sum_{i=1}^{N}\alpha_{n,i})(y_n-q)\|^2 \\
&\leq (1-\sum_{i=1}^{N}\alpha_{n,i})^2\|y_n-q\|^2+2\sum_{i=1}^{N}\alpha_{n,i}\langle u-q,J(x_{n+1}-q)\rangle \\
&\leq (1-\sum_{i=1}^{N}\alpha_{n,i})^2\|x_n-q\|^2+2\sum_{i=1}^{N}\alpha_{n,i}\langle u-q,J(x_{n+1}-q)\rangle \\
&\leq (1-\alpha_{n,i})^2\|x_n-q\|^2+2\sum_{i=1}^{N}\alpha_{n,i}\langle u-q,J(x_{n+1}-q)\rangle \\
&\leq (1-\alpha_{n,i})\|x_n-q\|^2+2\sum_{i=1}^{N}\alpha_{n,i}\langle u-q,J(x_{n+1}-q)\rangle, i\in N. \qquad (3.34)
\end{aligned}$$

Now, we show that $x_n \to q$ as $n \to \infty$. To do this, we consider two possible cases. Case A. Suppose $\{\|x_n - q\|\}_{n\in N}$ is a monotonically decreasing sequence, then there exists $n_0 \in N$ such that $\|x_n - q\|^2 - \|x_{n+1} - q\|^2 \to 0$ as $n \to \infty$. Thus, from (3.32) and conditions [(i) and (iii)], we have

$$\lim_{n\to\infty} \sum_{i=1}^{N} \beta_{n,i}(1-\beta_{n,i})\phi(\|S_i x_n - T_i x_n\|) = 0 \tag{3.35}$$

and from the properties of ϕ, we get

$$\lim_{n\to\infty} \|S_i x_n - T_i x_n\| = 0, \tag{3.36}$$

for each $i \in N$. Since $\{x_n\}_{n\in N}$ is bounded, there exists a subsequence $\{x_{n_k}\}_{k\in N}$ of $\{x_n\}_{n\in N}$ such that $x_{n_k} \rightharpoonup p \in D$ as $k \to \infty$. Furthermore, from (3.36), we have

$$\lim_{k\to\infty} \|S_i x_{n_k} - T_i x_{n_k}\| = 0, i \in N. \tag{3.37}$$

Again, since for each $i \in N$ (S_i, T_i) is assumed to satisfy jointly demiclosedness principle, it follows from (3.36) that $p \in \mathcal{F}$. Now, assume that there exists another subsequence $\{x_{n_i}\}_{k\in N}$ of $\{x_n\}_{n\in N}$ such that $x_{n_i} \rightharpoonup q \in D$ as $i \to \infty$ with $p \neq q$, where $q \in \mathcal{F}$. Then, (3.34) implies

$$\|p - q\|^2 \le (1-\alpha_n)\|p - q\|^2 + 2\sum_{i=1}^{N} \alpha_{n,i}\langle u - q, J(p - q)\rangle \tag{3.38}$$

Similarly,

$$\|q - p\|^2 \le (1-\alpha_n)\|q - p\|^2 + 2\sum_{i=1}^{N} \alpha_{n,i}\langle u - q, J(q - p)\rangle \tag{3.39}$$

Adding (3.38) and (3.39), we obtain

$$\|p - q\| \le \|q - p\|,$$

which is a contradiction. Hence, $p = q$.

Using the above information and Lemma 2.5, we get

$$\begin{aligned}\limsup_{n\to\infty}\langle u - q, J(x_{n+1} - q)\rangle &= \lim_{k\to\infty}\langle u - q, J(x_{n_k+1} - q)\rangle \\ &= \langle u - q, J(p - q)\rangle \\ &\le 0 \end{aligned} \tag{3.40}$$

Hence, putting $\delta_n = \langle u - q, J(x_{n+1} - q)\rangle, \gamma_n = \sum_{i=1}^{N} \alpha_{n,i}, s_n = \|x_n - q\|^2$, then it follows from (3.34), (3.40) and Lemma 2.1 that $\lim_{n\to} \|x_n - q\| = 0$, which is the desired result.

