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A Large Averaging Modified Implicit Iterative Scheme with Errors for Finite Family of Asymptotically Φ−Demicontractive Mappings

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Abstract

In this paper, a three-step implicit iteration process with errors is introduced and we prove strong convergence theorem of the new iterative scheme with some mild conditions on the real sequences to a common fixed point of finite family of asymptotically ϕ-demicontractive mappings defined on a closed convex subset of a Banach space. The new iterative scheme includes several well known explicit and implicit iterative schemes. The results in this paper generalize several strong convergence results in the literature.

Keywords: Fixed point, Banach space, Implicit iteration process, strong convergence asymptotically, ϕ-demicontractive mappings.
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1. Introduction

Let E be a real Banach space and $J : E \to 2^{E^*}$ denote the normalized duality mapping defined by

$$J(\zeta) = \{f^* \in E^* : \langle \zeta, f^* \rangle = \|\zeta\|^2 = \|f^*\|^2\}, \ \forall \zeta \in E, \tag{1.1}$$

where E^* denotes the dual space of E and $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing of E and E^*. In the sequel, we shall use j to denote the single-valued duality mapping and $F(R)$ denotes the set of fixed points of mapping R, i.e., $F(R) = \{\zeta \in E : R\zeta = \zeta\}$.

Definition 1.1. Let K be a nonempty subset of real Banach space E. A mapping $R : K \to K$ is said to be:

- *L-Lipschitzian* if there exists a constant $L \geq 0$ such that

  $$\|R\zeta - R\mu\| \leq L\|\zeta - \mu\|, \tag{1.2}$$

  for all $\zeta, \mu \in K$. R is said to be a contraction if $L \in [0, 1)$ and T is said to be *nonexpansive* if $L = 1$;

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- *uniformly L-Lipschitzian* if there exists a constant $L \geq 0$ such that
$$\|R^n\zeta - R^n\mu\| \leq L\|\zeta - \mu\|, \ \forall \zeta, \mu \in K \text{ and } n \geq 1; \tag{1.3}$$
.
- *asymptotically nonexpansive* if there exists a sequence $\{h_n\} \subseteq [1, \infty)$ with $\lim_{n\to\infty} h_n = 1$ such that
$$\|R^n\zeta - R^n\mu\| \leq h_n\|\zeta - \mu\|, \ \forall \zeta, \mu \in K, n \geq 1; \tag{1.4}$$
- *k-strictly pseudocontractive* if for all $\zeta, \mu \in K$, there exist a constant $k \in (0, 1]$ and $j(\zeta-\mu) \in J(\zeta-\mu)$ such that
$$\langle R\zeta - R\mu, j(\zeta - \mu)\rangle \leq \|\zeta - \mu\|^2 - k\|\zeta - \mu - (R\zeta - R\mu)\|^2. \tag{1.5}$$
If I denotes the identity mapping, then (1.5) can be written as
$$\langle (I - R)\zeta - (I - R)\mu, j(\zeta - \mu)\rangle \geq k\|(I - R)\zeta - (I - R)\mu\|^2; \tag{1.6}$$
- *demicontractive* if $F(R) \neq \emptyset$ and for each $\zeta \in K$, $p \in F(R)$, there exist a constant $k \in (0, 1]$ and $j(\zeta - \mu) \in J(\zeta - \mu)$ such that
$$\langle \zeta - R\zeta, j(\zeta - p)\rangle \geq k\|\zeta - R\zeta\|^2. \tag{1.7}$$
It easy to see that every k-strictly pseudocontractive map with a nonempty fixed point set is a demicontractive map (see Hick and Cubicek [20]);
- *asymptotically k-strictly pseudocontractive* if there exists a constant $k \in [0, 1)$ and a sequence $\{h_n\} \subseteq [1, \infty)$ with $\lim_{n\to\infty} h_n = 1$ such that
$$\begin{aligned}\langle (I - R^n)\zeta - (I - R^n), j(\zeta - \mu)\rangle \ \geq \ & \frac{1}{2}(1 - k)\|(I - R^n)\zeta - (I - R^n)\mu\|^2 \\ & -\frac{1}{2}(h_n^2 - 1)\|\zeta - \mu\|^2,\end{aligned} \tag{1.8}$$
for all $\zeta, \mu \in K$ and $n \geq 1$;
- *asymptotically demicontractive* if there exists a sequence $\{h_n\} \subseteq [1, \infty)$ with $\lim_{n\to\infty} h_n = 1$, if $F(R) \neq \emptyset$ and there exists a constant $k \in [0, 1)$ and $j(\zeta - p) \in J(\zeta - p)$ such that
$$\begin{aligned}\langle \zeta - R^n\zeta, j(\zeta - p)\rangle \ \geq \ & \frac{1}{2}(1 - k)\|\zeta - R^n\zeta\|^2 \\ & -\frac{1}{2}(h_n^2 - 1)\|\zeta - p\|^2,\end{aligned} \tag{1.9}$$
for all $\zeta, \mu \in K$, $p \in F(R)$ and $n \geq 1$.

The class of asymptotically k-strictly pseudocontractive mappings and asymptotically demicontractive mappings were introduced in Hilbert space by Qihou [41]. By virtue of the normalized duality mapping, Osilike [36], first extended the concept of k-strictly asymptotically pseudocontractive and asymptotically demicontractive mappings from Hilbert spaces to general Banach spaces. Observe that every k-strictly asymptotically pseudocontractive mapping with a nonempty fixed point set is asymptotically demicontractive mapping. It is proved in [40] that the class of $k-$strictly asymptotically pseudocontractive mappings and the class of k-strictly pseudocontractive mapping are independent.

We now give an example of a mapping which is an asymptotically demicontractive mapping, but not a demicontractive mapping.

Example 1.2 [41]. Let $E = \Re$ with the absolute value norm and $K = [0, 1]$. Define $R : K \to K$ by
$$R\zeta = (1 - \zeta^{\frac{1}{3}})^3, \ \forall \zeta \in K. \tag{1.10}$$

First, we shall show that R is asymptotically demicontractive mapping for all $k \in (0,1)$. It is clear that $\frac{1}{8}$ is fixed point of R. But $RoR = I$ and R is monotonically decreasing, it follows that

$$|R^n - p|^2 \le h_n^2|\zeta - p|^2 + k|\zeta - R^n\zeta|^2, \tag{1.11}$$

where $h_n \in [1,\infty)$ and $k \in (0,1)$. Hence, R is asymptotically demicontractive mapping for all $k \in (0,1)$. Next, we shall show that T is not a demicontractive mapping for all $k \in (0,1)$. Let $p = \frac{1}{8}$ and $\zeta = \frac{1}{27}$, then

$$|R\zeta - p|^2 > |\zeta - p|^2 + k|\zeta - R^n\zeta|^2, \tag{1.12}$$

for infinitely many $k \in (0,1)$. Therefore, R is not a demicontractive mapping for $k \in (0,1)$. Hence, the class of demicontractive mappings is a proper subclass of the class of asymptotically demicontractive mapping.

- *asymptotically ϕ-demicontractive* if there exists a sequence $\{h_n\} \subseteq [1,\infty)$ with $\lim_{n\to\infty} h_n = 1$, $F(R) \neq \emptyset$ and a strictly increasing continuous function $\phi : [0,\infty) \to [0,\infty)$ with $\phi(0) = 0$ and there exists $j(\zeta - p) \in J(\zeta - p)$ such that

$$\langle \zeta - R^n\zeta, j(\zeta - p)\rangle \ge \phi(\|\zeta - R^n\zeta\|) - \frac{1}{2}(h_n^2 - 1)\|\zeta - p\|^2, \tag{1.13}$$

for all $\zeta \in K$, $p \in F(R)$ and $n \ge 1$. The class asymptotically ϕ-demicontractive mappings was first introduced by Osilike and Isiogugu [39]. It is proved in [39] that the class of asymptotically demicontractive mapping is a proper subclass of the class of asymptotically ϕ-demicontractive mapping.

The convergence of iterative scheme to the fixed of these operators have been studied by several authors (see for example, [23, 24, 25, 26, 27, 28, 30, 31, 36, 38, 39, 41]).

