Available online at http://www.cna-journal.com
Commun. Nonlinear Anal. 2 (2020), 1–8
Research Article

Communications in Nonlinear Analysis
Journal Homepage: www.cna-journal.com

Some results for uniformly Lipchizian mappings in convex modular spaces

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Abstract

In this paper, we introduce the concept of generalized uniformly L-Lipchizian mappings with constant u and a strong convergence theorem for a pair of generalized uniformly L-Lipchizian mappings in convex modular spaces. Our work generalizes and extends a good number of results in this area of research.

Keywords: uniformly L-Lipchizian, asymptotically nonexpansive, asymptotically pseudo-contractive mapping, convex modular space, Fixed point.
2010 MSC: 37C25, 47H10, 54H25, 55M20.

1. Introduction and preliminaries

Nakano [10] initiated the theory of modular space in connection with the theory of order spaces and this space was later generalized by Musielak and Orlicz [9]. This space generalizes the normed linear space. By defining a norm, particular Banach spaces of functions can be considered. Metric fixed theory for these Banach spaces of functions has been widely studied [3,5]. Many problems in fixed point theory for nonexpansive mappings can be formulated in modular spaces.
The purpose of this work is to improve and extend the work of Shih, Yeol and Jong and many other related work in literature (see[1-13]). Throughout this work, we assume that X_ρ is a modular space, X_ρ^* is the dual space of X_ρ,K is a nonempty closed convex subset of X_ρ and $J : X_\rho \to 2^{X_\rho^*}$ is the normalized duality mapping defined by

$$J(x) = \{f \in X_\rho^* : \langle x, f\rangle = \rho(x)^2 = \rho(f)^2, \rho(x) = \rho(f)\} \forall x \in X_\rho$$

where $\langle ., .\rangle$ denotes the duality pairing between X_ρ and X_ρ^*. The single -valued normalized duality mapping is denoted by j.

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Received 2020-04-12

Let X be a modular space, K, a nonempty closed convex subset of X and $T : K \to K$ a mapping,

(i) T is said to be uniformly L-Lipchizian if there exists $L > 0$ such that for any $x, y \in K$

$$\rho(T^n x - T^n y) = L\rho(x - y)$$

,
for all $n \geq 1$ and generalized uniformly L-Lipchizian if

$$\rho(T^n x - T^n y) = L(\rho(x - y) + u)$$

,
where $u = \{a \in Z : 0 \leq a \leq 1\}$ for all $n \geq 1$;

(ii) T is said to be asymptotically nonexpansive if there exists a sequence $\{k_n\}$ in $[0, \infty)$ with $k_n \to 1$ such that for any given $x, y \in K$,

$$\rho(T^n x - T^n y) = k_n \rho(x - y), \forall n \geq 1$$

;

(iii) T is said to be asymptotically pseudo-contractive if there exists a sequence $\{k_n\}$ in $[0, \infty)$ with $k_n \to 1$ such that for any given $x, y \in K$, there exists $j(x, y) \in J(x, y)$ such that

$$\langle T^n x - T^n y, j(x - y)\rangle = k_n \rho(x - y), \forall n \geq 1$$

;

Remark 1. It is easy to see that if T is asymptotically nonexpansive mapping, then T is uniformly L-Lipchizian mapping, where $L = Sup_{n\geq 1} k_n$ and every asymptotically nonexpansive mapping is asymptotically pseudo-contractive but the inverse is not true. The following definitions are already in literature.

Definition 1.1[6]. Let X be an arbitrary vector space over $K(= \mathbb{R}\ or\ \mathbb{C})$. A functional $\rho : X \to [0, \infty)$ is called modular if:

(i) $\rho(x) = 0$ if and only if $x = 0$.

(ii) $\rho(\alpha x) = \rho(x)$ for $\alpha \in K$ with $|\alpha| = 1$ for all $x \in X$.

(iii) $\rho(\alpha x + \beta y) = \rho(x) + \rho(y)$ if $\alpha, \beta \geq 0$, $\alpha + \beta = 1$ for all $x, y \in X$.

Definition 1.2[6]. Let X be an arbitrary vector space over $K(= \mathbb{R}\ or\ \mathbb{C})$. A functional $\rho : X \to [0, \infty)$ is called convex modular if:

(i) $\rho(x) = 0$ if and only if $x = 0$.

(ii) $\rho(\alpha x) = \rho(x)$ for $\alpha \in K$ with $|\alpha| = 1$ for all $x \in X$.