Case B. If $\{\|x_n - q\|\}_{n\in N}$ is not monotonically decreasing sequence, then there exists a nondecreasing sequence $\{n_i\}_{i\in N}$ of $\{n\}_{n\in N}$ such that

$$\|x_{n_i} - q\| < \|x_{n_i+1} - q\|. \tag{3.41}$$

Thus, by Lemma 2.4, there exists a nondecreasing sequence $\{x_{n_j}\}_{j\in N}$ such that

$$m_j \to \infty, \|x_{n_j} - q\| \le \|x_{m_j+1} - q\| \quad \text{and} \quad \|x_j - q\| \le \|x_{m_j+1} - q\| \tag{3.42}$$

By substituting m_j for n in (3.32), using the first part of the last inequality, we get

$$\begin{aligned}\sum_{i=1}^{N} \beta_{m_j,i}(1-\beta_{m_j,i})\phi(\|Sx_{m_j} - Tx_{m_j}\|) &\le \|x_{m_j} - q\|^2 - \|x_{m_j+1} - q\|^2 + \sum_{i=1}^{N} \alpha_{m_j,i}M \\ &\le \sum_{i=1}^{N} \alpha_{m_j,i}M, \forall j \in N. \end{aligned} \tag{3.43}$$

Thus, by conditions [(i) and (iii)]and the properties of ϕ, we get

$$\lim_{j\to\infty} \|S_i x_{m_j} - T_i x_{m_j}\| = 0, i \in N. \tag{3.44}$$

Using similar argument as in Case A, it is easy to show that

$$\begin{aligned} \limsup_{n\to\infty} \langle u-q, J(x_{n+1}-q)\rangle &= \lim_{j\to\infty} \langle u-q, J(x_{m_j+1}-q)\rangle \\ &\leq 0 \end{aligned} \tag{3.45}$$

Again, substituting m_j for n in (3.34), we have

$$\|x_{m_j+1} - q\|^2 \leq (1-\alpha_{m_j,i})\|x_{m_j} - q\|^2 + 2\sum_{i=1}^{N} \alpha_{m_j,i}\langle u-q, J(x_{m_j+1}-q)\rangle. \tag{3.46}$$

Using the last inequality with $\alpha_{m_j,i} \in (0,1)$, we obtain

$$0 \leq \|x_{m_j} - q\|^2 - |x_{m_j} - q\|^2 \leq 2\sum_{i=1}^{N} \alpha_{m_j,i}[\langle u-q, J(x_{m_j+1}-q)\rangle - \|x_{m_j} - q\|].$$

Hence, from (3.45), we have

$$\lim_{j\to\infty} \|x_{m_j} - q\| = 0. \tag{3.47}$$

Also, from (3.46) and (3.47), we have

$$\lim_{j\to\infty} \|x_{m_j+1} - q\| = 0. \tag{3.48}$$

Finally, from (3.48) and the second part of the inequalities in (3.42), for all $j \in N$, we have $x_j \to q$ as $j \to \infty$.Thus, we have $x_n \to q$ as $n \to \infty$ as desired. This completes the proof. This completes the proof. □

Corollary 3.3. *Let E be a Banach space and D a nonempty, closed and convex subset of E and $v \in D$. Let $S_i : D \longrightarrow D$, $i = 1, 2, \cdots, N$, be a finite family of nonspreading mapping and $T_i : D \longrightarrow D$, $i = 1, 2, \cdots, N$, be a finite family of nonexpansive mapping such that $\mathcal{F} = \cap_{i=1}^{N} F(S_i) \cap \cap_{i=1}^{N} F(T) \neq \emptyset$ is closed and convex. Let (S_i, T_i), $i = 1, 2, \cdots, N$, satisfies jointly demiclosedness principle on D and $\{x_n\}$ be the sequence defined by*

$$x_1 = x \in D; \tag{3.49}$$

$$x_{n+1} = \sum_{i=1}^{N} \alpha_{n,i} u + (1 - \sum_{i=1}^{N} \alpha_{n,i}) T_i y_n; \tag{3.50}$$

$$y_n = (1 - \sum_{i=1}^{N} \beta_{n,i}) S_i x_n + \sum_{i=1}^{N} \beta_n T_i x_n, \forall n \in N \tag{3.51}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences in $(0,1)$. If the conditions of Theorem 3.1 holds, then, the sequence defined in (3.52) converges strongly to $Q_{\mathcal{F}} v$, where $Q_{\mathcal{F}}$ is the sunny nonexpansive retaction from E onto $\mathcal{F}$.