In [39], Osilike and Isiogugu proved the convergence of the modified averaging iteration process of Mann [33] to the fixed points of asymptotically ϕ-demicontractive mappings. In particular, they proved the following:

Theorem 1.1 ([39], p. 65). *Let E be real Banach space and K a nonempty closed convex subset of E. Let $R : K \to K$ be a completely continuous uniformly L-Lipschitzian asymptotically ϕ-demicontractive mapping with a sequence $\{h_n\}_{n=1}^{\infty} \subseteq [1,\infty)$, such that $\sum(h_n^2 - 1) < \infty$. Let a_n be a real sequence satisfying (i) $0 < a_n < 1$ (ii) $\sum a_n = \infty$ (iii) $\sum a_n^2 < \infty$*
Then the sequence $\{\zeta_n\}_{n=1}^{\infty}$ generated from arbitrary $\zeta_1 \in K$ by the modified averaging Mann iteration process

$$\zeta_{n+1} = (1 - a_n)\zeta_n + \alpha_n R^n\zeta_n, \;\; n \ge 1 \tag{1.14}$$

converges strongly to a common fixed point of of R.

Fixed point theory provides a suitable framework to investigate various nonlinear phenomena arising in applied sciences including complex graphics, geometry, biology and physics (see for example, [5], [22], [42] and [48]). Complex graphical shapes such as fractals, were discovered as fixed points as contain in [5]. A wide range of problems of applied sciences and engineering are usually formulated as functional equations. Such equations can be written in the form of fixed point equations. Operator equations representing phenomena occurring in different fields, such as steady state temperature, distribution, chemical reactions, neutron transport theory, economy theories and epidemics, often require appropriate and adequate solutions. Thus, the aim of finding solution to these equations is to locate the fixed point and approximate it value. However, once we ensure the existence of fixed point of some mapping, then it is always desirable to develop such methods which can efficiently be used to approximate that fixed point. Iterative processes are one of the fundamental tools that can be used to locate a fixed point. In the last few decades, various authors have introduced numerous iterative schemes which have been utilized widely to approximate the fixed point of operators. The celebrated Banach contraction theorem [4] which is one of the most widely and extensively utilized result uses the Picard iteration process for locating the fixed point. Owing to the importance of iteration processes, many new iteration schemes have been constructed in the last few decades to approximate fixed points of certain mappings in different spaces. Some well know iterative schemes are Mann [33], Ishikawa [29], Noor [34], Argawal et al. [3], Normal S-iteration [43], Abass and Nazir [2] and so on.

In the past few decades fixed point theory has been kept a watchful eye as many authors tend to use fixed point approach to solve several problems in applied science as enumerated earlier on.

Recently, Abbas et al. [1] presented an application of fixed point iterative precess in generation of fractals namely Julia and Mandelbrot sets for the complex polynomials of the form $R(\zeta) = \zeta^n + m\zeta + r$, $m, r \in \mathbb{C}$ and $n \geq 2$. Fractals represent the phenomena of expanding or unfolding symmetries which exhibit similar patterns displayed at every scale. They proved some escape time results for generation of Julia and Mandelbrot sets using a Picard Ishikawa type iterative process. A visualization of the Julia and Mandelbrot set for certain complex polynomials and their graphical behavior was examined. Further, they discussed the effects of parameters on the color variation and shape of fractal. Let $\mathbb{C}$ be a complex space and $R_{\mathbb{C}} : \mathbb{C} \to \mathbb{C}$ be a complex polynomial with complex coefficients. Precisely, Abbas et al [1] considered the following Picard Ishikawa type iterative process in achieving their results.

$$\begin{cases} \zeta_0 \in \mathbb{C}, \\ \zeta_{n+1} = (1-a)\mu_n + aR\mu_n, \\ \mu_n = R_{\mathbb{C}}\gamma_n, \\ \gamma_n = R_{\mathbb{C}}\gamma_n t_n, \\ t_n = (1-a')\zeta_n + a'R\zeta_n, \end{cases} \tag{1.15}$$

where $n = 0, 1, 2, ...$ and $a, a' \in (0, 1]$.

Interestingly, De la Sen [14] considered a modified Ishikawa iteration scheme and illustrated that the parameterizing sequences might be vectors of distinct components and admitted that auxiliary self-mapping which supports the iterative scheme is asymptotically demicontractive.

Let $\mathcal{H}$ and $\mathcal{G}$ be nonempty subsets of a normed linear space $\mathcal{X}$. A mapping $R : \mathcal{H} \cup \mathcal{G} \to \mathcal{H} \cup \mathcal{G}$ is said to be a noncyclic if $R(\mathcal{H}) \subset \mathcal{H}$,$R(\mathcal{G}) \subset \mathcal{G}$ and $\|R\zeta - R\mu\| \leq \|\zeta - \mu\|$ for all $(\zeta, \mu) \in \mathcal{H} \times \mathcal{G}$. A best proximity pair for such mapping R is a point $(q_1, q_2) \in \mathcal{H} \times \mathcal{G}$ such that $q_1 = Rq_1$, $q_2 = Rq_2$ and $d(q_1, q_2) = dist(\mathcal{H}, \mathcal{G})$.

With the beauty of fixed point theory, Gabeleh [15] introduced a geometric notion of proximal Opial's condition on a nonempty, closed and convex pair of subsets of strictly convex Banach spaces. By using their new geometric notion, they studied the strong and weak convergence of the Ishikawa iterative scheme for noncyclic relatively nonexpansive mappings in uniformly convex Banach spaces. Furthermore, they established a best proximity pair theorem for noncyclic contraction type mappings in the setting of strictly convex Banach spaces. Inarguably, fixed point theory has helped in solving numerous problems in applied sciences and Engineering.

In 1974, Ishikawa [29] introduced an iteration process $\{\zeta_n\}$ defined by

$$\begin{cases} \zeta_0 \in K, \\ \zeta_{n+1} = (1-a_n)\zeta_n + a_n R\mu_n, \qquad \forall n \geq 1, \\ \mu_n = (1-a'_n)\zeta_n + a'_n R\zeta_n \end{cases} \tag{1.16}$$

where $\{a_n\}$ and $\{a'_n\}$ are sequences in [0,1]. This iteration process reduces to Mann iteration [33] if $a'_n = 0$ for all $n \geq 1$ as follows:

$$\begin{cases} \zeta_0 \in K, \\ \zeta_{n+1} = (1-a_n)\zeta_n + a_n R\zeta_n, \end{cases} \quad \forall n \geq 1, \tag{1.17}$$

where $\{a_n\}$ is a sequence in [0,1].

In 1991, Schu [44] introduced the following Mann-type iterative process for an asymptotically nonexpansive in Hilbert spaces

$$\begin{cases} \zeta_0 \in K, \\ \zeta_{n+1} = (1-a_n)\zeta_n + a_n R^n\zeta_n, \end{cases} \quad \forall n \geq 1, \tag{1.18}$$

where $\{a_n\}$ is a sequences in [0,1].

In 2001, Xu and Ori [54] introduced the following implicit iteration process for finite family of nonexpansive self-mapping in Hilbert spaces.

$$\begin{cases} \zeta_0 \in K, \\ \zeta_n = \alpha_n\zeta_{n-1} + (1-\alpha_n)R_n\zeta_n, \end{cases} \quad \forall n \geq 1, \tag{1.19}$$

where $\{\alpha_n\}$ is a sequence in [0,1] and $R_n = R_{n(\,mod\,N)}$. They proved in [54] that the sequence $\{\zeta_n\}$ converges to a common fixed point of $\{R_n\}_{n=1}^N$.

Later on, Osilike and Akuchu [37] and Chen et al. [11] extended the iteration process (1.19) to a finite family of asymptotically pseudocontractive mapping and a finite family of continuous pseudocontractive self-mapping, respectively.

In 2003, Sun [50] modified the implicit iteration of Xu and Ori [54] and applied the modified averaging iteration process for the approximation of fixed points of asymptotically quasi-nonexpansive mappings. Sun introduced the following implicit iteration process for common fixed points of a finite family in Banach spaces:

$$\begin{cases} \zeta_0 \in K, \\ \zeta_n = \alpha_n\zeta_{n-1} + (1-\alpha_n)R_{i(n)}^{k(n)}\zeta_n, \end{cases} \quad \forall n \geq 1, \tag{1.20}$$

where $\{a_n\}$ is a sequence in [0,1], $n = (k-1)N + i$, $i = n(i) \in I = \{1, 2, ..., N\}$.