(iii) $\rho(\alpha x + \beta y) = \alpha\rho(x) + \beta\rho(y)$ if $\alpha, \beta \geq 0$, $\alpha + \beta = 1$ for all $x, y \in X$.

Definition 1.3[6]. Let X be an arbitrary vector space over $K(= \mathbb{R}\ or\ \mathbb{C})$. A functional $\rho : X \to [0, \infty)$ is called s-convex modular if:

(i) $\rho(x) = 0$ if and only if $x = 0$.

(ii) $\rho(\alpha x) = \rho(x)$ for $\alpha \in K$ with $|\alpha| = 1$ for all $x \in X$.

(iii) $\rho(\alpha x + \beta y) = \alpha^s\rho(x) + \beta^s\rho(y)$ if $\alpha, \beta \geq 0$, $\alpha^s + \beta^s = 1$ with $s \in [0, 1)$ for all $x, y \in X$.

Lemma 1.4. Let $\{\theta_n\}_{n\geq 0}$ be a nonnegative sequence which satisfies the following inequality

$$\theta_{n+1} \quad \leq \quad (1-\lambda_n)\theta_n + \beta_n, \;\; n \geq 0,$$

where $\lambda_n \in (0,1), \; \sum_{n=1}^{\infty} \lambda_n = \infty, \; \beta_n = o(\lambda_n),$ then $lim_{n\to\infty}\theta_n = 0$

Lemma 1.5. Let X_ρ be a modular space and $J : X_\rho \to 2^{X_\rho^*}$ be the nomalized duality mapping, then for any $x, y \in X_\rho$,

$$\rho(x+y)^2 \leq \rho(x)^2 + 2\langle y, j(x+y)\rangle, \forall j(x+y) \in J(x+y)$$

.

Lemma 1.6. Let $\{a_n\}$ and $\{b_n\}$ be two nonnegative real sequences satisfying the following conditions:

$$a_{n+1} \leq (1+\lambda_n)a_n + b_n, \forall n \geq n_0$$

where $\{\lambda_n\}$ is a sequence in $(0,1)$ with $\sum_{n=0}^{\infty} \lambda_n < \infty$. If $\sum_{n=0}^{\infty} b_n < \infty$, then $Lim_{n\to\infty}a_n$ exists.

Lemma 1.7. Let $\phi : [0,\infty) \to [0,\infty)$ be an increasing function with $\phi(0) = 0$ and $\{b_n\}_{n=0}^{\infty}$ be a positive real sequence satisfying:

$$\sum_{n=0}^{\infty} b_n = +\infty$$

and

$$Lim_{n\to 0}b_n = 0$$

Suppose that $\{a_n\}_{n=0}^{\infty}$ is a nonnegative real sequence. If there exists an integer $n_0 > 0$ satisfying

$$a_{n+1}^2 < a_n^2 + o(b_n) - b_n\phi(a_{n+1}), \forall n \geq n_0$$

where

$$\lim_{n\to\infty} \frac{o(b_n)}{b_n} = 0, \; \lim_{n\to\infty} a_n = 0.$$

2. Main Results

Theorem 2.1.
Let X_ρ be a convex modular space, K a nonempty closed convex subset of X_ρ, and $T_i : K \to K$, $i = 1, 2$ be two generalized uniformly L-Lipchizian mappings with $F(T_1)\bigcap F(T_2) \neq \emptyset$ where $F(T_i)$ is the fixed point of T_i in K and x^* a point in $F(T_1)\bigcap F(T_2)$. Let k_n in $[1,\infty)$ be a sequence with $k_n \to 1$ and let α_n and β_n be two sequences in $[0,1]$ satisfying the following conditions:

(i)$\sum_{n=0}^{\infty} \alpha_n = \infty$.

(ii)$\sum_{n=0}^{\infty} \alpha_n^2 < \infty$.

(ii)$\sum_{n=0}^{\infty} \beta_n < \infty$.

(iv)$\sum_{n=0}^{\infty} \alpha_n(k_n - 1) < \infty$.
For any $x_0 \in K$, let (x_n) be the iterative sequence defined by

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T_1^n y_n \tag{2.1}$$
$$y_n = (1-\beta_n)x_n + \beta_n T_2^n x_n \forall n \geq 0.$$

If there exists a strict increasing function $\phi : [0,\infty) \to [0,\infty)$ with $\phi(0) = 0$ such that

$$\langle T^n x - x^*, j(x-x^*)\rangle = k_n\rho(x-x^*)^2 - \phi(\rho(x-x^*)) + u, \forall n \geq 0$$

;
then $\{x_n\}$ converges strongly to x^*.