Corollary 3.4. *Let E be a Banach space and D a nonempty, closed and convex subset of E and $v \in D$. Let $S : D \longrightarrow D$, $i = 1, 2, \cdots, N$, be a finite family of nonspreading mapping and $T_i : D \longrightarrow D$, $i = 1, 2, \cdots, N$, be a finite family of nonexpansive mapping such that $\mathcal{F} = \cap_{i=1}^{N} F(S_i) \cap \cap_{i=1}^{N} F(T) \neq \emptyset$ is closed and convex.*

Let (S_i, T_i), $i = 1, 2, \cdots, N$, satisfies jointly demiclosedness principle on D and $\{x_n\}$ be the sequence defined by

$$x_1 = x \in D; \tag{3.52}$$

$$x_{n+1} = \sum_{i=1}^{N} \alpha_{n,i} u + (1 - \sum_{i=1}^{N} \alpha_{n,i}) y_n; \tag{3.53}$$

$$y_n = (1 - \sum_{i=1}^{N} \beta_{n,i}) S_i x_n + \sum_{i=1}^{N} \beta_n T_i x_n, \forall n \in \mathbb{N} \tag{3.54}$$

*where $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences in $(0, 1)$. If the conditions of Theorem 3.1 holds, then, the sequence defined in (**??**) converges strongly to $Q_{\mathcal{F}} v$, where $Q_{\mathcal{F}}$ is the sunny nonexpansive retaction from E onto $\mathcal{F}$.*

In conclusion, we make the following remarks:

Remark 3.5.
1. We propose two iteration schemes and establish strong convergence results of the schemes to the common fixed points for finite family of two quasi-nonexpansive mappings. Our main results generalise the results in [1] from single mapping to finite family of mappings.
2. Corollary 3.3 and Corollary 3.4 provide a partial answers to the open Question 1.1 raised in [1] in a more general setting.

Acknowledgment

The author is grateful to Prof. Donatus lkechukwu lgbokwe (Department of Mathematics, Michael Okpara University of Agriculture, Umudike, Nigeria) for his mentorship, devotion and guidance in all aspect of functional analysis. For the reviewers who carefully read through the paper and made invaluable suggestions which improved the quality of this work, l say thank you so much.

Competing Interest

The author declares that there is no conflict of interest.

References

[1] E. Naraghirad, *Approximation of common fixed points of nonlinear mappings satisfying jointly demiclosedness principle in Banach spaces,* Mediterr. J. Math. 14(2017),162. 1.3, 1, 1, 1, 1, 1, 1.8, 1.9, 1.9, 2.3, 2.4, 2.5, 2.6, 2.7, 1, 2

[2] R. E. Bruck, *Properties of fixed point sets of nonexpansive mappings in Banach spaces*, J. Amer. Math. Soc., 179(1973), 251-262. 1, 1

[3] K. Goebel, S. Reich, *Uniform convexity, hyperbolic geometry and nonexpansive mappings*, Monograph and Textbooks in Pure and Applied Mathematics, Vol. 83, Marcel Dekker, New York, 1984. 1.3, 1

[4] A. Aleyner, S. Reich, *An explicit construction of sunny nonexpansive retractions in Banach spaces*, Fixed Point Theory and Appl. 3(2005), 295-365. 1

[5] S. H. Khan, W. Takahashi, *Approximating common fixed points of two asymptotically nonexpansive mappings*, Sci., Math. Japon, 53(1)(2001), 143-148.