In 2006, Su and Li [49] introduced the following implicit Ishikwa-type iteration scheme and called it composite implicit iteration process and applied the iteration process for the approximation of common fixed point of a finite family of strictly pseudocontractive maps:

$$\begin{cases} \zeta_0 \in K, \\ \zeta_n = \alpha_n\zeta_{n-1} + (1-\alpha_n)R_n\mu_n, \\ \mu_n = \alpha'_n\zeta_{n-1} + (1-\alpha'_n)R_n\zeta_n \end{cases} \quad \forall n \geq 1, \tag{1.21}$$

where $\{\alpha_n\}$ and $\{\alpha'_n\}$ are sequences in [0,1] and $R_n = R_{n\,(mod)\,N}$.

In 2011, Igbokwe and Ini [25] modified and improved the composite implicit iteration process of Su and Li [49] for the approximation of common fixed point of finite family of $k-$strictly asymptotically pseudocontractive mappings in Banach spaces. Precisely, they considered the following modified averaging composite iteration process:

$$\begin{cases} \zeta_0 \in K, \\ \zeta_n = \alpha_n\zeta_{n-1} + (1-\alpha_n)R_{i(n)}^{k(n)}\mu_n, \\ \mu_n = \alpha'_n\zeta_{n-1} + (1-\alpha'_n)R_{i(n)}^{k(n)}\zeta_n \end{cases} \quad \forall n \geq 1, \tag{1.22}$$

where $\{\alpha_n\}$ and $\{\alpha'_n\}$ are sequences in [0,1] and $n = (k-1)N + i$, $i = i(n) \in \{1, 2, ..., N\}$, $k = k(n) \geq 1$ is some positive integers and $k(n) \to \infty$ as $n \to \infty$.
Igbokwe and Jim [26]-[27] extended the result of Igbokwe and Ini [25] from the class of finite family of k-strictly asymptotically pseudocontractive mappings to the more general class of asymptotically ϕ-demicontractive mappings in Hilbert and Banach spaces, respectively.

In 2007, Gu [18] introduced a composite implicit iteration process with errors for a finite family of strictly pseudocontractive mappings in Banach spaces as follows:

$$\begin{cases} \zeta_0 \in K, \\ \zeta_n = (1 - a_n - b_n)\zeta_{n-1} + a_n R_n \mu_n + b_n u_n, \qquad \forall n \geq 1, \\ \mu_n = (1 - a'_n - b'_n)\zeta_n + a'_n R_n \zeta_n + b'_n v_n \end{cases} \tag{1.23}$$

where $R_n = R_{n(\,mod\,n)}$, $\{a_n\}$, $\{b_n\}$, $\{a'_n\}$, $\{b'_n\}$, are four real sequences in $[0, 1]$, $\{u_n\}$ and $\{v_n\}$ are bounded sequences in K.

In 2007, Thahur [52] proposed the following composite implicit iteration process for a finite family of asymptotically nonexpansive mappings as follows:

$$\begin{cases} \zeta_0 \in K, \\ \zeta_n = (1 - a_n)\zeta_{n-1} + a_n R_{i(n)}^{k(n)} \mu_n, \qquad \forall n \geq 1, \\ \mu_n = (1 - a'_n)\zeta_n + a'_n R_{i(n)}^{k(n)} \zeta_n \end{cases} \tag{1.24}$$

where $\{a_n\}$ and $\{a'_n\}$ are sequences in [0,1] and $n = (k-1)N + i$, $i = i(n) \in \{1, 2, ..., N\}$, $k = k(n) \geq 1$ is some positive integers and $k(n) \to \infty$ as $n \to \infty$.

In Yang [56] and Cianciaruso [12], they considered a two-step implicit iteration process with errors for a finite family of asymptotically nonexpansive and asymptotically demicontractive mappings defined as follows:

$$\begin{cases} \zeta_0 \in K, \\ \zeta_n = (1 - a_n - b_n)\zeta_{n-1} + a_n R_{i(n)}^{k(n)} \mu_n + b_n u_n, \qquad \forall n \geq 1, \\ \mu_n = (1 - a'_n - b'_n)\zeta_n + a'_n R_{i(n)}^{k(n)} \zeta_n + b'_n v_n \end{cases} \tag{1.25}$$

where$\{a_n\}$, $\{b_n\}$, $\{a'_n\}$, $\{b'_n\}$, are four real sequences in $[0, 1]$, $\{u_n\}$ and $\{v_n\}$ are bounded sequences in K and $n = (k-1)N+i$, $i = i(n) \in \{1, 2, ..., N\}$, $k = k(n) \geq 1$ is some positive integers and $k(n) \to \infty$ as $n \to \infty$.

In 2010, Gu [19] introduced another composite implicit iteration process with errors for a finite family of strictly pseudocontractive mappings in Banach spaces as follows:

$$\begin{cases} \zeta_0 \in K, \\ \zeta_n = (1 - a_n - b_n)\zeta_{n-1} + a_n R_n \mu_n + b_n u_n, \qquad \forall n \geq 1, \\ \mu_n = (1 - a'_n - b'_n)\zeta_{n-1} + a'_n R_n \zeta_n + b'_n v_n \end{cases} \tag{1.26}$$

where $R_n = R_{n(\,mod\,n)}$, $\{a_n\}$, $\{b_n\}$, $\{a'_n\}$, $\{b'_n\}$, are four real sequences in $[0, 1]$, $\{u_n\}$ and $\{v_n\}$ are bounded sequences in K.

In 2012, Jim [30] extended the results of Gu [19] from the class of strictly pseudocontractive mappings to the more general class of ϕ-demicontractive mappings in Hilbert spaces.

Furthermore, Jim et al. [31] improved and modified the composite implicit iteration process of Gu [19] for a finite family of asymptotically ϕ-demicontractive maps in Banach spaces as follows:

$$\begin{cases} \zeta_0 \in K, \\ \zeta_n = (1 - a_n - b_n)\zeta_{n-1} + a_n R_{i(n)}^{k(n)} \mu_n + b_n u_n, \qquad \forall n \geq 1, \\ \mu_n = (1 - a'_n - b'_n)\zeta_{n-1} + a'_n R_{i(n)}^{k(n)} \zeta_n + b'_n v_n \end{cases} \tag{1.27}$$

where $\{a_n\}$, $\{b_n\}$, $\{a'_n\}$, $\{b'_n\}$, are four real sequences in $[0, 1]$, $\{u_n\}$ and $\{v_n\}$ are bounded sequences in K and $n = (k-1)N+i$, $i = i(n) \in \{1, 2, ..., N\}$, $k = k(n) \geq 1$ is some positive integers and $k(n) \to \infty$ as $n \to \infty$.

Noor et al. [34] introduced and studied the following three-step iteration process for solving non-linear operator equations in real Banach spaces:

$$\begin{cases} \zeta_0 \in K, \\ \zeta_{n+1} = (1 - \alpha_n)\zeta_n + \alpha_n R\mu_n, \qquad \forall n \geq 1, \\ \mu_n = (1 - a'_n)\zeta_n + a'_n R\gamma_n \\ \gamma_n = (1 - a''_n)\zeta_n + a''_n R\zeta_n \end{cases} \tag{1.28}$$

(1.29)

where $\{a_n\}$, $\{a'_n\}$ and $\{a''_n\}$ are sequences in [0,1].

Since then, Noor iteration scheme has been applied to study the strong and weak convergence of several mappings (see, e.g., [13], [51] [55]). It was proved by Bnouhachem et al. [6] that three-step method performs better than two-step and one-step methods for solving variational inequalities. Moreover, three-step schemes are natural generalizations of the splitting methods to solve partial differential equations, (see [45], [47], [51]).

On the other hand, Glowinski and Le-Tallec [16] used a three-step iterative method to solve elasto-viscoplasticity, liquid crystal and eigenvalue problems. They also established that three-step iterative scheme performs better than one-step (Mann) and two-step (Ishikawa) iterative schemes. Haubruge et al. [21] studied the convergence analysis of the three-step iterative processes of Glowinski and Le-Tallec [16] and used the three-step iteration to obtain some new splitting type algorithms for solving variational inequalities, separable convex programming and minimization of a sum of convex functions. They also proved that three-step iteration also lead to highly parallelized algorithms under certain conditions. Hence, three-step iterative scheme play an important role in solving various problems in pure and applied sciences.

Although, implicit methods are more complex to programme and require more computational effort in each iteration or solution step, they are used because many physical problems arising in practice are stiff, for which the use of explicit method requires small time steps to keep the errors in the result bounded. For example, in numerical stability, which has to do with behaviour of the solution as the time–step is increased, if the solution remains well behaved for arbitrary large values of time step, the method is said to be unconditionally stable. This situation never occurs with explicit methods which are always conditionally stable. Therefore, for stiff problems, to achieve given accuracy, it takes much less computational time to use an implicit method with larger time steps. Implicit iterative schemes have been studied recently by several authors (see for example, [11] [37], [50], [54] and the references there in).