Proof:
It follows from (2.1) and Lemma 1.3 that .

$$\begin{aligned}
\rho(x_{n+1}-x^*)^2 &= \rho((1-\alpha_n)x_n + \alpha_n T_1^n y_n - x^*)^2 \\
&= \rho((1-\alpha_n)(x_n - x^*) + \alpha_n(T_1^n y_n - x^*))^2 \\
&\leq (1-\alpha_n)^2\rho(x_n-x^*)^2 + 2\alpha_n\langle T_1^n y_n - x^*, j(x_{n+1}-x^*)\rangle \\
&\leq (1-\alpha_n)^2\rho(x_n-x^*)^2 + 2\alpha_n\langle T_1^n x_{n+1} - x^*, j(x_{n+1}-x^*)\rangle \\
&\quad +2\alpha_n\langle T_1^n y_n - T_1^* x_{n+1}, j(x_{n+1}-x^*)\rangle \\
&\leq (1-\alpha_n)^2\rho(x_n-x^*)^2 + 2\alpha_n(k_n\rho(x_{n+1}-x^*)^2 \\
&\quad -\phi(\rho(x_{n+1}-x^*)) + u) + 2\alpha_n(L\rho(y_n - x_{n+1}) \\
&\quad +uL)\rho(x_{n+1}-x^*)
\end{aligned}$$

but

$$\begin{aligned}
\rho(x_{n+1}-y_n) &= \rho((1-\alpha_n)x_n + \alpha_n T_1^n y_n - y_n) \\
&= \rho((1-\alpha_n)(x_n - y_n) + \alpha_n(T_1^n y_n - y_n)) \\
&\leq (1-\alpha_n)\rho(x_n-y_n) + \alpha_n\rho(T_1^n y_n - x^*) + \alpha_n\rho(y_n - x^*) \\
&\leq (1-\alpha_n)\rho(x_n-y_n) + \alpha_n(1+L)\rho(y_n-x^*) + \alpha_n uL \\
&\leq (1-\alpha_n)\rho(x_n-y_n) + \alpha_n(1+L)(\rho(y_n-x_n) + \rho(x_n-x^*)) \\
&\quad +\alpha_n uL \\
&\leq (1+\alpha_n L)\rho(x_n-y_n) + \alpha_n(1+L)\rho(x_n-x^*) + \alpha_n uL \\
&\leq (1+\alpha_n L)\rho(\beta_n x_n - \beta_n T_2^n x_n) + \alpha_n(1+L)\rho(x_n-x^*) \\
&\quad +\alpha_n uL \\
&\leq (1+\alpha_n L)\beta_n(\rho(T_2^n x_n - x^*) + \rho(x_n-x^*)) \\
&\quad +\alpha_n(1+L)\rho(x_n-x^*) + \alpha_n uL \\
&\leq (1+\alpha_n L)\beta_n(L\rho(x_n-x^*) + uL) \\
&\quad +(1+\alpha_n L)\beta_n\rho(x_n-x^*) \\
&\quad +\alpha_n(1+L)\rho(x_n-x^*) + \alpha_n uL \\
&\leq (1+\alpha_n L)\beta_n L\rho(x_n-x^*) + (1+\alpha_n L)\beta_n uL \\
&\quad +(1+\alpha_n L)\beta_n\rho(x_n-x^*) \\
&\quad +\alpha_n(1+L)\rho(x_n-x^*) + \alpha_n uL \\
&\leq (1+L)[(1+\alpha_n L)\beta_n + \alpha_n]\rho(x_n-x^*) + uL[(1+\alpha_n L)\beta_n + \alpha_n] \\
&\leq (1+L)f_n\rho(x_n-x^*) + uLf_n
\end{aligned}$$

where $f_n = [(1+\alpha_n L)\beta_n + \alpha_n]$.
By the conditions (i), (ii) and (iii), we have