[6] Z. Opial, *Weak convergence of the sequence of successive approximation for nonexpansive mappings* , Bull Amer: Math Soc., 73(1967), 591-597. 1, 1, 1

[7] B. E. Rhoades, *Fixed point iteration for certain nonlinear mappings*, J. Math. Anal. Appl., 183(1994), 118-120.

[8] W. Takahashi, G. E. Kim, *Approximating fixed points of nonexpansive mappings in Banach spaces*, Math. Japonoica, 48(1)(1998), 1-9. 1, 1

[9] K. K. Tan, H. K. Xu, *Approximating fixed point of nonexpansive mappings by the lshikawa iteration process*, J. Math. Anal. Appl., 178(1993),301-308. 1

[10] D. I. Igbokwe, S. J. Uko, *Weak and strong convergence theorems for approximating fixed points of nonexpansive mappings using composite hybrid iteration method,* J Nig Math Soc., 33(2014), 129-144. 1, 1

[11] M. Eslamian, *Weak and strong convergence theorems of iterative process for two finite families of mappings,* Sci. Bull. Politeh. Univ. Buchar. ; Ser. A. Appl. Math. Phys., 75(4)(2013), 81-90.

[12] J. Garcia-Falset, E. Llorens-Fuster, T. Suzuki, *Fixed point theory for a class of generalized nonexpansive mappings,* J. Math. Anal. Appl.,375(2011), 185-195.

[13] F. E. Browder, *Fixed point theorems for noncompact mappings in Hilbert space,* Proc. Nut. Sci., 43 (1965), 1272-1276. 1

[14] F. E. Browder, *Nonexpansive nonlinear operators in a Banach space*, Proc. Nut. Sci. , 43 (1965), 1041-1044. 1

[15] D. Gohde, *Zum prinzip der kontraktiven abbildung*, Math. Nach., 30 (1965), 251-258. 1

[16] W. A. Kirk: *A fixed point theorem for mappings which do not increase distances*, Amer. Math. Monthly, 72 (1965), 1004-1006. 1

[17] J. B. . Baillon and R. Schoneberg, *Asymptotic normal structure and fixed points of nonexpansive mappings,* Proc, Arner. Math. Soc., 81 (1981), 257-264. 1

[18] B. Halpern, *Fixed points of nonexpansive mappings,* Bull. Amer. Math. Soc., 73(1967), 957-961. 1

[19] F. Kohasaka, W. Takahashi, *Fixed point theorems for a class of nonlinear mappings related to maximal monotone operator in Banach spaces* , Arch. Math., 91(2008), 166-177. 1.5

[20] F. Kohasaka, W. Takahashi, *Existence and approximationn of fixed points of firmly nonexpansive-type mappings in Banach spaces,* SIAM. J. Optim., 19(2008), 824-835. 1.5

[21] W. P. Cholamjiak, S. Suantai, Y. J. Cho, *Fixed points for nonspreading-type multivalued mappings: Existence and convergence results,* Ann. Acad. Rom. Sci. Ser. Math. Apll., 10(2)(2018), 838-844.

[22] W. R. Mann, *Mean value method in iteration*, Proc. Ame. Math. Soc., 4(1953), 506-510.

[23] S. Reich, *Weak convergence theorems for nonexpansive mappings in Banach spaces,* J. Math. Anal. Appl., 67(2)(1979), 274-276. 1

[24] A. Genel, J. Lindenstrauss, *An example concerning fixed points,* lsr. J. Math.,22(1975),81-86.

[25] C. E. Chidume, C. O. Chidume, *lterative approximation of fixed points of nonexpansive mappings*, J. Math. Anal. Appl., 318(1)(2006), 288-295. 1

[26] P. L. Lions, *Approximation de points fixes de contraction*, C. R. Acad. Sci., Ser. A-B Paris, 284(1977), 1357-1359. 1

[27] S. lshikawa, *Fixed points by a new iteration method*, Proc. Amer. Math. Soc., 44(1974), 147-150. 1

[28] R. P. Agarwal, D. O'Regan, D. R. Sahu, *lterative construction of fixed points of nearly asymptotically nonexpansive mappings*, Int. J. Nonlinear Convex Anal., 8(1)(2007), 61-69. 1