Recently, Okeke and Olaleru [32] introduced the following modified three-step iterative scheme with errors for approximation of the unique common fixed point of a family of strongly pseudocontractive maps:

$$\begin{cases} \zeta_0 \in K, \\ \zeta_{n+1} = (1 - \alpha_n - \beta_n - e_n)\zeta_n + \alpha_n R\mu_n + \beta_n R\gamma_n + e_n u_n, \qquad \forall n \geq 1, \\ \mu_n = (1 - a_n - b_n - e'_n)\zeta_n + a_n S\gamma_n + b_n S\zeta_n + e'_n v_n, \\ \gamma_n = (1 - c_n - e''_n)\zeta_n + c_n H\zeta_n + e''_n w_n \end{cases} \tag{1.30}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, $\{e_n\}$, $\{a_n\}$, $\{b_n\}$, $\{e'_n\}$, $\{c_n\}$, $\{e''_n\}$ are real sequences in $[0,1]$, $\{u_n\}$, $\{v_n\}$ and $\{w_n\}$ are bounded sequences in K.

Motivated and inspired by the above results, we introduce a new modified three-step composite implicit iteration process with errors for approximating common fixed points of a finite family of N-asymptotically ϕ-demicontractive mappings in Banach spaces as follows:

$$\begin{cases} \zeta_0 \in K, \\ \zeta_n = (1-a_n-b_n-c_n)\zeta_{n-1} + a_n R_{i(n)}^{k(n)}\mu_n + b_n R_{i(n)}^{k(n)}\gamma_n + c_n u_n, \\ \mu_n = (1-a'_n-b'_n-c'_n)\zeta_{n-1} + a'_n R_{i(n)}^{k(n)}\gamma_n + b'_n R_{i(n)}^{k(n)}\zeta_n + c'_n v_n, \\ \gamma_n = (1-a''_n-b''_n)\zeta_n + a''_n R_{i(n)}^{k(n)}\zeta_n + b''_n w_n \end{cases} \quad \forall n \geq 1, \tag{1.31}$$

where $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{a'_n\}$, $\{b'_n\}$, $\{c'_n\}$, $\{a''_n\}$, $\{b''_n\}$ are real sequences in $[0,1]$ satisfying $a_n+b_n+c_n \leq 1$, $a'_n+b'_n+c'_n \leq 1$ and $a''_n+b''_n \leq 1$, $\{u_n\}$, $\{v_n\}$ and $\{w_n\}$ are bounded sequences in K and $n=(k-1)N+i$, $i=i(n) \in \{1,2,...,N\}$, $k=k(n) \geq 1$ is some positive integers and $k(n) \to \infty$ as $n \to \infty$.

The iteration process (1.31) reduces to:

- (1.17) when $b_n = c_n = a'_n = b'_n = c'_n = a''_n = b''_n{=}0$, $R^n = R$, $N = 1$.
- (1.18) when $b_n = c_n = a'_n = b'_n = c'_n = a''_n = b''_n{=}0$, $N = 1$.
- (1.19) when $a_n = c_n = a'_n = b'_n = c'_n = a''_n = b''_n{=}0$, $R^n = R$, $(1-b_n) = \alpha_n$.
- (1.20) when $a_n = c_n = a'_n = b'_n = c'_n = a''_n = b''_n{=}0$, $(1-b_n) = \alpha_n$.
- (1.21) when $b_n = c_n = b'_n = c'_n = a''_n = b''_n{=}0$, $1-a_n = \alpha_n$, $1-a'_n = \alpha'_n$, $R^n = R$.
- (1.22) when $b_n = c_n = b'_n = c'_n = a''_n = b''_n{=}0$, $1-a_n = \alpha_n$, $1-a'_n = \alpha'_n$.
- (1.23) when $a_n = a'_n = b'_n = c'_n{=}0$, $R^n = R$, $N = 1$.
- (1.24) when $a_n = c_n = a'_n = b'_n = c'_n = b''_n = 0$.
- (1.25) when $a_n = a'_n = b'_n = c'_n = 0$.
- (1.26) when $b_n = b'_n = a''_n = b''_n = 0$, $R^n = R$, $N = 1$.
- (1.27) when $b_n = b'_n = a''_n = b''_n = 0$.

Hence, the new iteration process (1.31) properly includes the iteration processes (1.17)-(1.26).

The purpose of this paper is to use a simple and quite different method, to study the strong convergence of our new implicit iterative sequence $\{\zeta_n\}$ defined by (1.31) to a common fixed points of finite family of asymptotically ϕ-hemicontractive mappings in a real Banach space. Our results extend and improve some recent results in Su and Li [49], Sun [50], Xu and Ori [54], Osilike and Isiogugu [39], Schu [44], Yang [56], Cianciaruso [12], Igbokwe and Ini [25], Igbokwe and Jim [26]-[27], Gu [18], Thahur [52], Jim [30] and Jim et al. [31].

2. Preliminaries

In order to prove our main results, we also need the following lemmas.

Lemma 2.1 (see [7]). *Let $J : E \to 2^{E^*}$ be the normalized duality mapping. Then for any $\zeta, \mu \in E$, one has*

$$\|\zeta+\mu\|^2 \leq \|\zeta\|^2 + 2\langle \mu, j(\zeta+\mu)\rangle, \ \ \forall j(\zeta+\mu) \in J(\zeta+\mu).$$

Lemma 2.2 (see [53]). *Let ρ_n, ω_n and ϖ_n be a nonnegative sequences satisfying*

$$\rho_{n+1} \leq (1-\theta_n)\rho_n + \omega_n + \varpi_n$$

where $\theta_n \in [0,1]$, $\sum_{n\geq 1}\theta_n = \infty$, $\omega_n = o(\theta_n)$ and $\sum_{n\geq 1}\varpi_n < \infty$. Then

$$\lim_{n\to\infty} \rho_n = 0.$$

3. Main Resuls

Theorem 3.1. *Let K be a nonempty closed convex subset of a real Banach space E. Let $N \geq 1$ be a positive integer and $I = \{1, 2, 3, ..., N\}$. Let $R_i : K \to K$ be a finite family asymptotically ϕ-demicontractive mappings with sequence $\{\lambda_{in}\} \subset [1, \infty)$, where $\lambda_{in} \to 1$ as $n \to \infty$, for each $i \in I$. Furthermore, let $R_i(K)$ be bounded and R_i be uniformly continuous for each $i \in I$. Assume that $\boldsymbol{F} = \bigcap_{i=1}^{N} F(R_i) \neq \emptyset$. Let $\{u_n\}$, $\{v_n\}$, $\{w_n\}$ be bounded in K and $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{a'_n\}$, $\{b'_n\}$, $\{c'_n\}$, $\{a''_n\}$ and $\{b''_n\}$ be sequences in [0,1] such that $a_n + b_n + c_n \leq 1$, $a'_n + b'_n + c'_n \leq 1$ and $a''_n + b''_n \leq 1$, for each $n \geq 1$. Let $\{\zeta_n\}$ be a sequence generated in (1.31). Assume that the following conditions are satisfied:*

(i) $\lim_{n\to\infty}(a_n + b_n) = 0 = \lim_{n\to\infty} c_n = \lim_{n\to\infty} a'_n = \lim_{n\to\infty} b'_n = \lim_{n\to\infty} c'_n = \lim_{n\to\infty} a''_n = \lim_{n\to\infty} b''_n$;

(ii) $\sum_{n=1}^{\infty}(a_n + b_n) = \infty$;

(iii) $\sum_{n=1}^{\infty} c_n < \infty$.

Then the sequence $\{\zeta_n\}$ converges strongly to a point in $\mathbf{F}$.

Proof. Fixing $p \in \mathbf{F}$. Since R_i has a bounded range, we let

$$\begin{aligned} M_1 &= \|\zeta_0 - p\| + \sup_{n\geq 1}\|R_{i(n)}^{k(n)}\mu_n - p\| + \sup_{n\geq 1}\|R_{i(n)}^{k(n)}\gamma_n - p\| \\ &\quad + \sup_{n\geq 1}\|R_{i(n)}^{k(n)}\zeta_n - p\| + \sup_{n\geq 1}\|u_n - p\| + \sup_{n\geq 1}\|v_n - p\| \\ &\quad + \sup_{n\geq 1}\|w_n - p\|. \end{aligned} \tag{3.1}$$

Obviously, $M_1 < \infty$. It is clear that $\|\zeta_0 - p\| \leq M_1$. Let $\|\zeta_{n-1} - p\| \leq M_1$. Next we prove that $\|\zeta_n - p\| \leq M_1$.