$$\sum_{n=0}^{\infty} \alpha_n f_n < \infty \tag{2.2}$$

Substitute (2.3) in (2.2),

$$\begin{aligned}
\rho(x_{n+1}-x^*)^2 &= (1-\alpha_n)^2\rho(x_n-x^*)^2 + 2\alpha_n(k_n\rho(x_{n+1}-x^*)^2 \\
&\quad -\phi(\rho(x_{n+1}-x^*)) + u) + 2\alpha_n(L\rho(y_n - x_{n+1}) \\
&\quad +uL)\rho(x_{n+1}-x^*) \\
&\leq (1-\alpha_n)^2\rho(x_n-x^*)^2 + 2\alpha_n k_n\rho(x_{n+1}-x^*)^2 \\
&\quad -2\alpha_n\phi(\rho(x_{n+1}-x^*)) + 2u\alpha_n \\
&\quad +2\alpha_n L[(1+L)f_n\rho(x_n-x^*) + uLf_n]\rho(x_{n+1}-x^*) \\
&\quad +2uL\alpha_n\rho(x_{n+1}-x^*) \\
&\leq \frac{G_n}{H_n}\rho(x_n-x^*)^2 + \frac{2u\alpha_n}{H_n} - \frac{2\alpha_n\phi(\rho(x_{n+1}-x^*))}{H_n} \\
&\quad +\frac{2u\alpha_n(f_nL^2+L)}{H_n}\rho(x_{n+1}-x^*) \\
&\leq \{1+\frac{2\alpha_n(k_n-1)+\alpha_n^2}{H_n}\}\rho(x_n-x^*)^2 + \frac{2u\alpha_n}{H_n} \\
&\quad -\frac{2\alpha_n\phi(\rho(x_{n+1}-x^*))}{H_n} + \frac{2u\alpha_n(f_nL^2+L)}{H_n}\rho(x_{n+1}-x^*)
\end{aligned}$$

where
$G_n = 1-2\alpha_n+\alpha_n^2+\alpha_n L(1+L)f_n$ and $H_n = 1-2\alpha_n k_n - \alpha_n L(1+L)f_n$
Since $\alpha_n \to 0$ as $n \to \infty$ there exists a positive integer n_0 such that $\frac{1}{2} < H_n \leq 1\ \forall n \geq n_0$. it follows from (2.5) that:

$$\begin{aligned}
\rho(x_{n+1}-x^*)^2 &\leq \{1+2[2\alpha_n(k_n-1)+\alpha_n^2]\}\rho(x_n-x^*)^2 + \{2u\alpha_n\} \\
&\quad -\{2\alpha_n\phi(\rho(x_{n+1}-x^*))\} \\
&\quad +\{2u\alpha_n(f_nL^2+L)\}\rho(x_{n+1}-x^*)
\end{aligned}$$

and so

$$\rho(x_{n+1}-x^*)^2 \leq \{1+2[2\alpha_n(k_n-1)+\alpha_n^2]\}\rho(x_n-x^*)^2$$

Using the conditions (ii) and (iv)

$$2\sum_{n=0}^{\infty}[2\alpha_n(k_n-1)+\alpha_n^2] < \infty$$

It follows from Lemma (1.4) that $\rho(x_n-x^*)$ exists. Hence $\rho(x_n-x^*)$ is bounded. That is $\rho(x_n-x^*)^2 \leq M$ where M is a positive constant.
Considering (6) and setting
$\theta_n = \rho(x_n-x^*)$, $\lambda_n = 2\alpha_n$ and

$$o_n = 2[2\alpha_n(k_n-1)+\alpha_n^2]M + \{2u\alpha_n\} + \{2u\alpha_n(f_nL^2+L)\}M,$$

we have

$$\theta_{n+1}^2 \leq \theta_n^2 - \lambda_n \phi(\theta_{n+1}) + o_n, \forall n \geq n_0$$

. Hence, the conditions in Lemma (1.5) are satisfied.
Therefore, $\rho(x_n - x^*) \to 0$ that is $x_n \to 0$ as $n \to \infty$. This ends the proof.
In Theorem 2.1, if u = 0, the conclusion is as follows:

Corollary 2.2.
Let X_ρ be a convex modular space, K a nonempty closed convex subset of X_ρ, and $T_i : K \to K$, $i = 1, 2$ be two uniformly L-Lipchizian mappings with $F(T_1) \bigcap F(T_2) \neq \emptyset$ where $F(T_i)$ is the fixed point of T_i in K and x^* a point in $F(T_1) \bigcap F(T_2)$. Let k_n in $[1, \infty)$ be a sequence with $k_n \to 1$ and let α_n and β_n be two sequences in $[0, 1]$ satisfying the following conditions:

(i)$\sum_{n=0}^{\infty} \alpha_n = \infty$.

(ii)$\sum_{n=0}^{\infty} \alpha_n^2 < \infty$.

(ii)$\sum_{n=0}^{\infty} \beta_n < \infty$.

(iv)$\sum_{n=0}^{\infty} \alpha_n (k_n - 1) < \infty$.