Using (1.31) we obtain that

$$\begin{aligned} \|\zeta_n - p\| &= \|(1 - a_n - b_n - c_n)\zeta_{n-1} + a_n R_{i(n)}^{k(n)}\mu_n + b_n R_{i(n)}^{k(n)}\gamma_n + c_n u_n - p\| \\ &= \|(1 - a_n - b_n - c_n)(\zeta_{n-1} - p) + a_n(R_{i(n)}^{k(n)}\mu_n - p) \\ &\quad + b_n(R_{i(n)}^{k(n)}\gamma_n - p) + c_n(u_n - p)\| \\ &\leq (1 - a_n - b_n - c_n)\|\zeta_{n-1} - p\| + a_n\|R_{i(n)}^{k(n)}\mu_n - p\| \\ &\quad + b_n\|R_{i(n)}^{k(n)}\gamma_n - p\| + c_n\|u_n - p\| \\ &\leq (1 - a_n - b_n - c_n)M_1 + a_n M_1 + b_n M_1 + c_n M_1 \\ &= M_1. \end{aligned}$$

Hence, the sequence $\{\|\zeta_n - p\|\}$ is bounded. Let $M_2 = \sup_{n\geq 1}\|\zeta_n - p\|$.

Denote

$$M = M_1 + M_2. \text{ Clearly } M < \infty. \tag{3.2}$$

Using (1.31) and Lemma 2.1 we obtain

$$\begin{aligned}
\|\zeta_n - p\|^2 &= \|(1-a_n-b_n-c_n)\zeta_{n-1} + a_n R_{i(n)}^{k(n)}\mu_n + b_n R_{i(n)}^{k(n)}\gamma_n + c_n u_n - p\|^2 \\
&\leq (1-a_n-b_n-c_n)^2\|\zeta_{n-1}-p\|^2 + 2\langle a_n(R_{i(n)}^{k(n)}\mu_n - p) \\
&\quad + b_n(R_{i(n)}^{k(n)}\gamma_n - p) + c_n(u_n - p), j(\zeta_n - p)\rangle \\
&= (1-a_n-b_n-c_n)^2\|\zeta_{n-1}-p\|^2 + 2a_n\langle R_{i(n)}^{k(n)}\mu_n - p, j(\zeta_n - p)\rangle \\
&\quad + 2b_n\langle R_{i(n)}^{k(n)}\gamma_n - p, j(\zeta_n - p)\rangle + 2c_n\langle u_n - p, j(\zeta_n - p)\rangle \\
&= (1-a_n-b_n-c_n)^2\|\zeta_{n-1}-p\|^2 \\
&\quad + 2a_n\langle R_{i(n)}^{k(n)}\mu_n - R_{i(n)}^{k(n)}\zeta_n, j(\zeta_n - p)\rangle \\
&\quad + 2a_n\langle R_{i(n)}^{k(n)}\zeta_n - p, j(\zeta_n - p)\rangle \\
&\quad + 2b_n\langle R_{i(n)}^{k(n)}\gamma_n - R_{i(n)}^{k(n)}\zeta_n, j(\zeta_n - p)\rangle \\
&\quad + 2b_n\langle R_{i(n)}^{k(n)}\zeta_n - p, j(\zeta_n - p)\rangle \\
&\quad + 2c_n\langle u_n - p, j(\zeta_n - p)\rangle \\
&= (1-a_n-b_n-c_n)^2\|\zeta_{n-1}-p\|^2 \\
&\quad + 2a_n\langle R_{i(n)}^{k(n)}\mu_n - R_{i(n)}^{k(n)}\zeta_n, j(x_n - p)\rangle \\
&\quad + 2b_n\langle R_{i(n)}^{k(n)}\gamma_n - R_{i(n)}^{k(n)}\zeta_n, j(\zeta_n - p)\rangle \\
&\quad - 2a_n\langle \zeta_n - R_{i(n)}^{k(n)}\zeta_n, j(\zeta_n - p)\rangle \\
&\quad - 2b_n\langle \zeta_n - R_{i(n)}^{k(n)}\zeta_n, j(\zeta_n - p)\rangle \\
&\quad + 2a_n\langle \zeta_n - p, j(\zeta_n - p)\rangle \\
&\quad + 2b_n\langle \zeta_n - p, j(\zeta_n - p)\rangle \\
&\quad + 2c_n\langle u_n - p, j(\zeta_n - p)\rangle. \qquad (3.3)
\end{aligned}$$

Since each $R_i : K \to K$, $i \in I = \{1, 2, ...N\}$ is an asymptotically ϕ-demicontractive mapping with sequence $\{\lambda_{in}\} \subset [1, \infty)$, where $\lambda_{in} \to 1$ as $n \to \infty$, for each $i \in I$. Then for all $\zeta \in K$ and $p \in F(R)$, there exists $j(\zeta - p) \in J(\zeta - p)$ such that

$$\langle \zeta_n - R_{i(n)}^{k(n)}\zeta_n, j(\zeta_n - p)\rangle \geq \phi_i(\|\zeta_n - R_{i(n)}^{k(n)}\zeta_n\|) - \frac{1}{2}(\lambda_{in}^2 - 1)\|\zeta_n - p\|^2, \forall i \in I. \qquad (3.4)$$

Let $h_n = \max\{\lambda_{in} : i \in I\}$ and $\phi(\wp) = \max\{\phi_i(\wp) : i \in I\}$, for each $\wp \geq 0$, then

$$\langle \zeta_n - R_{i(n)}^{k(n)}\zeta_n, j(\zeta_n - p)\rangle \geq \phi(\|\zeta_n - R_{i(n)}^{k(n)}\zeta_n\|) - \frac{1}{2}(h_{k(n)}^2 - 1)\|\zeta_n - p\|^2, \forall i \in I. \qquad (3.5)$$

substituting (3.5) into (3.3), we obtain

$$\begin{aligned}
\|\zeta - p\|^2 &\leq (1 - a_n - b_n - c_n)^2\|\zeta_{n-1} - p\|^2 \\
&+2a_n\|R^{k(n)}_{i(n)}\mu_n - R^{k(n)}_{i(n)}\zeta_n\|\|\zeta_n - p\| \\
&+2b_n\|R^{k(n)}_{i(n)}\gamma_n - R^{k(n)}_{i(n)}\zeta_n\|\|\zeta_n - p\| \\
&-2a_n\{\phi(\|\zeta_n - R^{k(n)}_{i(n)}\zeta_n\|) - \frac{1}{2}(h^2_{k(n)} - 1)\|\zeta_n - p\|^2\} \\
&-2b_n\{\phi(\|\zeta_n - R^{k(n)}_{i(n)}\zeta_n\|) - \frac{1}{2}(h^2_{k(n)} - 1)\|\zeta_n - p\|^2\} \\
&+2a_n\|\zeta_n - p\|^2 + 2b_n\|\zeta_n - p\|^2 + 2c_n\|u_n - p\|\|\zeta_n - p\| \qquad (3.6)\\
&\leq (1 - a_n - b_n - c_n)^2\|\zeta_{n-1} - p\|^2 + 2M(a_n\nu^i_n + b_n\delta^i_n) \\
&-2(a_n + b_n)\phi(\|\zeta_n - R^{k(n)}_{i(n)}\zeta_n\|) + (a_n + b_n)(h^2_{k(n)} - 1)\|\zeta_n - p\|^2 \\
&+2(a_n + b_n)\|\zeta_n - p\|^2 + 2c_n\|u_n - p\|\|\zeta_n - p\| \\
&\leq (1 - a_n - b_n)^2\|\zeta_{n-1} - p\|^2 - 2(a_n + b_n)\phi(\|\zeta_n - R^{k(n)}_{i(n)}\zeta_n\|) \\
&+[(a_n + b_n)(h^2_{k(n)} - 1) + 2(a_n + b_n)]\|\zeta_n - p\|^2 \\
&+2c_nM^2 + 2M(a_n + b_n)\max\{\nu^i_n, \delta^i_n\}, \qquad (3.7)
\end{aligned}$$

where

$$\begin{aligned}
\nu^i_n &= \|R^{k(n)}_{i(n)}\mu_n - R^{k(n)}_{i(n)}\zeta_n\| \\
\delta^i_n &= \|R^{k(n)}_{i(n)}\gamma_n - R^{k(n)}_{i(n)}\zeta_n\|.
\end{aligned}$$

Using (1.31), we have

$$\begin{aligned}
\|\mu_n - \zeta_n\| &= \|\mu_n - \zeta_{n-1} + \zeta_{n-1} - \zeta_n\| \\
&\leq \|\mu_n - \zeta_{n-1}\| + \|\zeta_{n-1} - \zeta_n\| \\
&= \|(1 - a'_n - b'_n - c'_n)\zeta_{n-1} + a'_nR^{k(n)}_{i(n)}\gamma_n \\
&+b'_nR^{k(n)}_{i(n)}\zeta_n + c'_nv_n - \zeta_{n-1}\| \\
&+\|\zeta_{n-1} - [(1 - a_n - b_n - c_n)\zeta_{n-1} \\
&+a_nR^{k(n)}_{i(n)}\mu_n + b_nR^{k(n)}_{i(n)}\gamma_n + c_nu_n]\| \\
&= \|a'_n(R^{k(n)}_{i(n)}\gamma_n - \zeta_{n-1}) + b'_n(R^{k(n)}_{i(n)}\zeta_n - \zeta_{n-1}) + c'_n(v_n - \zeta_{n-1})\| \\
&+\|a_n(\zeta_{n-1} - R^{k(n)}_{i(n)}\mu_n) + b_n(\zeta_{n-1} - R^{k(n)}_{i(n)}\gamma_n) + c_n(\zeta_{n-1} - u_n)\| \\
&\leq a'_n(\|R^{k(n)}_{i(n)}\gamma_n - p\| + \|\zeta_{n-1} - p\|) + b'_n(\|R^{k(n)}_{i(n)}\zeta_n - p\| + \|\zeta_{n-1} - p\|) \\
&+c'_n(\|v_n - p\| + \|\zeta_{n-1} - p\|) + a_n(\|R^{k(n)}_{i(n)}\mu_n - p\| + \|\zeta_{n-1} - p\|) \\
&+b_n(\|R^{k(n)}_{i(n)}\gamma_n - p\| + \|\zeta_{n-1} - p\|) + c_n(\|u_n - p\| + \|\zeta_{n-1} - p\|) \\
&\leq 2a'_nM + 2b'_nM + 2c'_nM + 2a_nM + 2b_nM + 2c_nM \\
&= 2M(a'_n + b'_n + c'_n + a_n + b_n + c_n). \qquad (3.8)
\end{aligned}$$

From the condition (i) and (3.8), we obtain

$$\lim_{n\to\infty}\|\mu_n - \zeta_n\| = 0, \qquad (3.9)$$

and the uniform continuity of R_i leads to

$$\lim_{n\to\infty}\|R^{k(n)}_{i(n)}\mu_n - R^{k(n)}_{i(n)}\zeta_n\| = 0,$$

thus, we have

$$\lim_{n\to\infty} \nu_n^i = 0. \tag{3.10}$$

Again from (1.31) we have

$$\begin{aligned}
\|\gamma_n - \zeta_n\| &= \|(1-a''-b'')\zeta_n + a_n'' R_{i(n)}^{k(n)}\zeta_n + b_n'' w_n - \zeta_n\| \\
&= \|a_n''(R_{i(n)}^{k(n)}\zeta_n - \zeta_n) + b_n''(w_n - \zeta_n)\| \\
&= \|a_n''(R_{i(n)}^{k(n)}x_n - p + p - \zeta_n) + b_n''(w_n - p + p - \zeta_n)\| \\
&\le a_n''(\|R_{i(n)}^{k(n)}\zeta_n - p\| + \|\zeta_n - p\|) + b_n''(\|w_n - p\| + \|\zeta_n - p\|) \\
&\le a_n''(M+M) + b_n''(M+M) \\
&= 2M(a_n'' + b_n''). \qquad (3.11)
\end{aligned}$$

From the condition (i) and (3.11), we obtain

$$\lim_{n\to\infty} \|\gamma_n - \zeta_n\| = 0, \tag{3.12}$$

and the uniform continuity of R_i leads to

$$\lim_{n\to\infty} \|R_{i(n)}^{k(n)}\gamma_n - R_{i(n)}^{k(n)}\zeta_n\| = 0, \tag{3.13}$$

thus, we have

$$\lim_{n\to\infty} \delta_n^i = 0. \tag{3.14}$$

From (3.7) we obtain

$$\begin{aligned}
\|\zeta_n - p\|^2 &\le \frac{(1-a_n-b_n)^2}{1-[(a_n+b_n)(h_{k(n)}^2-1)+2(a_n+b_n)]}\|\zeta_{n-1}-p\|^2 \\
&\quad - \frac{2(a_n+b_n)}{1-[(a_n+b_n)(h_{k(n)}^2-1)+2(a_n+b_n)]}\phi(\|\zeta_n - R_{i(n)}^{k(n)}x_n\|) \\
&\quad + \frac{2c_nM^2}{1-[(a_n+b_n)(h_{k(n)}^2-1)+2(a_n+b_n)]} \\
&\quad + \frac{2M(a_n+b_n)\max\{\nu_n^i,\delta_n^i\}}{1-[(a_n+b_n)(h_{k(n)}^2-1)+2(a_n+b_n)]} \\
&= \left[1 + \frac{-2(a_n+b_n)+(a_n+b_n)^2+(a_n+b_n)(h_{k(n)}^2-1)+2(b_n+b_n)}{1-[(a_n+b_n)(h_{k(n)}^2-1)+2(a_n+b_n)]}\right]\|\zeta_{n-1}-p\|^2 \\
&\quad + \frac{2c_nM^2}{1-[(a_n+b_n)(k_n^2-1)+2(a_n+b_n)]} \\
&\quad + \frac{2M(a_n+b_n)\max\{\nu_n^i,\delta_n^i\}}{1-[(a_n+b_n)(h_{k(n)}^2-1)+2(a_n+b_n)]} \\
&\quad - \frac{2(a_n+b_n)}{1-[(a_n+b_n)(h_{k(n)}^2-1)+2(a_n+b_n)]}\phi(\|x_n - R_{i(n)}^{k(n)}x_n\|). \qquad (3.15)
\end{aligned}$$

Since $a_n + b_n \to 0$ and $h_{k(n)} \to 1$ then there exists a natural number n_0 such that

$$1 - [(a_n+b_n)(k_n^2-1)+2(a_n+b_n)] \ge \frac{1}{2} \ \ \forall n \ge n_0.$$

From (3.15), we have

$$\begin{aligned}
\|\zeta_n - p\|^2 &\leq [1 + 2\{-2(a_n + b_n) + (a_n + b_n)^2 + (a_n + b_n)(h^2_{k(n)} - 1) + 2(a_n + b_n)\}] \\
&\quad \times\|\zeta_{n-1} - p\|^2 + 4M(a_n + b_n)\max\{\nu^i_n, \delta^i_n\} + 4c_nM^2 \\
&\quad -2(a_n + b_n)\phi(\|\zeta_n - R^{k(n)}_{i(n)}\zeta_n\|) \\
&= [1 - 2(a_n + b_n)]\|\zeta_{n-1} - p\|^2 + 2[(a_n + b_n)^2 + (a_n + b_n)h^2_{k(n)}]\|\zeta_{n-1} - p\|^2 \\
&\quad +4M(a_n + b_n)\max\{\nu^i_n, \delta^i_n\} + 4c_nM^2 \\
&\quad -2(a_n + b_n)\phi(\|\zeta_n - R^{k(n)}_{i(n)}\zeta_n\|) \qquad (3.16)
\end{aligned}$$

Since $\phi(q) \geq 0$ for all $q \geq 0$, then for all $n \geq n_0$, it follows from (3.16) that

$$\begin{aligned}
\|\zeta_n - p\|^2 &\leq [1 - 2(a_n + b_n)]\|\zeta_{n-1} - p\|^2 + 2[(a_n + b_n)^2 + (a_n + b_n)h^2_{k(n)}]M^2 \\
&\quad +4M(a_n + b_n)\max\{\nu^i_n, \delta^i_n\} + 4c_nM^2 \\
&= [1 - 2(a_n + b_n)]\|\zeta_{n-1} - p\|^2 + 2(a_n + b_n)\{[(a_n + b_n) + h^2_{k(n)}]M^2 \\
&\quad +2\max\{\nu^i_n, \delta^i_n\}]\} + 4c_nM^2.
\end{aligned}$$

For all $n \geq 1$, put

$$\begin{aligned}
\rho_n &= \|\zeta_{n-1} - p\|, \\
\theta_n &= 2(a_n + b_n), \\
\omega_n &= 2(a_n + b_n)\{[(a_n + b_n) + h^2_{k(n)}]M^2 \\
&\quad +2\max\{\nu^i_n, \delta^i_n\}]\} \text{ and} \\
\varpi_n &= 4c_nM^2
\end{aligned}$$

then by Lemma 2.2, we obtain that

$$\lim_{n\to\infty} \|\zeta_n - p\| = 0. \qquad (3.17)$$

This completes the prove of Theorem 3.1.