For any $x_0 \in K$, let (x_n) be the iterative sequence defined by

$$x_{n+1} = (1 - \alpha_n) x_n + \alpha_n T_1^n y_n$$

$$y_n = (1 - \beta_n) x_n + \beta_n T_2^n x_n \forall n \geq 0.$$

If there exists a strict increasing function $\phi : [0, \infty) \to [0, \infty)$ with $\phi(0) = 0$ such that

$$\langle T^n x - x^*, j(x - x^*) \rangle = k_n \rho(x - x^*)^2 - \phi(\rho(x - x^*)), \forall n \geq 0$$

;
then $\{x_n\}$ converges strongly to x^*.
In Theorem 2.1, if u = 1, we have:

Corollary 2.3.
Let X_ρ be a convex modular space, K a nonempty closed convex subset of X_ρ, and $T_i : K \to K$, $i = 1, 2$ be two generalized uniformly L-Lipchizian mappings with $F(T_1) \bigcap F(T_2) \neq \emptyset$ where $F(T_i)$ is the fixed point of T_i in K and x^* a point in $F(T_1) \bigcap F(T_2)$. Let k_n in $[1, \infty)$ be a sequence with $k_n \to 1$ and let α_n and β_n be two sequences in $[0, 1]$ satisfying the following conditions:

(i)$\sum_{n=0}^{\infty} \alpha_n = \infty$.

(ii)$\sum_{n=0}^{\infty} \alpha_n^2 < \infty$.

(ii)$\sum_{n=0}^{\infty} \beta_n < \infty$.

(iv)$\sum_{n=0}^{\infty} \alpha_n (k_n - 1) < \infty$.

For any $x_0 \in K$, let (x_n) be the iterative sequence defined by

$$x_{n+1} = (1 - \alpha_n) x_n + \alpha_n T_1^n y_n$$

$$y_n = (1 - \beta_n) x_n + \beta_n T_2^n x_n \forall n \geq 0.$$

If there exists a strict increasing function $\phi : [0,\infty) \to [0,\infty)$ with $\phi(0) = 0$ such that

$$\langle T^n x - x^*, j(x - x^*)\rangle = k_n\rho(x - x^*)^2 - \phi(\rho(x - x^*)) + 1, \forall n \geq 0$$

;
then $\{x_n\}$ converges strongly to x^*.

Remark 2.

(1) Theorem 2.1 extends and improve the results of S. S. Chang, Y. J. Cho, J. K. Kim [4].

(2) If $u = 0$ and $\rho(x) = \|x\|$ in Theorem 2.1, we have the result of the authors in [4] but if $u = 1$, our result will be a generalization of some results in this area of research.

(3) Undersuitable conditions, the sequence $\{x_n\}$ in Theorem 2.1 can also be generalized to the iterative scheme with errors.

Example 2.4.
Let $X = R$, $K = [0.1]$ and $T : K \to K$ be a map defined by

$$Tx = \frac{x}{4}$$

.
Clearly, T is a generalized uniformly L-Lipchizian with $F(T) = 0$.
Define $\rho(x) = |x|$ and $\phi : [0,+\infty) \to [0,+\infty)$ by

$$\phi(t) = \frac{t^2}{4},$$

then ϕ is a strictly increasing function with $\phi(0) = 0$, $\forall x \in K$, $x^* \in F(T)$ we get

$$\begin{aligned}
\langle T^n x - T^n x^*, j(x - x^*)\rangle &= \langle \frac{x}{4^n} - 0, j(x - 0)\rangle \\
&= \langle \frac{x}{4^n} - 0, x\rangle \\
&\leq \frac{x}{4^n} \\
&\leq x^2 - \frac{x^2}{4} \\
&\leq x^2 - \phi(|x|)
\end{aligned}$$

obviously, T completes

$$\langle T^n x - x^*, j(x - x^*)\rangle = k_n\rho(x - x^*)^2 - \phi(\rho(x - x^*)) + u, \forall n \geq 0$$

with sequence $\{k_n\} = 1$ and u = 0. If we take $\alpha_n = \beta_n = \frac{1}{n+1} \forall n \geq 1$, for arbitrary $x_1 \in K$, the sequence $\{x_n\}_{n=1}^{\infty}$ in K defined by (1) converges strongly to the unique fixed point $x^* \in F(T)$.

Competing interests.
The authors declare that they have no competing interests.

Acknowledgment.
The authors are grateful to referees and editor for their valuable comments and corrections.

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