Theorem 3.1 extends, generalizes and and improves the corresponding results of Su and Li [49], Sun [50], Xu and Ori [54], Osilike and Isiogugu [39], Schu [44], Yang [56], Cianciaruso [12], Igbokwe and Ini [25], Igbokwe and Jim [26]-[27], Gu [18], Thahur [52], Jim [30] and Jim et al. [31] and several others in the existing literatures.

Using the method of proof in Theorem 3.1, we have the following results.

Corollary 3.2. *Let K be a nonempty closed convex subset of a real Banach space E. Let $N \geq 1$ be a positive integer and $I = \{1, 2, 3, ..., N\}$. Let $R_i : K \to K$ be a finite family of asymptotically ϕ-demicontractive mappings with sequence $\{\lambda_{in}\} \subset [1, \infty)$, where $\lambda_{in} \to 1$ as $n \to \infty$, for each $i \in I$. Furthermore, let $R_i(K)$ be bounded and R_i be uniformly continuous for each $i \in I$. Assume that $\boldsymbol{F} = \bigcap_{i=1}^N F(R_i) \neq \emptyset$. Let $\{u_n\}$, $\{v_n\}$ be bounded in K and $\{a_n\}$, $\{c_n\}$, $\{a'_n\}$, $\{c'_n\}$ be sequences in [0,1] such that $a_n + c_n \leq 1$, $a'_n + c'_n \leq 1$, for each $n \geq 1$. Assume that the following conditions are satisfied:*

(i) $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} c_n = \lim\limits_{n\to\infty} a'_n = \lim\limits_{n\to\infty} c'_n = 0$

(ii) $\sum\limits_{n=1}^{\infty} a_n = \infty$;

(iii) $\sum_{n=1}^{\infty} c_n < \infty$.

Let $\{\zeta_n\}$ be a sequence generated by

$$\begin{cases} \zeta_0 \in K, \\ \zeta_n = (1 - a_n - c_n)\zeta_{n-1} + a_n R_{i(n)}^{k(n)} \mu_n + c_n u_n, \qquad \forall n \geq 1. \\ \mu_n = (1 - a'_n - c'_n)\zeta_{n-1} + a'_n R_{i(n)}^{k(n)} \zeta_n + c'_n v_n \end{cases} \tag{3.18}$$

Then the sequence $\{\zeta_n\}$ converges strongly to a point in $\mathbf{F}$

Proof. Take $b_n = b'_n = a''_n = b''_n = 0$ in Theorem 3.1.

Corollary 3.3. *Let K be a nonempty closed convex subset of a real Banach space E. Let $N \geq 1$ be a positive integer and $I = \{1, 2, 3, ..., N\}$. Let $R_i : K \to K$ be a finite family of asymptotically ϕ-demicontractive mappings with sequence $\{\lambda_{in}\} \subset [1, \infty)$, where $\lambda_{in} \to 1$ as $n \to \infty$, for each $i \in I$. Furthermore, let $R_i(K)$ be bounded and R_i be uniformly continuous for each $i \in I$. Assume that $\boldsymbol{F} = \bigcap_{i=1}^{N} F(R_i) \neq \emptyset$. Let $\{a_n\}$ and $\{a'_n\}$ be sequences in [0,1], for each $n \geq 1$. Assume that the following conditions are satisfied:*

(i) $\lim_{n\to\infty} a_n = \lim_{n\to\infty} a'_n = 0$

(ii) $\sum_{n=1}^{\infty} a_n = \infty$.

Let $\{\zeta_n\}$ be a sequence generated by

$$\begin{cases} \zeta_0 \in K, \\ \zeta_n = (1 - a_n)\zeta_{n-1} + a_n R_{i(n)}^{k(n)} \mu_n, \qquad \forall n \geq 1, \\ \mu_n = (1 - a'_n)\zeta_{n-1} + a'_n R_{i(n)}^{k(n)} \zeta_n \end{cases} \tag{3.19}$$

Then the sequence $\{\zeta_n\}$ converges strongly to a point in $\mathbf{F}$

Proof. Set $c_n = c'_n = 0$ in Corollary 3.2

Corollary 3.4. *Let K be a nonempty closed convex subset of a real Banach space E. Let $N \geq 1$ be a positive integer and $I = \{1, 2, 3, ..., N\}$. Let $R_i : K \to K$ be a finite family of asymptotically ϕ-demicontractive mappings with sequence $\{\lambda_{in}\} \subset [1, \infty)$, where $\lambda_{in} \to 1$ as $n \to \infty$, for each $i \in I$. Furthermore, let $R_i(K)$ be bounded and R_i be uniformly continuous for each $i \in I$. Assume that $\boldsymbol{F} = \bigcap_{i=1}^{N} F(R_i) \neq \emptyset$. Let $\{u_n\}$ be bounded in K and $\{a_n\}$, $\{c_n\}$ be sequences in [0,1] such that $a_n + c_n \leq 1$, for each $n \geq 1$. Assume that the following conditions are satisfied:*

(i) $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = 0$

(ii) $\sum_{n=1}^{\infty} a_n = \infty$;

(iii) $\sum_{n=1}^{\infty} c_n < \infty$.

Let $\{\zeta_n\}$ be a sequence generated by

$$\begin{cases} \zeta_0 \in K, \\ \zeta_n = (1 - a_n - c_n)\zeta_{n-1} + a_n R_{i(n)}^{k(n)} \zeta_{n-1} + c_n u_n, \qquad \forall n \geq 1, \end{cases} \tag{3.20}$$

Then the sequence $\{\zeta_n\}$ converges strongly to a point in $\mathbf{F}$

Proof. Take $a'_n = c'_n = 0$ in Corollary 3. 2

Corollary 3.5. *Let K be a nonempty closed convex subset of a real Banach space E. Let $N \geq 1$ be a positive integer and $I = \{1, 2, 3, ..., N\}$. Let $R_i : K \to K$ be a finite family of asymptotically ϕ-demicontractive mappings with sequence $\{\lambda_{in}\} \subset [1, \infty)$, where $\lambda_{in} \to 1$ as $n \to \infty$, for each $i \in I$. Furthermore, let $R_i(K)$ be bounded and R_i be uniformly continuous for each $i \in I$. Assume that $\boldsymbol{F} = \bigcap_{i=1}^{N} F(R_i) \neq \emptyset$. Let $\{a_n\}$ be a sequences in [0,1], for each $n \geq 1$. Assume that the following conditions are satisfied:*

(i) $\lim\limits_{n\to\infty} a_n = 0$

(ii) $\sum\limits_{n=1}^{\infty} a_n = \infty$.

Let $\{\zeta_n\}$ be a sequence generated by

$$\begin{cases} \zeta_0 \in K, \\ \zeta_n = (1 - a_n)\zeta_{n-1} + a_n R_{i(n)}^{k(n)} \zeta_{n-1}, \end{cases} \qquad \forall n \geq 1, \tag{3.21}$$

Then the sequence $\{\zeta_n\}$ converges strongly to a point in $\mathbf{F}$

Proof. Take $c_n = 0$ in Corollary 3.4

Corollary 3.6. *Let K be a nonempty closed convex subset of a real Banach space E. Let $N \geq 1$ be a positive integer and $I = \{1, 2, 3, ..., N\}$. Let $R_i : K \to K$ be a finite family of asymptotically ϕ-demicontractive mappings with sequence $\{\lambda_{in}\} \subset [1, \infty)$, where $\lambda_{in} \to 1$ as $n \to \infty$, for each $i \in I$. Furthermore, let $R_i(K)$ be bounded and R_i be uniformly continuous for each $i \in I$. Assume that $\boldsymbol{F} = \bigcap_{i=1}^{N} F(R_i) \neq \emptyset$. Let $\{u_n\}$, $\{w_n\}$ be bounded in K and $\{b_n\}$, $\{c_n\}$, $\{a''_n\}$ and $\{b''_n\}$ be sequences in [0,1] such that $b_n + c_n \leq 1$ and $a''_n + b''_n \leq 1$, for each $n \geq 1$. Assume that the following conditions are satisfied:*

(i) $\lim\limits_{n\to\infty} b_n = \lim\limits_{n\to\infty} c_n = \lim\limits_{n\to\infty} a''_n = \lim\limits_{n\to\infty} b''_n = 0$;

(ii) $\sum\limits_{n=1}^{\infty} b_n = \infty$;

(iii) $\sum\limits_{n=1}^{\infty} c_n < \infty$.

Let $\{\zeta_n\}$ be a sequence generated by

$$\begin{cases} \zeta_0 \in K, \\ \zeta_n = (1 - b_n - c_n)x_{n-1} + b_n R_{i(n)}^{k(n)} \gamma_n + c_n u_n, \\ \gamma_n = (1 - a''_n - b''_n)\zeta_n + a''_n R_{i(n)}^{k(n)} \zeta_n + b''_n w_n \end{cases} \qquad \forall n \geq 1, \tag{3.22}$$

Then the sequence $\{\zeta_n\}$ converges strongly to a point in $\mathbf{F}$.

Proof. Take $a_n = a'_n = b'_n = c'_n = 0$ in Theorem 3.1.

Corollary 3.7. *Let K be a nonempty closed convex subset of a real Banach space E. Let $N \geq 1$ be a positive integer and $I = \{1, 2, 3, ..., N\}$. Let $R_i : K \to K$ be a finite family of asymptotically ϕ-demicontractive mappings with sequence $\{\lambda_{in}\} \subset [1, \infty)$, where $\lambda_{in} \to 1$ as $n \to \infty$, for each $i \in I$. Furthermore, let $R_i(K)$ be bounded and R_i be uniformly continuous for each $i \in I$. Assume that $\boldsymbol{F} = \bigcap_{i=1}^{N} F(R_i) \neq \emptyset$. Let $\{b_n\}$ and $\{a''_n\}$ be sequences in [0,1], for each $n \geq 1$. Assume that the following conditions are satisfied:*

(i) $\lim\limits_{n\to\infty} b_n = \lim\limits_{n\to\infty} a''_n = 0$;

(ii) $\sum\limits_{n=1}^{\infty} b_n = \infty$;

Let $\{\zeta_n\}$ be a sequence generated by

$$\begin{cases} \zeta_0 \in K, \\ \zeta_n = (1-b_n)\zeta_{n-1} + b_n R_{i(n)}^{k(n)}\gamma_n & \forall n \geq 1, \\ \gamma_n = (1-a_n'')\zeta_n + a_n'' R_{i(n)}^{k(n)}\zeta_n \end{cases} \tag{3.23}$$

Then the sequence $\{\zeta_n\}$ converges strongly to a point in $\mathbf{F}$.

Proof. Take $c_n = b_n'' = 0$ in Corollary 3.6.

Corollary 3.8. *Let K be a nonempty closed convex subset of a real Banach space E. Let $N \geq 1$ be a positive integer and $I = \{1, 2, 3, ..., N\}$. Let $R_i : K \to K$ be a finite family of asymptotically ϕ-demicontractive mappings with sequence $\{\lambda_{in}\} \subset [1, \infty)$, where $\lambda_{in} \to 1$ as $n \to \infty$, for each $i \in I$. Furthermore, let $R_i(K)$ be bounded and R_i be uniformly continuous for each $i \in I$. Assume that $\boldsymbol{F} = \bigcap_{i=1}^{N} F(R_i) \neq \emptyset$. Let $\{u_n\}$ be bounded in K and $\{b_n\}$, $\{c_n\}$ be sequences in [0,1] such that $b_n + c_n \leq 1$, for each $n \geq 1$. Assume that the following conditions are satisfied:*

(i) $\lim_{n\to\infty} b_n = \lim_{n\to\infty} c_n = 0$;

(ii) $\sum_{n=1}^{\infty} b_n = \infty$;

(iii) $\sum_{n=1}^{\infty} c_n < \infty$.

Let $\{\zeta_n\}$ be a sequence generated by

$$\begin{cases} \zeta_0 \in K, \\ \zeta_n = (1-b_n-c_n)\zeta_{n-1} + b_n R_{i(n)}^{k(n)}\zeta_n + c_n u_n, & \forall n \geq 1, \end{cases} \tag{3.24}$$

Then the sequence $\{\zeta_n\}$ converges strongly to a point in $\mathbf{F}$.

Proof. Take $a_n'' = b_n'' = 0$ in Corollary 3.6.

Corollary 3.9. *Let K be a nonempty closed convex subset of a real Banach space E. Let $N \geq 1$ be a positive integer and $I = \{1, 2, 3, ..., N\}$. Let $T_i : K \to K$ be a finite family of asymptotically ϕ-demicontractive mappings with sequence $\{\lambda_{in}\} \subset [1, \infty)$, where $\lambda_{in} \to 1$ as $n \to \infty$, for each $i \in I$. Furthermore, let $R_i(K)$ be bounded and T_i be uniformly continuous for each $i \in I$. Assume that $\boldsymbol{F} = \bigcap_{i=1}^{N} F(R_i) \neq \emptyset$. Let $\{b_n\}$ be a sequence in [0,1] for each $n \geq 1$. Assume that the following conditions are satisfied:*

(i) $\lim_{n\to\infty} b_n = 0$;

(ii) $\sum_{n=1}^{\infty} b_n = \infty$;

Let $\{\zeta_n\}$ be a sequence generated by

$$\begin{cases} \zeta_0 \in K, \\ \zeta_n = (1-b_n)\zeta_{n-1} + b_n R_{i(n)}^{k(n)}\zeta_n & \forall n \geq 1, \end{cases} \tag{3.25}$$

Then the sequence $\{\zeta_n\}$ converges strongly to a point in $\mathbf{F}$.

Proof. Set $c_n = 0$ in Corollary 3.8.

This is just to state but a few of the numerous results that can be obtained from Theorem 3.1.

Remark 3.10.
If we drop the bounded range condition, it can be proved that Theorem 3.1 and the related Corollaries are valid for the class of Lipschitz asymptotically ϕ-demicontractive mappings.

Example 3.11. Let E be the real line with the usual norm $|\cdot|$ and $K = [-1, 1]$. For $N = 2$, Define $R_1, R_2 : K \to K$ by

$$\begin{aligned} R_1\zeta &= \sin\zeta, \forall\zeta \in K \\ R_2\zeta &= \sin(-\zeta), \forall\zeta \in K. \end{aligned}$$

Then the following are satisfied:

(i) R_1 and R_2 are quasi-nonexpansive mappings, it follows that R_1 and R_2 are asymptotically quasi-nonexpansive mappings with constant sequence $\{h_n\} = \{1\}$. Hence, they are asymptotically ϕ-demicontractive mappings. Clearly, R_1 and R_2 have bounded ranges and are also uniformly continuous on [-1,1].

(ii) Obviously, $R_1(0) = 0$, $R_2(0) = 0$, that is, 0 is the common fixed point of R_1 and R_2, that is, $\mathbf{F} = F(R_1) \bigcap F(R_2) = \{0\}$.

Put

$$a_n = b_n = \frac{1}{n+2},\ c_n = \frac{1}{n^2},\ a'_n = b'_n = c'_n = \frac{1}{2(n+1)},\ a''_n = b''_n = \frac{1}{n+1}.$$

Observe that all the conditions of Theorem 3.1 are satisfied. Hence, Theorem 3.1 is applicable.

4. Conclusion

Our three-step implicit iteration process properly includes the iteration processes (1.17)-(1.26) and also the class of asymptotically ϕ-demicontractive mappings is more general than those mentioned the literature. Hence, our result extends, generalizes and improves the corresponding results of Su and Li [49], Sun [50], Xu and Ori [54], Osilike and Isiogugu [39], Schu [44], Yang [56], Cianciaruso [12], Igbokwe and Ini [25], Igbokwe and Jim [26]-[27], Gu [18], Thahur [52], Jim [30] and Jim et al. [31].

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